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Some examples of almost complex manifolds with Norden metric (**)

1 - Almost complex manifolds with a Norden metric

Let (M, J) be an almost complex manifold, $\dim M = 2n$. A *metric* g on M is said to be *Norden* if, at any point, the complex structure J is an antiisometry of the tangent space, i.e.

$$g(JX, JY) = -g(X, Y) \qquad \forall X, Y \in \chi(M).$$

The metric g is necessarily indefinite of signature (n, n) and (M, g, J) is said to be an *almost complex manifold with a Norden metric* (*Norden manifold*).

A J-basis on (M, g, J) is a basis of each tangent space T_pM $\{x_1, \ldots, x_n, Jx_1, \ldots, Jx_n\}$ such that the matrix associated to the metric g is given by

$$(g_{ij}) = \begin{pmatrix} I_n & \\ & -I_n \end{pmatrix}$$

where I_n denotes the identity matrix $n \times n$.

It is known [4] that a J-basis always exists on such manifolds.

If ∇ denotes the metric connection associated to g, let's consider on M the tensor field of type (0, 3)

$$F(X, Y, Z) = g((\nabla_X J) Y, Z) \qquad \forall X, Y, Z \in \chi(M)$$

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and the associated 1-form ψ on M

$$\psi(X) = g^{ij} F(e_i, e_j, X)$$

where X is a tangent vector at the point $p \in M$, $\{e_i\}_{i=1\dots 2n}$ a basis of the tangent space T_pM and (g^{ij}) the inverse of the matrix associated to g.

The following identities hold true

$$F(X, Y, Z) = F(X, Z, Y) = F(X, JY, JZ) \quad \forall X, Y, Z \in \chi(M).$$

Ganchev and Borisov [3] gave a *classification for Norden manifolds* obtaining the following eight classes:

(1.1) *Kähler manifolds with Norden metric*

$$F(X, Y, Z) = 0.$$

(1.2) *Conformally Kähler manifolds with Norden metric* (ω_1*-manifold*)

$$F(X, Y, Z)$$

$$= \frac{1}{2n}\{g(X, Y)\psi(Z) + g(X, Z)\psi(Y) + g(X, JY)\psi(JZ) + g(X, JZ)\psi(JY)\}.$$

(1.3) *Special complex manifolds with Norden metric* (ω_2 *manifolds*)

(i) $F(X, Y, JZ) + F(Y, Z, JX) + F(Z, X, JY) = 0$ (ii) $\psi = 0$.

(1.4) *Quasi-Kähler manifolds with Norden metric* (ω_3*-manifolds*)

$$F(X, Y, Z) + F(Y, Z, X) + F(Z, X, Y) = 0.$$

(1.5) *Complex manifolds with Norden metric* ($\omega_1 \oplus \omega_2$*-manifolds*)

$$F(X, Y, JZ) + F(Y, Z, JX) + F(Z, X, JY) = 0.$$

(1.6) *Semi-Kähler manifolds* ($\omega_2 \oplus \omega_3$*-manifolds*) $\psi = 0$.

(1.7) $\omega_1 \oplus \omega_3$*-manifolds*

$$F(X, Y, Z) + F(Y, Z, X) + F(Z, X, Y)$$

$$= \frac{1}{n} \{ g(X, Y)\psi(Z) + g(Y, Z)\psi(X) + g(Z, X)\psi(Y)$$

$$+ g(X, JY)\psi(JZ) + g(Y, JZ)\psi(JX) + g(Z, JX)\psi(JY) \}.$$

(1.8) *Almost complex manifolds with Norden metric without special conditions.*

2 - Conformal transformations on Norden manifolds

Let (M, g) and (M^0, g^0) be semi-Riemannian manifolds, and $\varnothing: M \to M^0$ a *conformal diffeomorphism*, that is, there exists a function $\sigma \in \mathfrak{F}(M)$ such that

$$g^0(X^0, Y^0) = \{ e^{2\sigma} g(X, Y) \}^0 \qquad \forall X, Y \in \chi(M) \tag{2.1}$$

where X^0 denotes the induced vector field on M^0 by X.

If (M, g) and (M^0, g^0) are semi-Riemannian conformally equivalent manifolds, and ∇, ∇^0 denote their metric connections, we have:

$$\nabla^0_{X^0} Y^0 = \{ \nabla_X Y + X(\sigma) Y + Y(\sigma) X - g(X, Y) \operatorname{grad} \sigma \}^0$$

for X^0, Y^0 arbitrary vector fields on M^0, being $\operatorname{grad} \sigma$ the vector field on M determined by

$$X(\sigma) = g(\operatorname{grad} \sigma, X) \qquad \forall X \in \chi(M).$$

Let us consider now that (M, g, J) is a Norden manifold, then by considering on M^0 the almost complex structure induced by $\varnothing$

$$J^0(X^0) = (JX)^0 \qquad \forall X^0 \in \chi(M^0)$$

it results that (M^0, g^0, J^0) is a Norden manifold and

$$(\nabla^0_{X^0} J^0) Y^0 = \{ (\nabla_X J) Y + JY(\sigma) X - Y(\sigma) JX \tag{2.2}$$

$$- g(X, JY) \operatorname{grad} \sigma + g(X, Y) J \operatorname{grad} \sigma \}^0.$$

Proposition 2.1. *If (M, g, J) and (M^0, g^0, J^0) are conformally equivalent Norden manifolds, the tensor fields F^0 and ψ^0 are determined by*

(a)
$$F^0(X^0, Y^0, Z^0) = \{e^{2\sigma}(F(X, Y, Z)$$
$$+ g(X, Z)JY(\sigma) + g(X, Y)JZ(\sigma) - g(X, JZ)Y(\sigma) - g(X, JY)Z(\sigma))\}^0$$

(b)
$$\psi^0(X^0) = \{\psi(X) + 2nJX(\sigma)\}^0 \qquad \forall X^0, Y^0, Z^0 \in \chi(M^0).$$

Proof. (a) It is a consequence of the definition of F^0 and of the identity (2.2).

(b) By considering on M the J-basis $\{E_1, \ldots, E_n, JE_1, \ldots, JE_n\}$ and on M^0 the J^0-induced basis $\{E_1^0, \ldots, E_n^0, J^0E_1^0, \ldots, J^0E_n^0\}$, we get that the matrix associated to g^0 with respect to this basis is

$$(g_{ij}^0) = (e^{2\sigma})^0 \begin{pmatrix} I_n & \\ & -I_n \end{pmatrix}$$

and that the expression of the 1-form ψ^0 is

$$\psi^0(X^0) = (e^{-2\sigma})^0 \sum_{i=1}^{n} \{F^0(E_i^0, E_i^0, X^0) - F^0(J^0E_i^0, J^0E_i^0, X^0)\}.$$

Thus the proof follows from (a).

Proposition 2.2. *If (M, g, J) is a Kähler Norden manifold, then (M^0, g^0, J^0) is a ω_1-manifold. Moreover, (M^0, g^0, J^0) is Kähler Norden if and only if the function σ is constant.*

Proof. If M is Kählerian, we have $F = 0$, $\psi = 0$. So we obtain

$$F(X, Y, Z)^0 = \frac{1}{2n}\{g(X, Y)\psi(Z) + g(X, Z)\psi(Y)$$
$$+ g(X, JY)\psi(JZ) + g(X, JZ)\psi(JY)\}^0$$

i.e., M^0 is a ω_1-manifold.

Conversely, if the Kähler class is conformally invariant, we have

$$g(X, Z)JY(\sigma)+g(X, Y)JZ(\sigma)-g(X, JZ)Y(\sigma)-g(X, JY)Z(\sigma)=0$$

and taking a J-basis, it results $\operatorname{grad}\sigma=0$.

On the other hand, using again the Proposition 2.1, we obtain

Proposition 2.3. *The subspace of the ω_1-manifolds is invariant by conformal changes on the metric.*

Proposition 2.4. *If (M, g, J) is a special complex Norden manifold, then (M^0, g^0, J^0) is a complex Norden manifold, which is not conformally Kähler. The ω_2-class is conformally invariant if and only if σ is a constant function.*

Proof. Using equation

$$F^0(X^0, Y^0, J^0Z^0)+F^0(Y^0, Z^0, J^0X^0)+F^0(Z^0, X^0, J^0Y^0)$$

$$=(e^{2\sigma})^0\{F(X, Y, JZ)+F(Y, Z, JX)+F(Z, X, JY)\}^0$$

we conclude that if M is a ω_2-manifold, then M^0 is a $\omega_1\oplus\omega_2$-manifold, which is not ω_1, by the proposition above.

If moreover we suppose that σ is constant, the proposition (2.1) shows that $\psi^0=0$, and conversely, if we suppose that the ω_2-class is conformally invariant, then

$$JX(\sigma)=0 \qquad \forall X\in\chi(M)$$

and so, σ is constant.

Proposition 2.5. *If (M, g, J) is a ω_3-manifold, then (M^0, g^0, J^0) is a $(\omega_1\oplus\omega_3)-\omega_2$ manifold. The ω_3 class is conformally invariant if and only if σ is a constant function.*

Proof. The first assertion is a direct consequence of the Proposition 2.1 and the definition of ω_3-manifold, since if M is a ω_3-manifold, then $\psi=0$ and so $JX(\sigma)^0=\frac{1}{2n}\{\psi^0(X^0)\}$.

On the other hand, if σ is constant, we have

$$F^0(X^0,\ Y^0,\ Z^0) = \{e^{2\sigma} F(X,\ Y,\ Z)\}^0 \qquad \forall X^0,\ Y^0,\ Z^0 \in \chi(M^0)$$

and then, if M is a ω_3-manifold, so is M^0.

If the ω_3-class is conformally invariant,

$$g(X,\ Y) JZ(\sigma) + g(Y,\ Z) JX(\sigma) + g(Z,\ X) JY(\sigma)$$

$$- g(X,\ JY) Z(\sigma) - g(Y,\ JZ) X(\sigma) - g(Z,\ JX) Y(\sigma) = 0$$

and using a J-basis on M, it results that σ is constant.

Using Proposition 2.4 and Proposition 2.5 a straightforward computation leads to

Proposition 2.6. *If $(M,\ g,\ J)$ is a $\omega_1 \oplus \omega_2$-manifold (resp. $\omega_1 \oplus \omega_3$), then $(M^0,\ g^0,\ J^0)$ is a Norden manifold corresponding to class $(\omega_1 \oplus \omega_2) - (\omega_1 \cup \omega_2)$ (resp. $(\omega_1 \oplus \omega_3) - (\omega_1 \cup \omega_3)$).*

Proposition 2.7. *The class $\omega_2 \oplus \omega_3$ is invariant by conformal transformations on the metric if and only if σ is constant. In general, if $(M,\ g,\ J)$ belongs to class $\omega_2 \oplus \omega_3$, then $(M^0,\ g^0,\ J^0)$ belongs to the eighth class of the Borisov-Ganchev's classification.*

Proof. The first assertion is an immediate consequence of the relation

$$\psi^0(X^0) = \{\psi(X) + 2nJX(\sigma)\}^0 .$$

If σ is not constant, the manifold $(M^0,\ g^0,\ J^0)$ can not be either $\omega_1 \oplus \omega_2$, or $\omega_1 \oplus \omega_3$ manifold, because in that case, by using the inverse transformation of $\varnothing$, we would have a contradiction with Proposition 2.6.

3 - Examples

In this section we present some examples of Norden manifolds corresponding to the classes Kähler, ω_1 and ω_2.

3.1 - Let $\mathbb{R}^{2n}$ be the $2n$-dimensional euclidean space, with the canonical complex structure J

$$J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i} \qquad J\left(\frac{\partial}{\partial y^i}\right) = -\frac{\partial}{\partial x^i} \qquad i = 1 \dots n$$

being $(x_1, \dots, x_n, y_1, \dots, y_n)$ the canonical system of global coordinates.

Considering the metric $g = \sum_{i=1}^{n} (dx^i \otimes dx^i - dy^i \otimes dy^i)$ it results that $(\mathbb{R}^{2n}, g, J)$ is a Kähler Norden manifold.

3.2 - Given the Lie Group

$$H = \{A = \begin{pmatrix} 1 & 0 \\ x & y \end{pmatrix};\ x,\ y \in \mathbb{R},\ y > 0\}$$

if we denote by (x, y) the system of global coordinates $x(A) = x$, $y(A) = y$, a basis of the left invariant vector fields on H is

$$X = y\frac{\partial}{\partial x} \qquad Y = y\frac{\partial}{\partial y}.$$

Let $\{\alpha, \beta\}$ be the dual basis of the left invariant 1-forms, and define the metric $g = \alpha \otimes \alpha - \beta \otimes \beta$ and the complex structure $J(X) = Y$, $J(Y) = -X$.

Then, the tensor field F is given by $F(X, X, X) = F(X, Y, Y) = -2$ and zero in other case.

It is not difficult to check that (H, g, J) is a Norden manifold conformally Kähler, but not Kähler.

We can generalize this example. Let consider the Lie Group

$$H(p) = \{A = \begin{pmatrix} I_{2p-1} & 0 \\ v_1 \dots v_p & w_1 \dots w_p \end{pmatrix};\ v_i,\ w_i \in \mathbb{R},\ 1 \leq i \leq p,\ \omega_p > 0\}$$

and the global system of coordinates $(x_1 \dots x_p,\ y_1 \dots y_p)$

$$x_i(A) = v_i \qquad y_i(A) = w_i \qquad 1 \leq i \leq p.$$

Then a basis of the left invariant vector fields is given by

$$X_i = y_p \frac{\partial}{\partial x^i} \qquad Y_i = y_p \frac{\partial}{\partial y^i} \qquad 1 \leq i \leq p .$$

If we define: $J(X_i) = Y_i$, $J(Y_i) = -X_i$, $1 \leq i \leq p$ $g = \sum_{i=1}^{p} (\alpha_i \otimes \alpha_i - \beta_i \otimes \beta_i)$ where $\{\alpha_i, \ldots, \alpha_p, \beta_1, \ldots, \beta_p\}$ denotes the dual basis of the left invariant 1-forms, another example of not Kähler ω_1-manifold is obtained.

3.3 - The generalized Heisenberg group $H(r, 1)$ is the group of matrices of real numbers of the form

$$A = \begin{pmatrix} 1 & X & z \\ 0 & I_r & {}^tY \\ 0 & 0 & 1 \end{pmatrix}$$

where $X, Y \in \mathbb{R}^r$, $z \in \mathbb{R}$. $H(r, 1)$ is a connected, simply connected, nilpotent Lie group of dimension $2r+1$.

A global system of coordinates (x_i, y_i, z), $1 \leq i \leq r$ on $H(r, 1)$ is given by

$$x_i(A) = x_i \qquad y_i(A) = y_i \qquad z(A) = z \qquad 1 \leq i \leq r$$

being $X = (x_1 \ldots x_r)$, $Y = (y_1 \ldots y_r)$.

Then a basis for the left invariant 1-forms on $H(r, 1)$ is given by

$$\alpha_i = \mathrm{d}x_i \qquad \beta_i = \mathrm{d}y_i \qquad \gamma = \mathrm{d}z - \sum_{k=1}^{r} x_k \, \mathrm{d}y_k \qquad 1 \leq i \leq r .$$

If we denote by $\{X_i, Y_i, Z;\ i = 1 \ldots r\}$ the global basis of the left invariant vector fields, dual of the basis of left invariant 1-forms above, we obtain $[X_i, Y_i] = Z$ $1 \leq i \leq r$ the other brackets being zero.

Now let $\Gamma(r, 1) \subset H(r, 1)$ be the discrete subgroup of matrices with integer entries, $M(r, 1) = \Gamma(r, 1) \diagdown H(r, 1)$ the space of right cosets and $\pi: H(r, 1) \to M(r, 1)$ the projection. Then, the 1-forms α_i, β_i, γ descend to $M(r, 1)$.

Denote by $\tilde{\alpha}_i$, $\tilde{\beta}_i$, $\tilde{\gamma}$, $\tilde{X}_i$, $\tilde{Y}_i$, $\tilde{Z}$ the 1-forms and the vector fields induced on $M(r, 1)$.

Let us consider the product manifold $M(r, 1) \times S^1$. Let t be the coordinate of S^1, T the vector field dual of the 1-form $dt = \eta$, and define the complex structure

$$J(\tilde{X}_{2i-1}) = \tilde{X}_{2i} \qquad J(\tilde{Y}_{2i-1}) = \tilde{Y}_{2i} \qquad J(\tilde{X}_{2i}) = -\tilde{X}_{2i-1} \qquad J(\tilde{Y}_{2i}) = -\tilde{Y}_{2i-1}$$

$$J(\tilde{X}_{2r+1}) = \tilde{Y}_{2r+1} \qquad J(\tilde{Y}_{2r+1}) = -\tilde{X}_{2r+1} \qquad J(\tilde{Z}) = T \qquad J(T) = -\tilde{Z}$$

where $1 \leqslant i \leqslant r$ and the metric

$$g = \sum_{i=1}^{r} [(\tilde{\alpha}_{2i-1} \otimes \tilde{\alpha}_{2i-1} - \tilde{\alpha}_{2i} \otimes \tilde{\alpha}_{2i}) + (\tilde{\beta}_{2i-1} \otimes \tilde{\beta}_{2i-1} - \tilde{\beta}_{2i} \otimes \tilde{\beta}_{2i})]$$
$$+ (\tilde{\alpha}_{2r+1} \otimes \tilde{\alpha}_{2r+1} - \tilde{\beta}_{2r+1} \otimes \tilde{\beta}_{2r+1}) + (\tilde{\gamma} \otimes \tilde{\gamma} - \eta \otimes \eta).$$

The Norden manifold $(M(r, 1) \times S^1, g, J)$ provides an example of ω_2-manifold.

Remark. Examples living in the class $\omega_1 \oplus \omega_2$ can be easily constructed by using conformal transformations on the metric g for the manifold of the Example 3.3.

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Summary

An almost complex manifold (M, J) with a metric such that J is an antiisometry of the tangent space at each point is said to be a Norden manifold. In this paper we study conformal transformations on this kind of manifolds and give some examples of Norden manifolds.

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