Riv. Mat. Univ. Parma (4) **15** (1989), 165-173

R. KHAZAL and N. P. MUKHERJEE (*)

A note on maximal subgroups in finite groups (**)

Introduction

In this note we examine cases when maximal subgroups of a finite group G have certain common properties. They are related to the notion of normal index of a maximal subgroup. These considerations lead to characterizations of supersolvable and nilpotent groups and under certain sets of conditions G is found respectively p-supersolvable and p-nilpotent.

For a maximal subgroup M of a group G the order of a chief factor X/Y of G, where X is minimal in the set of normal supplements of M in G, is unique and is known as the normal index M in G [3]. It is denoted by $\eta(G:M)$.

Here we shall associate with each $M \lessdot G$ the set

$$F(M) = \{(H,\ K) | H \lhd G,\ K \lhd G,\ K \subsetneqq H \text{ and (i) } \langle M,\ H\rangle = G, \text{ (ii) } \langle M,\ K\rangle = M\}.$$

An element $(H,\ K)$ in $F(M)$ will be called a *minimal element* if H/K is a chief factor. In that event H/K is a minimal normal supplement to $\frac{M}{K} \lessdot \frac{G}{K}$ and it follows that $\eta[\frac{G}{K}:\frac{M}{K}] = |\frac{H}{K}|$. This will imply $\eta(G:M) = \eta[\frac{G}{K}:\frac{M}{K}] = |\frac{H}{K}|$ [2].

$Z_\infty(G)$ and $Q^*(G)$ will denote repectively the hypercenter and the hyperquasi

(*) Indirizzo degli AA.: R. KHAZAL, Department of Mathematics, Kuwait University, P.O. Box 5969, KWT-13060 Safat; N. P. MUKHERJEE, School of Computer and Systems Sciences, Jawaharlal Nehru University, IND-110067 New Delhi.
(**) Ricevuto: 18-V-1989.

center of a group G. The quasicenter $Q(G)=Q_1(G)$ of a group G is the characteristic subgroup generated by all cyclic quasinormal subgroups of G. The hyperquasicenter $Q^*(G)$ is the largest term of the chain of subgroups $Q_0(G)=\{1\} \subseteq Q_1(G)=Q(G) \subseteq Q_2(G)$, where

$$\frac{Q_i(G)}{Q_{i-1}(G)}=Q[\frac{G}{Q_{i-1}(G)}] \qquad \text{for all } i>0 .$$

Clearly $Q(G) \supseteq Z(G)$ and $Q^*(G)$ is characteristic subgroup of G which of course contains $Z_\infty(G)$. For the sake of completeness we quote the following results which will be used.

Theorem ([5]$_1$, p. 24). If x is a QC-element of a group G then x^r is also a QC-element of G for every integer r.

Remark. x is called a *QC-element* of G if $\langle x \rangle$ is quasinormal in G.

Theorem ([7], p. 31). Let N be a normal subgroup of a group G and $N \subseteq Q^*(G)$. Then $Q^*(\frac{G}{N})=\frac{Q^*(G)}{N}$.

An analogous result is also valid when $Q^*(G)$ is replaced by $Z_\infty(G)$.

Theorem ([7], p. 32). The hyperquasicenter $Q^*(G)$ of a group G is the largest supersolvably embedded subgroup of G.

The notations used in the note are all standard. The groups are all standard. The groups considered throghout are all finite.

1 - Some solvability conditions

Set $J_p(G)=\{M \lessdot G | [G:M]_p=1 \text{ and } [G:M] \text{ is composite}\}$, p is a prime and

$$S_p(G)= \cap \{M \lessdot G | M \in J_p\} \qquad \varphi_p(G)= \cap \{M \lessdot G | [G:M]_p=1\} .$$

$\varphi_p(G)$ is solvable (Theorem 7, [6]). If p is the largest prime divisor of $|G|$, then

$S_p(G)$ is solvable (Theorem 8, [6]) and in that event if $J_p = \varphi$, then $S_p(G) = G$ is solvable.

Theorem 1. *A group G is solvable if and only if each $F(M)$, $M \in J_p(G)$ where p is the largest prime divisor of $|G|$, contains a minimal element (H, K) such that H/K is solvable.*

Proof. If G is solvable then the result follows trivially. Conversely, note that G is not simple. Let N be a minimal normal subgroup of G and consider G/N. If p divides $|\frac{G}{N}|$ then $\frac{M}{N} \in J_p(\frac{G}{N})$ implies that $M \in J_p(G)$. On the other hand if $p \nmid |\frac{G}{N}|$ and q is the largest prime divisor of $|\frac{G}{N}|$ then $\frac{M}{N} \in J_q(\frac{G}{N})$ implies that $M \in J_p(G)$.

If (H, K) is a minimal element in $F(M)$ such that H/K is solvable then $(H/N, K/N)$ is a minimal element in $F(M/N)$ with similar property if $N \subset K$. If $N \not\subset K$ then $(\frac{HN}{N}, \frac{KN}{N})$ is a minimal element in $F(\frac{M}{N})$ and $\frac{HN}{N}/\frac{KN}{N}$ is solvable. By induction it follows that G/N is solvable and N may be viewed as the unique minimal normal subgroup of G. If $N \subset \Phi_p(G)$ then clearly G is solvable. Suppose $N \not\subset \Phi_p(G)$ so that $G = MN$, $[G:M]_p = 1$. If $[G:M] =$ composite then $M \in J_p$ and if (H, K) is the minimal element in $F(M)$ for which H/K is solvable then evidently $|\frac{H}{K}| =$ a power of a prime. Consequently, $\eta(G:M) = |N| = |\frac{H}{K}| =$ a power of a prime and N is solvable which however implies that G is solvable. We may therefore assume $[G:M] =$ a prime $= s$ and note that M is core free. By representing G on the s cosets of M it follows that $|G|$ must divide $s!$ which is impossible. Hence $N \subset \Phi_p(G)$ and the theorem is proved.

Theorem 2. *A group G is solvable if and only if for each minimal element (H, K) in $F(M)$, $M \lessdot G$, $C_G(\frac{H}{K}) \neq 1$.*

Proof. If G is solvable then the result follows. Conversely, observe that G is not simple. Let N be a minimal normal subgroup of G and consider G/N. By induction G/N is solvable and N may be considered a unique minimal normal subgroup. If $N \not\subset \Phi(G)$ then $G = MN$ for $M \lessdot G$, $(N, 1)$ is a minimal element in $F(M)$ and $C_G(N) \neq 1$. Since N is unique and $C_G(N) \lhd G$, $N \subset C_G(N)$ and therefore N is solvable. It now follows that G is solvable.

2 - Conditions for supersolvability and nilpotency

Proposition 1. *A group G is supersolvable if and only if for some minimal element (H, K) in $F(M)$ $\forall$ maximal subgroup M of G*

$$H/K \cap Q^*(\frac{G}{K}) \neq \overline{1}.$$

Proof. If G is supersolvable then trivially the assertion follows. Conversely, if $H/K \cap Q^*(\frac{G}{K}) \neq \overline{1}$, then $\frac{H}{K} \subseteq Q^*(\frac{G}{K})$ = the largest supersolvably embedded subgroup of G/K. Therefore H/K is of prime order and it now follows from $\frac{G}{K} = \frac{M}{K} \cdot \frac{H}{K}$ that $[\frac{G}{K} : \frac{M}{K}] = [G:M] = |\frac{H}{K}|$ = a prime for each maximal subgroup M of G. Hence G is supersolvable.

Lemma 1. *Let M be a maximal subgroup and $Q^*(G)$ be the hyperquasicenter, respectively, of a group G. Then $G = MQ^*(G)$implies that the index of M is prime in G.*

Prof. Suppose the quasicenter $Q(G) \subsetneqq M$. Then $G = MQ(G)$ and therefore $G = M\langle x \rangle$ where x is a quasicentral p-element for some prime divisor p of $|G|$ ([5]$_1$, p. 24). Consequently, $[G:M] = p$ since $\langle x^i \rangle$, i an integer, is quasinormal for all i ([5]$_1$, p. 24). The result now follows by induction since the hyperquasicenter of $\frac{G}{Q(G)}$ equals $\frac{Q^*(G)}{Q(G)}$ ([7], p. 31).

Lemma 2. *Let M be a maximal subgroup of a group G. If for each minimal element (C, D) in $F(M)$ it implies that $Q^*(\frac{G}{D}) \neq \overline{1}$, then M is of prime index in G.*

Proof. If core $M = 1$, then a minimal element in $F(M)$ is of the form $(C, 1)$ and by hypothesis $Q^*(G) \neq 1$. Therefore $G = MQ^*(G)$ and by Lemma 1 it follows that $[G:M]$ = a prime.

Now suppose core $M \neq 1$ and let N be a minimal normal subgroup of G contained in M and consider G/N. If $(\frac{x}{N}, \frac{y}{N})$ is a minimal element in $F(M/N)$, then observe (X, Y) is a minimal element in $F(M)$ and $Q^*(\frac{G}{Y}) \neq \overline{1}$ which however implies that $Q^*(\frac{G/N}{Y/N}) \neq \bar{1}$. It now follows by induction that $[\frac{G}{N}:\frac{M}{N}] = [G:M]$ = a prime.

Theorem 3. *A group G is supersolvable if and only if for each minimal element (C, D) in $F(M)$ $\forall$ maximal subgroup M of G it implies that $Q^*(\frac{G}{D}) \neq \bar{1}$.*

Proof. If G is supersolvable then clearly the assertion is true since $Q^*(X) = X$ for every supersolvable group X. Conversely, by Lemma 2, $[G:M]$ = a prime $\forall$ maximal subgroup M of G and therefore G is supersolvable.

Corollary. *Let p be the largest prime divisor of $|G|$ and q be one of its other divisors. Then G is supersolvable iff $\forall$ minimal element (C, D) in $F(M)$, $M \in J_p(G) \cup J_q(G)$ it implies that $Q^*(\frac{G}{D}) \neq \bar{1}$.*

Proof. If G is supersolvable then the result follows. Conversely, by Lemma 2 it follows that $J_p(G) = J_q(G) = \Phi$. Hence $G = S_p(G) = S_q(G)$ and by Theorem 11 ([**6**], p. 610) G is supersolvable.

Proposition 2. *A group G is supersolvable if and only if for any pair of maximal subgroups M_1, M_2 in G for which $F(M_1)$ and $F(M_2)$ have a common minimal element (H, K) it follows that $[G:M_1] = [G:M_2]$ and H/K is cyclic.*

Proof. The result clearly follows if G is supersolvable. Conversely, observe that G is not simple and let N be a minimal normal subgroup of G. By induction G/N is supersolvable and N may be viewed as the unique minimal normal subgroup of G. The theorem stands proved if $N \subset \Phi(G)$. Suppose $N \not\subseteq \Phi(G)$ and let $M \lessdot G$ such that $G = MN$. If $\hat{M}$ is another core free maximal subgroup of G then $G = \hat{M} \cdot N$ and $(N, 1)$ is minimal element common to $F(M)$ and $F(\hat{M})$. Therefore $N/1$ is cyclic and G is supersolvable.

Results which follow next relate to the *characterizations of nilpotent groups*. In most cases proofs are similar to the ones given for analogous characterizations of supersolvable groups and are therefore omitted.

Proposition 3. *A solvable group G is nilpotent if and only if $(G, \operatorname{core} M)$ is a minimal element in $F(M)$ $\forall$ maximal subgroup M of G.*

Theorem 4. *Let G be a solvable group and (C, D) be a minimal element common to $F(M)$ $\forall$ maximal subgroup M of G. Then G is a p-group.*

Proof. Let N be a minimal normal subgroup of G and $\frac{M}{N} \lessdot \frac{G}{N}$. If $N \subset D$ then $(\frac{C}{N}, \frac{D}{N})$ is a minimal element in $F(\frac{M}{N})$ $\forall \frac{M}{N} \lessdot \frac{G}{N}$. Otherwise, $N \not\subset D$ implies that $(\frac{CN}{N}, \frac{DN}{N})$ is a minimal element common to $F(\frac{M}{N})$ $\forall \frac{M}{N} \lessdot \frac{G}{N}$. (Note that $N \subset C$ will imply $DN = C$ since C/D is a chief factor and in that case $\langle M, C\rangle = \langle M, DN\rangle = M$, a contradiction).

By induction G/N is nilpotent and N may be viewed as the unique minimal subgroup of G. If $W \lessdot G$ and $N \not\subset W$ then $G = WN$ and W is corefree. Since (C, D) is a minimal element in $F(W)$ it follows that $D = 1$ and $C/1$ is minimal normal in G. Hence $C = N$ and $(N, 1)$ is the minimal element common to all $F(M)$, $M \lessdot G$. However if Y is a maximal subgroup of G containing N, then $F(Y)$ cannot contain $(N, 1)$. Consequently, $N \subseteq \Phi(G)$ and G is nilpotent. Each maximal subgroup M of G is therefore normal and by Theorem 1 in $[5]_2$, $\eta(G:M) = [G:M]$. By hypothesis for any $X \lessdot G$, $\eta(G:X) = |\frac{C}{D}|$ and so every maximal subgroup has the same index. It now clearly follows that G is a p-group.

Proposition 4. *A group G is nilpotent if and only if for $M \lessdot G$, $F(M)$ contains a minimal pair (H, K) such that $\frac{H}{K} \cap Z_\infty(\frac{G}{K}) \neq \bar{1}$.*

Proof. Follows immediately form the fact that a minimal normal subgroup contained in the hypercenter of a group is central.

Proofs of the following results are omitted since they are identical to the proofs of Lemma 1 and Lemma 2 respectively.

Lemma 3. *A maximal subgroup M of a group G is normal if $G = MZ_\infty(G)$ where $Z_\infty(G)$ is the hyperquasicenter of G.*

Lemma 4. *Let M be a maximal subgroup of a group G. If for each minimal element (C, D) in $F(M)$ it follows that $Z_\infty(\frac{G}{D}) \neq \bar{1}$, then $M \trianglelefteq G$.*

The next theorem is now an immediate consequence of the above Lemma.

Theorem 5. *A group G is nilpotent if and only if for each minimal element $(C, D) \in F(M)$, $M \lessdot G$ it implies that $Z_\infty(\frac{G}{D}) \neq \bar{1}$.*

3 - p-supersolvable and p-nilpotent structures

Theorem 6. *Let p be the smallest prie divisor of the order of a group G. G is p-supersolvable if for any maximal subgroup M of G, $\eta(G:M)_p \neq \bar{1}$ implies that for some minimal element (H, K) in $F(M)$ the subgroups of order p in G/K are self centralizing.*

Proof. Observe that a Sylow p-subgroup of $G/K = \bar{G}$ is of order p and $|\frac{H}{K}|_p \neq 1$. Set $\bar{W} = \frac{H}{K}$ and let $\langle \bar{u} \rangle$ be a Sylow p-subgroup of $\bar{W}$. If $\bar{x} \in \bar{G} \setminus \langle \bar{u} \rangle$ normalizes $\langle \bar{u} \rangle$ then $\langle \bar{u} \rangle = \langle \bar{u} \rangle^{\bar{x}}$ where $\langle \bar{x} \rangle = \langle \bar{x}_1 \rangle \times \langle \bar{x}_2 \rangle \times \ldots \times \langle \bar{x}_n \rangle$, $\langle \bar{x}_i \rangle$, $i = 1, 2, \ldots, n-1$ is the Sylow p_i-subgroup and $\langle \bar{x}_n \rangle$ is the Sylow p-subgroup respectively of $\langle \bar{x} \rangle$. Note that $\bar{x}_i$ $\forall i = 1, 2, \ldots, n$ normalizes $\langle \bar{u} \rangle$ and $H_i = \langle \bar{x}_i \rangle \langle \bar{u} \rangle$, $i = 1, 2, \ldots, n-1$ being supersolvable $\langle \bar{x}_i \rangle$ is normalized by $\bar{u}$ and hence $\bar{x}_i \bar{u} = \bar{u} \bar{x}_i$, $i = 1, 2, \ldots, n-1$. Also $\langle \bar{x}_n \rangle \langle \bar{u} \rangle = \langle \bar{u} \rangle \langle \bar{x}_n \rangle$ is a group of order p^2 and is therefore abelian. Consequently, $\bar{x}_n \bar{u} = \bar{u} \bar{x}_n$ and $\bar{u}$ is indeed centralized by $\bar{x}$, a contradiction. Then $C_{\bar{G}}(\langle \bar{u} \rangle) = N_{\bar{G}}(\langle \bar{u} \rangle)$ and by Burnside's theorem $\bar{G} = \bar{F} \langle \bar{u} \rangle$ where $\bar{F}$ is a normal p-complement. Observe that $\bar{W} = (\bar{W} \cap \bar{F}) \langle \bar{u} \rangle$ and $\bar{W} = \langle \bar{u} \rangle$ if $\bar{W} \cap \bar{F} = \bar{1}$. On the other hand if $\bar{W} = \bar{W} \cap \bar{F}$ then $\bar{W} \subset \bar{F}$ and this is not possible since $p \nmid |\bar{F}|$. The order of $\bar{W}$ therefore is p.

Therefore for an arbitrary maximal subgroup X of G we have either $\eta(G:X) = p$ or $\eta(G:X)$ is prime to p. Since $[G:X]$ divides $\eta(G:X)$ it follows that $\eta(G:X) = [G:X]_p$ $\forall X \lessdot G$. By Theorem 2, [5]$_2$, G is p-solvable. This however implies that G is p-supersolvable by Theorem 9.3 ([4], p. 717) since every maximal subgroup is of index p or has an index prime to p.

The next two theorems give sets of necessary and sufficient conditions for a group to have p-nilpotent structure.

Theorem 7. *A group G is p-nilpotent if and only if for any maximal subgroup M of G, it implies that for each minimal pair (C, D) in $F(M)$, $Z(\frac{G}{D}) \neq \bar{1}$ if $[C:D]_p \neq 1$.*

Proof. Suppose G is p-nilpotent and (C, D) is a minimal element in $F(M)$, $M \lessdot G$ where $[C, D]_p \neq 1$. By Lemma 4.3 ([4], p. 428) it follows that C/D is central and so $Z(\frac{G}{D}) \neq \bar{1}$. Conversely, observe that G is not simple and let N be a minimal normal subgroup of G. Let $\frac{M}{N} \lessdot \frac{G}{N}$ and suppose $[\frac{C}{N}, \frac{D}{N}]$ is a minimal element in $F(\frac{M}{N})$ such that $[\frac{C}{N}, \frac{D}{N}]_p \neq 1$. This implies (C, D) is a minimal element in $F(M)$ and $[C:D]_p \neq 1$. Hence $Z(\frac{G}{D}) \neq \bar{1}$ so that $Z[\frac{G/N}{D/N}] \neq \bar{1}$.

By induction G/N is p-nilpotent and N may be viewed as the unique minimal normal subgroup of G. If $N \subseteq \Phi(G)$ then G is p-nilpotent and therefore suppose $G = XN$, $X \lessdot G$ and note that X is corefree. $(N, 1)$ is a minimal element in $F(X)$. If $[N:1]_p = 1$ then N is a p'-group and if $[N:1]_p \neq 1$ then by hypothesis $Z(G/1) \neq 1$ and therefore N is central and is of order p. In either of these cases it follows that G is p-nilpotent since G/N is p-nilpotent.

Theorem 8. *A group is p-nilpotent if and only if for any maximal subgroup M of G, it implies that $\forall$ minimal element (C, D) in $F(M)$, $C_G(\frac{C}{D}) \supset O^p(G)$ if $[C:D]_p \neq 1$.*

Proof. Let G be p-nilpotent and (C, D) be a minimal element in $F(M)$, $M \lessdot$ such that $[C:D]_p \neq 1$. Then C/D in central in G/D and consequently $Z(\frac{G}{D}) \neq \bar{1}$. Conversely, observe that G is not simple and let N be a minimal normal subgroup of G. It easily follows that G/N is p-nilpotent by induction and N may be wieved as a unique minimal normal subgroup of G. If $N \subseteq \Phi(G)$ then G is p-nilpotent and we may therefore assume $G = XN$, $X \lessdot G$. $(N, 1)$ is a minimal element in $F(X)$ and if $[N:1]_p = 1$ then N is a p'-group and therefore G is p-nilpotent. However if $[N:1]_p \neq 1$ then $C_G(N) \supset O^p(G)$ so that $N \subset C_G(O^p(G))$ and therefore N is an elementary abelian p-group. Let R/N be the normal p-complement in G/N. If T is a p-complement of N in R then T is centralized by N so that $R = T \times N$ and it now follows that T is the normal p-complement in G. We may thus assume $N \subseteq \Phi(G)$ and hence G is p-nilpotent.

It may be remarked that if for each minimal element (C, D) in $F(M)$, $[C:D]_p = 1$ for each $M \lessdot G$ then G is a p'-group.

References

[1] R. BAER, *Classes of finite groups and their properties*, Illinois J. Math. **1** (1957), 115-187.

[2] J. C. BEIDLEMAN and A. E. SPENCER, *The normal index of maximal subgroups in finite groups*, Illinois J. Math. **16** (1972), 95-101.

[3] W. E. DESKINS, *On maximal subgroups*, Proceedings of Symposium in Pure Mathematics, Amer. Math. Soc. **1** (1959), 100-104.

[4] B. HUPPERT, *Endlichen Gruppen* I, Springer, N. Y. 1967.

[5] N. P. MUKHERJEE: $[\bullet]_1$ *The hyperquasicenter of a finite group* II, Proc. Amer. Math. Soc. 32 (1972), 24-28; $[\bullet]_2$ *A note on normal index and maximal subgroups in finite groups*, Illinois J. Math. 2 (1975), 173-178.

[6] N. P. MUKHERJEE and P. BHATTACHARYA, *On the intersection of a class of maximal subgroups of a finite group*, Canad. J. Math. (39) 3 (1987), 603-611.

[7] M. WEINSTEIN, *Between nilpotent and solvable*, Polygonal Publishing House, 1982.

Summary

See Introduction.

***