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ANNA BENINI (*)

Near-rings on certain groups (**)

1 - Introduction

In $[4]_1$ Clay gives a method for constructing all near-rings on a given additive group.

We shall call *Clay function* of an additive group G a function F: $G \to \mathrm{End}(G)$ such that the multiplication «$\cdot$» inferred in G is associative, that is $[G, \cdot]$ is a left near-ring.

Now, let $N = A +_\varphi B$ be a semidirect sum of additive groups A and B with homomorphism φ. Obviously, by Clay method, every near-ring on N can be constructed but, generally, restrictions of Clay functions on A^0 and ${}^0B$ are not Clay functions, that is, from a multiplicative view-point, isomorphic images of semidirect summands are not even sub-structures.

Furthermore, for characterizing some classes of near-rings, it is better that such images are one-sided or two-sided ideals of the constructed near-ring or, at least, of its multiplicative semigroup.

For this reason it is necessary to find conditions on Clay functions so that A^0 and ${}^0B$ are support for given structures, in particular left ideals of the multiplicative semigroup of the near-ring.

A near-ring constructed by one of the last functions is called *Φ-sum* of A and B and among the near-rings characterizable as Φ-sums, those of Def. 1 and Def. 2 of [1] and as also some geometric exemples of [3], can be seen.

The class of left permutable zero-symmetric near-rings with an idempotent non-zero element is characterizable as Φ-sum too (see [2]) and generally, a near-

(*) Indirizzo: Dipartimento di Matematica, Facoltà di Ingegneria, V.le Europa 39, I-25060 Brescia.

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ring is a Φ-sum if and only if its additive group is a semidirect sum of the additive groups of a left ideal and a left N-subgroup, respectively.

Subsequently the definition of Λ-sum of near rings is given and such a structure results to be a near-ring.

Finally we can prove that the class of abstract affine near-rings is a particular Λ-sum and then we characterize the respective Clay functions.

2 - Preliminaries

Throughout the paper N stands for a left near-ring.

In general we adhere to the notation and therminology used in $[4]_2$. In particular a near-ring $N = N_0 + N_c$ with $N_0 \neq \{0\} \neq N_c$ is called *mixed near-ring*, the additive group and the multiplicative semigroup of N are denoted by N^+ and $N^\cdot$ respectively; $S \subseteq N$ is called *ideal* of $N^\cdot$ if $SN \subseteq S$ and $NS \subseteq S$; a subgroup of N^+ which is a left (right) ideal of $N^\cdot$ is called *left* (*right*) *N-subgroup* of N.

N_d denotes the set of distributive elements of N.

$r(x) = \{\bar{x} \in N / x\bar{x} = 0\}$ is the right annihilator of x and $r(S) = \bigcap_{x \in S} r(x)$.

If $A \subseteq N$, $\mathscr{F}(A) \subseteq \mathrm{Aut}(N^+)$ denotes the subset of automorphisms of N^+ which transforms A into itself. If A is a structure, O_A denotes the zero endomorphism of A. If f, g are functions from S to T and $H \subseteq T$, we write $f =_H g$ for $f(x) - g(x) \in H$ for every x belonging to S. Moreover $\gamma_a : x \to ax \quad \forall x \in N$ is a left translation of N determined by a.

$C(A)$ denote the centre of A.

If $G = A +_\varphi B$ and $A^0 = \{\langle a, 0\rangle / a \in A\}$, ${}^0B = \{\langle 0, b\rangle / b \in B\}$, then it follows that A^0 and ${}^0B$ are subgroups of $A +_\varphi B$; $A +_\varphi B = A^0 + {}^0B$; $A^0 \cap {}^0B = \{\langle 0, 0\rangle\}$; $A^0 \sim A$ and $A +_\varphi B / A^0 \sim B$.

3 - Φ-sum of near-rings

Proposition 1. *Let* $N = A +_\varphi B$, F: $A \times B \to \mathrm{End}(N)$ *be a Clay function and «·» denotes the multiplication inferred in* N, *then an additive subgroup* $S \subseteq N$, *by «·», turns to:*

(i) *a subnear-ring of* $[N, \cdot]$ *iff* $F(S) \subseteq \mathscr{F}(S)$;

(ii) *a left N-subgroup of* $[N, \cdot]$ *iff* $F(N) \subseteq \mathscr{F}(S)$;

(iii) *a right ideal of* $[N,\cdot]$ *iff it is a normal subgroup of* N^+ *and* $F_{a+\varphi_b(\bar{a}),b+\bar{b}} = {}_S F_{a,b}$ (*) *holds for every* $\langle a, b\rangle$ *in* N *and for every* $\langle \bar{a}, \bar{b}\rangle$ *in* S;
(iv) *a right* N*-subgroup of* $[N,\cdot]$ *iff* $F_{a,b}(N) \subseteq S$ *for every* $\langle a, b\rangle \in S$.

Proof. (i) Let S be a subgroup of $N = A +_\varphi B$ and $F(S) \subseteq \mathscr{F}(S)$, then

$$\langle a, b\rangle \cdot \langle a', b'\rangle = F_{a,b}(\langle a', b'\rangle) \in S \quad \forall \langle a, b\rangle, \langle a', b'\rangle \in S$$

so S is a subnear-ring of $[N,\cdot]$. Viceversa, let S be a subnear-ring of $[N,\cdot]$, then $\langle a, b\rangle \cdot \langle a', b'\rangle \in S$ for every $\langle a, b\rangle, \langle a', b'\rangle \in S$, so $F_{a,b}(\langle a', b'\rangle) \in S$ and $F(S) \subseteq \mathscr{F}(S)$.
(ii) and (iv) analogously to (i).
(iii) Let S^+ be a normal subgroup of N^+ and let (*) be true $\forall \langle a, b\rangle \in N$, $\forall \langle \bar{a}, \bar{b}\rangle \in S$, then

$$(\langle a, b\rangle + \langle \bar{a}, \bar{b}\rangle)\langle a', b'\rangle - \langle a, b\rangle\langle a', b'\rangle$$

$$= \langle a + \varphi_b(\bar{a}), b + \bar{b}\rangle\langle a', b'\rangle - \langle a, b\rangle\langle a', b'\rangle$$

$$= F_{a+\varphi_b(\bar{a}),b+\bar{b}}(\langle a', b'\rangle) - F_{a,b}(\langle a', b'\rangle) \in S \qquad \forall \langle a', b'\rangle \in N$$

so S is a right ideal of $[N,\cdot]$. The converse is analogous.

Proposition 2. *Let* $N = A +_\varphi B$, *then a multiplication on* N *makes* A^0 *and* ${}^0B$ *left ideals of* $N^\cdot$ *iff it is inferred by a Clay function defined as follows*

$$\forall \langle a, b\rangle \in A \times B \qquad F_{a,b}(\langle a', b'\rangle) = \langle f_{a,b}(a'), \bar{f}_{a,b}(b')\rangle$$

where $f_{a,b} = f(\langle a, b\rangle)$, $\bar{f}_{a,b} = \bar{f}(\langle a, b\rangle)$, $f\colon A \times B \to \mathrm{End}(A)$, $\bar{f}\colon A \times B \to \mathrm{End}(B)$ *are functions for which the following properties are true*:

$$(1) \quad f_{a,b} {}^0 f_{a',b'} = f_{f_{a,b}(a'),\bar{f}_{a,b}(b')} \qquad (2) \quad \bar{f}_{a,b} {}^0 \bar{f}_{a',b'} = \bar{f}_{f_{a,b}(a'),\bar{f}_{a,b}(b')}$$

$$(3) \quad f_{a,b} {}^0 \varphi_{b'} = \varphi_{\bar{f}_{a,b}(b')} {}^0 f_{a,b}.$$

Proof. $F_{a,b}$ defined above is an endomorphism of N $\forall \langle a, b\rangle$ belonging to $A \times B$ by (3) and the associativity of multiplication inferred in N arises from (1) and (2), so F is a Clay function.

It is easy to verify that now A^0 and ${}^0B$ are left ideals of $N^{\cdot}$. Viceversa, if $F\colon A\times B \to \mathrm{End}(N^+)$ is a Clay function and «·» denotes the multiplication inferred in N, then

$$\langle a,\ b\rangle \cdot \langle a',\ b'\rangle = F_{a,b}(\langle a',\ b'\rangle = F_{a,b}(\langle a',\ 0\rangle) + F_{a,b}(\langle 0,\ b'\rangle).$$

Now $\forall a' \in A \quad \forall b' \in B \quad \forall \langle a,\ b\rangle \in N$

$$F_{a,b}(\langle a',\ 0\rangle) = \langle a'',\ 0\rangle \in A^0 \qquad F_{a,b}(\langle 0,\ b'\rangle) = \langle 0,\ b''\rangle \in {}^0B$$

because A^0 and ${}^0B$ are left ideals of $N^{\cdot}$, so $F_{a,b}/_{A^0}$ and $F_{a,b}/_{{}^0B}$ are endomorphisms of A^0 and ${}^0B$ repectively. Take now $f_{a,b}(a') = a''$ and $\bar{f}_{a,b}(b') = b''$ $\forall \langle a,\ b\rangle \in A\times B$, then two functions from $A\times B$ to End(A) and End(B) respectively are defined, so we can write $\forall \langle a,\ b\rangle \in A\times B$.

$$F_{a,b}(\langle a',\ b'\rangle) = \langle f_{a,b}(a'),\ \bar{f}_{a,b}(b')\rangle\,.$$

The property (3) is true because $F_{a,b} \in \mathrm{End}(N^+)$ and (1) and (2) arise from associativity of «·».

From Proposition 2 we can see that $F/_{A^0}$ and $F/_{{}^0B}$ determine a Clay function of A and B respectively.

Obviously, particular remarkable cases come from choice of homomorphism φ and functions f and $\bar{f}$; for example:

(1) If $f(\langle 0,\ 0\rangle) = O_A$ and $\bar{f}(A\times B) = \{\mathrm{id}\}$, then we find Def. 1 of [1]. Besides, if also $\varphi(B) = \{\mathrm{id}\}$, A and B are non trivial abelian groups and $f(A\times B) \to \mathrm{End}(A)$ is a commutative subset, all the mixed semirings and only those arise, in fact the near-ring constructed is abelian, because additively it is a direct sum of abelian groups, and also left permutable; moreover $N_0 = A^0 \neq \{0\}$ and $N_c = {}^0B \neq \{0\}$, so it is a mixed semiring.

Viceversa, if N is a mixed semiring, then $N^+ = N_0^+ \oplus N_c^+$, moreover $(n_0 + n_c)(n_0' + n_c') = (n_0 + n_c)\,n_0' + n_c'$, so now

$$F_{n_0,n_c}(\langle n_0',\ n_c'\rangle) = F_{n_0,n_c}(\langle n_0',\ 0\rangle) + F_{n_0,n_c}(\langle 0,\ n_c'\rangle) = \langle f_{n_0,n_c}(n_0'),\ n_c'\rangle\,.$$

Finally, $f(N_0\times N_c)$ is obviously a commutative subset of $\mathrm{End}(N_0^+)$ and

$S\colon N \to N_0^+ \oplus N_c^+$ with $S(n_0+n_c)=\langle n_0, n_c\rangle$ is an isomorphism of mixed semi-rings.

(2) If $f(\langle 0, 0\rangle)=O_A$, $\bar{f}(\langle 0, 0\rangle)=O_B$ and $\varphi(B)=\{\mathrm{id}\}$, then we find Def. 2 of [1].

(3) If A is a near-ring, $A=B$, $\varphi(A)=\{\mathrm{id}\}$, $f(\langle a, b\rangle)=\gamma_a$, $\bar{f}(A\times B)=\{\mathrm{id}\}$ then we obtain the Example 2.13 of [3].

(4) If $A=B$, $\varphi(A)=\{\mathrm{id}\}$, $f_{a,b}=f_{0,b}=\gamma_b=\bar{f}_{a,b}$ $\forall\langle a, b\rangle\in A\times B$, then we obtain the Example 2.11 of [3].

(5) If A and B are vectorial spaces and A is normed, $\varphi(B)=\{\mathrm{id}\}$ $f(\langle a, b\rangle)(a')=|a|a'$ $\forall\langle a, b\rangle\in A\times B$ $\forall a'\in A$ and $\bar{f}(A\times B)=\{O_B\}$, then we obtain the Example 2.8 of [3].

We shall call *Φ-sum* of A and B a near-ring constructed as in Proposition 2.
If N is a Φ-sum of A and B, the multiplication of N infers a multiplication in A and B if we define

$$aa'=\Pi_A(\langle a, 0\rangle\langle a', 0\rangle)\qquad bb'=\Pi_B(\langle 0, b\rangle\langle 0, b'\rangle)$$

and with respect to such operations A and B are near-rings, isomorphic images of A^0 and ${}^0B$ respectively.

Proposition 3. *If N is a Φ-sum of A and B where A and B are near-rings, the multiplication inferred in A and B by multiplication of N and the multiplication of A and B coincide iff we define $f_{a,0}=\gamma_a$ $\forall a\in A$ and $\bar{f}_{0,b}=\gamma_b$ $\forall b\in B$.*

Proof. Easy verification.

In the following, *Φ-sum of near-rings* means that the assumption of Proposition 3 is given.

It can easily be seen that if N is a Φ-sum of A and B, then A^0 is a left ideal of $[N,\cdot]$ and ${}^0B$ is a left N-subgroup of $[N, \cdot]$; indeed:

Theorem 1. *Let N be a near-ring, then $N=I+K$ where I is a left ideal and K is a left N-subgroup and $I\cap K=\{0\}$ iff N is isomorphic to a Φ-sum of I and K.*

Proof. It follows immediately from Proposition 2.

Corollary 1. *In a near-ring N, N_0 is an ideal iff N is isomorphic to the Φ-sum of N_0 and N_c with $f(\langle 0,\ 0\rangle)=O_{N_0}$ and $\bar{f}(N_0\times N_c)=\{\mathrm{id}\}$.*

Proof. Obviously, since it is always $N=N_0+N_c$ with $N_0\cap N_c=\{0\}$ and now N_c is even a N-subgroup. Thus we find the Theorem 1 of [**1**].

Corollary 2. *If N is a near-ring, then $N=I+J$ where I and J are left ideals with $I\cap J=\{0\}$, iff N is isomorphic to the Φ-sum of I and J with $\varphi(I)=\{\mathrm{id}\}$.*

Proof. Obviously, because a left ideal is of course a left N-subgroup and I is a left ideal iff the sum $I+J$ is direct, that is $\varphi(I)=\{\mathrm{id}\}$.

In addition, if we assume $f(\langle 0,\ 0\rangle)=O_{I^+}$ and $\bar{f}(\langle 0,\ 0\rangle)=O_{J^+}$, then all and only zero-symmetric near-rings belonging to the class seen in Corollary 2 are found; thus we find the Theorem 2 of [**1**].

Proposition 4. *Let N be a Φ-sum of A and B, then:*

(i) *N is zero-symmetric iff $f(\langle 0,\ 0\rangle)=O_A$ and $\bar{f}(\langle 0,\ 0\rangle)=O_B$.*
(ii) *${}^0B$ is a N-subgroup iff $f({}^0B)=O_A$.*
(iii) *A^0 is a right ideal iff $\bar{f}_{a,b}=\bar{f}_{0,b}$ $\forall a\in A,\ \forall b\in B$.*

Proof. The verification of (i) and (ii) is routine.
(iii) Generally, if N is a Φ-sum of A and B, we have

$$(\langle a,\ b\rangle+\langle \bar{a},\ 0\rangle)\langle a',\ b'\rangle-\langle a,\ b\rangle\langle a',\ b'\rangle$$

$$=\langle a+\varphi_b(\bar{a}),\ b\rangle\langle a',\ b'\rangle-\langle f_{a,b}(a'),\ \bar{f}_{a,b}(b')\rangle$$

$$=\langle \dots,\ \bar{f}_{a+\varphi_b(\bar{a}),b}(b')\rangle+\langle \dots,\ -\bar{f}_{a,b}(b')\rangle=\langle \dots,\ \bar{f}_{a+\varphi_b(\bar{a}),b}(b')-\bar{f}_{a,b}(b')\rangle=(*);$$

then if $\bar{f}_{a,b} = \bar{f}_{0,b}$ $\forall a \in A$, $\forall b \in B$, we have

$$(*) = \langle \dots, \bar{f}_{0,b}(b') - \bar{f}_{0,b}(b') \rangle = \langle \dots, 0 \rangle$$

so A^0 is a right ideal. Viceversa: if A^0 is a right ideal then

$$\bar{f}_{a+\varphi_b(\bar{a}),b}(b') - \bar{f}_{a,b}(b') = 0 \qquad \forall a,\ \bar{a} \in A,\ \forall b,\ b' \in B;$$

in particular assume $a = 0$, so $\bar{f}_{\varphi_b(\bar{a}),b} = \bar{f}_{0,b}$ and $\bar{f}_{a,b} = \bar{f}_{0,b}$ $\forall a \in A$ because φ_b is an automorphism of A and $\varphi_b(A) = A$.

Proposition 5. *Let N be a Φ-sum of A and B, then N is a medial (left permutable) near-ring iff $f(A \times B) \subseteq \mathrm{End}(A)$ and $\bar{f}(A \times B) \subseteq \mathrm{End}(B)$ are right permutable (commutative) subsets.*

Proof. It is enough to recall (1) and (2) of Proposition 2.

Proposition 6. *Let N be a Φ-sum of A and B, then:*

(i) $r(\langle a,\ b \rangle) = \ker f_{a,b} X \ker \bar{f}_{a,b}$.

(ii) *If A^0 is a right ideal and $\bar{f}(\langle 0,\ 0 \rangle) = O_B$, then ${}^0B \subseteq r(A^0)$.*

Proof. (i) It is trivial. (ii) If A^0 is a right ideal, then $\bar{f}_{a,b} = \bar{f}_{0,b}$ $\forall a \in A$, $\forall b \in B$ from Proposition 2, so, in particular, $\bar{f}_{a,0} = \bar{f}_{0,0} = O_B$ thus $\ker \bar{f}_{a,0} = B$ and, from (i), $\{0\} \times B \subseteq \ker f_{a,0} \times \ker \bar{f}_{a,0}$ so ${}^0B \subseteq r(A^0)$.

4 - Λ-sum of near-rings

Consider now the near-rings on direct sums of additive groups, that is $\varphi(B) = \{\mathrm{id}\}$.

Proposition 7. *Let $N = A \oplus B$ be the direct sum of the additive groups A and B, and let B be an abelian group. The function F defined by: $\forall \langle a,\ b \rangle \in A \times B$ $\quad F(\langle a,\ b \rangle) = F_{a,b}$, with $F_{a,b}(\langle a',\ b' \rangle) = \langle f_a(a'),\ \lambda_{a'}(b) + b' \rangle$ $\forall \langle a',\ b' \rangle \in N$ where f is a Clay function of A, $\lambda_{a'} = \lambda(a')$ and λ: $A \to \mathrm{End}(B)$ is a group homomorphism for which $\lambda_{a'} \circ \lambda_a = \lambda_{f_a(a')}$ is true, is a Clay function of N.*

Proof. F defined as above is a function from $A \times B$ to End(N); in fact $F_{a,b}$ is an endomorphism of N $\forall \langle a, b \rangle \in A \times B$, moreover the multiplication inferred in N is associative, so F: $A \times B \to$ End(N) is a Clay function.

Proposition 8. *In a near-ring* $[N, \cdot]$ *constructed on* $N = A \oplus B$ *by Clay function like the one in Proposition 7,* A^0 *is a right ideal including* N_0 *and* ${}^0B$ *is an abelian ideal included in* N_c.

Proof. Obviously A^0 is a normal additive subgroup of N, moreover

$$F_{a+\bar{a},b+0}(\langle a', 0\rangle) - F_{a,b}(\langle a', 0\rangle) = \langle f_{a+\bar{a}}(a'), \lambda_{a'}(b)\rangle - \langle f_a(a'), \lambda_{a'}(B)\rangle$$

$$= \langle f_{a+\bar{a}}(a') + f_a(a'), 0\rangle \in A^0$$

so condition (iv) of Proposition 1 is true; ${}^0B$ is also a normal additive subgroup, moreover

$$F_{a,b+\bar{b}}(\langle 0, b'\rangle) - F_{a,b}(\langle 0, b'\rangle)$$

$$= \langle a_a(0), \lambda_0(b) + b'\rangle - \langle f_a(0), \lambda_0(b) + b'\rangle = \langle 0, 0\rangle \in {}^0B$$

which results to be a right ideal, and $F_{a,b}(\langle 0, b'\rangle) = \langle 0, \lambda_0(b) + b'\rangle$ belonging to ${}^0B$ $\forall \langle a, b\rangle \in N$, $\forall b' \in B$, so $F_{a,b}(N) \subseteq \mathscr{F}({}^0B)$ and condition (ii) of Proposition 1 is true, thus ${}^0B$ is a left ideal.

Finally, is routine to verify that $N_0 \subseteq {}^0A$ and ${}^0B \subseteq N_c$.

The structure described above will be called *Λ-sum* of A and B.

If N is a Λ-sum of A and B, the multiplication of N infers a multiplication in A and B if we define

$$aa' = \Pi_A(\langle a, 0\rangle\langle a', 0\rangle) \qquad bb' = \Pi_B(\langle 0, b\rangle\langle 0, b'\rangle)$$

and with respect to such operations A and B are near-rings, isomorphic images of A^0 and ${}^0B$ respectively.

Proposition 9. *If* N *is a* Λ*-sum of* A *and* B*, where* A *and* B *are near-rings, the multiplication inferred in* A *and* B *by multiplication of* N *and the multiplication of* A *and* B *coincide iff we define* $f_{a,0} = \gamma_a$ $\forall a \in A$ *and* B *is a constant nearing.*

Proof. Easy verification.

In the following *Λ-sum of near-rings* means that the assumption of Proposition 9 is given.

Proposition 10. *Let N be a Λ-sum of A and B, then:*

(i) $A^0 = N_d$ *iff* f *is a homomorphism.* (ii) *The following conditions are equivalent*

$$\text{(a)}\quad N_0 = A^0 \qquad \text{(b)}\quad N_c = {}^0B \qquad \text{(c)}\quad f(0) = O_A .$$

Proof. (i) It is trivial. (ii) $N_0 = A^0$ implies $N_c = {}^0B$, obviously. $N_c = {}^0B$ implies $f(0) = O_A$; in fact, if $N_c = {}^0B$, then $\langle 0, b\rangle\langle a', b'\rangle = \langle 0, \bar{b}\rangle$ so $f_0(a') = 0$ $\forall a' \in A$ and $f(0) = O_A$. $f(0) = O_A$ implies $N_0 = A^0$; in fact, if $f(0) = O_A$ then $\langle 0, 0\rangle\langle a, 0\rangle = \langle 0, 0\rangle$ $\forall a \in A$ so $A^0 \subseteq N_0$, moreover $\langle a, b\rangle \in N_0$ implies that $\langle 0, 0\rangle\langle a, b\rangle = \langle f_0(a), \lambda_a(0) + b\rangle = \langle 0, b\rangle = \langle 0, 0\rangle$ so $N_0 \subseteq A^0$.

The proof of the following theorems is a direct consequence of Propositions 7, 8, 9 and 10.

Theorem 2. *A near-ring N in which $N_0 = N_d$ and N_c is an abelian ideal, is isomorphic to a Λ-sum of N_0 and N_c.*

Theorem 3. *A near-ring N is abstract affine iff it is isomorphic to a Λ-sum of a ring A and a constant near-ring on an abelian group B.*

So the Clay functions which allow us the construction of abstract affine near-rings are characterized, in fact: *A near-ring N is abstract affine iff it is constructed on the direct sum $A \oplus B$ of abelian groups by a Clay function as in Proposition* 7, *where f is a homomorphism,* thus we find Theorem 5 of $[4]_2$.

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Summary

Particular classes of near-rings on direct or semidirect sums of additive groups are constructed, so that direct or semidirect summands are one-sided or two-sided ideals of the constructed near-ring or, at least, of its multiplicative semigroup.

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