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On contaction of Fourier series (IV) (**)

1 - Introduction

Let $\sum a_n$ be a given infinite series with s_n as its nth partial sum. If $\{p_n\}$ is a sequence of positive constants, and

$$P_n = p_0 + p_1 + \ldots + p_n \to \infty \quad \text{as } n \to \infty \qquad P_{-l} = p_{-l} = 0 \quad \text{for } l \geqslant 1$$

then the Riesz mean $\bar{t}_n$ of $\sum a_n$ is defined by

$$\bar{t}_n = \frac{1}{P_n} \sum_{l=0}^{n} p_l s_l \qquad (P_n \neq 0).$$

For a positive number k, if the series

$$\sum_{n=1}^{\infty} |P_n/p_n|^{k-1} |\bar{t}_n - \bar{t}_{n-1}|^k$$

converges, then the series $\sum a_n$ is said to be *summable* $|R, P_n, 1|_k$ or *summable* $|\bar{N}, p_n|_k$ (see [3]).

The case $k = 1$ is reduced to the absolute Riesz summability $|R, P_n, 1|$ and further, in the special case $p_n = 1/(n+1)$, the summability $|R, P_n, 1|$ is the same as the absolute logarithmic summability. Also, the summability $|R, e^n, 1|$ is the absolute convergence (see [6]).

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We note it follows from Sunouchi's theorem [9]$_1$ that

(i) for $0<\alpha<1$

$$|R,\ \exp n/(\log n)^{\alpha},\ 1| \subset |R,\ \exp n^{\alpha},\ 1| \subset |R,\ n^{\alpha},\ 1|$$
$$\subset |R,\ \exp(\log n)^{\alpha},\ 1| \subset |R,\ (\log n)^{\alpha},\ 1|$$

(ii) for $\alpha \geqslant 1$

$$|R,\ \exp n^{\alpha},\ 1| \subset |R,\ \exp n/(\log n)^{\alpha},\ 1| \subset |R,\ n^{\alpha},\ 1|$$
$$\subset |R,\ \exp(\log n)^{\alpha},\ 1| \subset |R,\ (\log n)^{\alpha},\ 1|$$

where, if every series summable $|A|$ is also summable $|B|$, we write $|A| \subset |B|$.
A denotes a positive absolute constant that is not always the same.

2 - Orthogonal series

Let $\{\varphi_n(x)\}$ be an orthonormal system defined in the interval $(a,\ b)$. For a function $f(x) \in L^2(a,\ b)$ such that

$$f(x) \sim \sum_{n=0}^{\infty} a_n \varphi_n(x)$$

we denote by $E_n^{(2)}(f)$ the best approximation to $f(x)$ in the metric of L^2 by means of polynomials $\varphi_0(x),\ \ldots,\ \varphi_{n-1}(x)$. Then we have

$$E_n^{(2)}(f) = \{\sum_{l=n}^{\infty} |a_l|^2\}^{1/2}.$$

Applying the theorem due to Okuyama [7]$_2$, we can obtain the following theorem on the absolute Riesz summability of orthogonal series.

Theorem A. Let α be a positive number.

(i) If the series $\sum |a_n|^2 \{n \log n (\log\log n)^{1+\varepsilon}\}^{2/k-1}$

converges for some $\varepsilon>0$ and $0<k<2$, then the series $\sum |a_n|^k$ converges.

(ii) If the series $\sum |a_n|^2 (\log n)^{(\alpha-1)(2/k-1)} \{\log n(\log\log n)^{1+\varepsilon}\}^{2/k-1}$

converges for some $\varepsilon>0$ and $1\leqq k\leqq 2$, then the series $\sum a_n \varphi_n(x)$ is summable $|R, \exp(\log n)^\alpha, 1|_k$ almost everywhere.

(iii) If the series $\sum |a_n|^2 (\log n)^{1-2/k} \{\log n(\log\log n)^{1+\varepsilon}\}^{2/k-1}$

converges for some $\varepsilon>0$ and $1\leqq k\leqq 2$, then the series $\sum a_n \varphi_n(x)$ is summable $|R, (\log n)^\alpha, 1|_k$ almost everywhere.

(iv) If the series $\sum |a_n|^2 n^{\alpha(2/k-1)} \{\log n(\log\log n)^{1+\varepsilon}\}^{2/k-1}$

converges for some $\varepsilon>0$ and $1\leqq k\leqq 2$, then the series $\sum a_n \varphi_n(x)$ is summable $|R, \exp n^\alpha, 1|_k$ almost everywhere.

(v) If the series $\sum |a_n|^2 \{\log n(\log\log n)^{1+\varepsilon})^{2/k-1}$

converges for some $\varepsilon>0$ and $1\leqq k\leqq 2$, then the series $\sum a_n \varphi_n(x)$ is summable $|R, n^\alpha, 1|_k$ almost everywhere.

(vi) If the series $\sum |a_n|^2 (n/\log n)^\alpha)^{2/k-1} \{\log n(\log\log n)^{1+\varepsilon}\}^{2/k-1}$

converges for some $\varepsilon>0$ and $1\leqq k\leqq 2$, then the series $\sum a_n \varphi_n(x)$ is summable $|R, \exp n/(\log n)^\alpha, 1|_k$ almost everywhere.

(vii) If the series $\sum |a_n|^2 \{\log n(\log\log n)^{1+\varepsilon}\}^{2/k-1}$

converges for some $\varepsilon>0$ and $1\leqq k\leqq 2$, then the series $\sum a_n \varphi_n(x)$ is summable $|R, n/(\log n)^\alpha, 1|_k$ almost everywhere.

The case $k=1$ of this theorem is due to Okuyama and Tsuchikura [8]. Leindler [5]$_2$ proved the equivalent theorem.

Theorem B. Let $\{\lambda_n\}$ be a monotone sequence of positive numbers, $\{a_n\}$ a sequence of nonnegative numbers, $0<\beta<\nu$, and denote $\Lambda_n=\sum_{l=1}^{n} 1/\lambda_l$. Then the condition

(i) $$\sum_{n=1}^{\infty} \lambda_n^{-1}\{\sum_{l=n}^{\infty} a_l^\nu\}^{\beta/\nu}<\infty$$

is equivalent to the fact that there exists a monotone nondecreasing sequence $\{\mu_n\}$ such that

(ii) $$\sum_{n=1}^{\infty} a_n^{\nu} \mu_n < \infty$$

(iii) $$\sum_{n=1}^{\infty} \Lambda_n^{\beta/(\nu-\beta)} / \lambda_n \mu_n^{\beta/(\nu-\beta)} < \infty .$$

Applying Theorem B for $\beta = k$ and $\nu = 2$, we can obtain the following theorem from Theorem A.

Theorem 1. *Let α be a positive number.*

(i) *If the series* $\sum n^{-k/2} \{E_n^{(2)}(f)\}^k$

converges for $0 < k < 2$, then the series $\sum |a_n|^k$ converges.

(ii) *If the series* $\sum n^{-1} (\log n)^{-1+\alpha(2-k)/2} \{E_n^{(2)}(f)\}^k$

converges for $1 \leqq k < 2$, then the series $\sum a_n \varphi_n(x)$ is summable $|R, \exp(\log n)^{\alpha}, 1|_k$ almost everywhere.

(iii) *If the series* $\sum n^{-1} (\log n)^{-1} (\log \log n)^{-k/2} \{E_n^{(2)}(f)\}^k$

converges for $1 \leqq k < 2$, then the series $\sum a_n \varphi_n(x)$ is summable $|R, (\log n)^{\alpha}, 1|_k$ almost everywhere.

(iv) *If the series* $\sum n^{-1-\alpha(k/2-1)} \{E_n^{(2)}(f)\}^k$

converges for $1 \leqq k < 2$, then the series $\sum a_n \varphi_n(x)$ is summable $|R, \exp n^{\alpha}, 1|_k$ almost everywhere.

(v) *If the series* $\sum n^{-1} (\log n)^{-k/2} \{E_n^{(2)}(f)\}^k$

converges for $1 \leqq k < 2$, then the series $\sum a_n \varphi_n(x)$ is summable $|R, n^{\alpha}, 1|_k$ almost everywhere.

(vi) *If the series* $\sum n^{-k/2}(\log n)^{\alpha(k/2-1)}\{E_2^{(n)}(f)\}^k$

converges for $1 \leqslant k < 2$, *then the series* $\sum a_n \varphi_n(x)$ *is summable* $|R, \exp n/(\log n)^\alpha, 1|_k$ *almost everywhere.*

(vii) *If the series* $\sum n^{-1}(\log n)^{-k/2}\{E_n^{(2)}(f)\}^k$

converges for $1 \leqslant k < 2$, *then the series* $\sum a_n \varphi_n(x)$ *is summable* $|R, n/(\log n)^\alpha, 1|_k$ *almost everywhere.*

The result (i) is due to Watari and Okuyama [**10**], and the case $k=1$ in the results (ii)-(vii) are due to Okuyama $[7]_3$.

Proof of Theorem 1. We treat only the result (ii), because the other results can be shown similarly. For this purpose, we put

$$\lambda_n = n(\log n)^{1-\alpha(2-k)/2} \qquad \mu_n = (\log n)^{(\alpha-1)(2/k-1)}\{\log n(\log\log n)^{1+\varepsilon}\}^{2/k-1}.$$

If we put $\beta = k$ and $\nu = 2$ in Theorem B, then we have

$$\Lambda_n = \sum_{l=2}^{n} \lambda_l^{-1} = \sum_{l=2}^{n} l^{-1}(\log l)^{-1+\alpha(2-k)/2} \leqslant A(\log n)^{\alpha(2-k)/2}$$

$$\sum_{n=2}^{\infty} \Lambda_n^{k/(2-k)}/\lambda_n \mu_n^{k/2(2-k)} \leqslant A \sum_{n=2}^{\infty} n^{-1}(\log n)^{-1}(\log\log n)^{-1-\varepsilon} < \infty .$$

Thus we can establish the result (ii) by Theorem A (ii) and Theorem B.

3 - Equivalence relations

Let $\{\nu_n\}$ be a sequence of non-negative numbers. Then we shall prove the following equivalent theorem.

Theorem 2. *For* $1 \leqslant k < 2$, *the equivalent relations hold.*

(i) *The convergence of two series*

$$\sum_{n=1}^{\infty} n^{-1-\alpha(k/2-1)} \{\sum_{l=n}^{\infty} \nu_l^2\}^{k/2} \qquad \sum_{n=1}^{\infty} n^{-1-k-\alpha(k/2-1)} \{\sum_{l=1}^{n} l^2 \nu_l^2\}^{k/2}$$

is mutually equivalent for $0<\alpha<2/(2-k)$.

(ii) *The convergence of two series*

$$\sum_{n=2}^{\infty} n^{-k/2}(\log n)^{\alpha(k/2-1)} \{\sum_{l=n}^{\infty} \nu_l^2\}^{k/2} \qquad \sum_{n=2}^{\infty} n^{-3k/2}(\log n)^{\alpha(k/2-1)} \{\sum_{l=1}^{n} l^2 \nu_l^2\}^{k/2}$$

is mutually equivalent for $\alpha>0$.

Proof. (i) If $0<\alpha<2/(2-k)$, then the equivalence between

$$\sum_{n=1}^{\infty} n^{-1-\alpha(k/2-1)} \{\sum_{l=n}^{\infty} \nu_l^2\}^{k/2}<\infty \qquad \sum_{n=1}^{\infty} 2^{\alpha(1-k/2)n} \{\sum_{l=2^n}^{\infty} \nu_l^2\}^{k/2}<\infty$$

is nothing but Cauchy's condensation theorem. On the other hand,

$$\sum_{n=1}^{\infty} n^{-1-k-\alpha(k/2-1)} \{\sum_{l=1}^{n} l^2 \nu_l^2\}^{k/2} = \sum_{j=1}^{\infty} \sum_{n=2^{j-1}}^{2^j-1} n^{-1-k-\alpha(k/2-1)} \{\sum_{l=1}^{n} l^2 \nu_l^2\}^{k/2}$$

$$\leqslant A \sum_{j=1}^{\infty} 2^{-(1+k+\alpha(k/2-1))j} \sum_{n=2^{j-1}}^{2^j} \{\sum_{l=1}^{n} l^2 \nu_l^2\}^{k/2} \leqslant A \sum_{j=1}^{\infty} 2^{-(k+\alpha(k/2-1))j} \{\sum_{l=1}^{2^j} l^2 \nu_l^2\}^{k/2}$$

and

$$\sum_{n=1}^{\infty} n^{-1-k-\alpha(k/2-1)} \{\sum_{l=1}^{n} l^2 \nu_l^2\}^{k/2} = \sum_{j=0}^{\infty} \sum_{n=2^j}^{2^{j+1}-1} n^{-1-k-\alpha(k/2-1)} \{\sum_{l=1}^{n} l^2 \nu_l^2\}^{k/2}$$

$$\geqslant A \sum_{j=0}^{\infty} 2^{-(1+k+\alpha(k/2-1))j} \sum_{n=2^j}^{2^{j+1}-1} \{\sum_{l=1}^{n} l^2 \nu_l^2\}^{k/2} \geqslant A \sum_{j=0}^{\infty} 2^{-(k+\alpha(k/2-1))j} \{\sum_{l=1}^{2^j} l^2 \nu_l^2\}^{k/2}.$$

Therefore it is sufficient to prove that the convergence of two series

$$\sum_{n=1}^{\infty} 2^{\alpha(1-k/2)n} \{\sum_{l=2^n}^{\infty} \nu_l^2\}^{k/2} \qquad \sum_{n=1}^{\infty} 2^{-(k+\alpha(k/2-1))n} \{\sum_{l=1}^{2^n} l^2 \nu_l^2\}^{k/2}$$

is mutually equivalent.

Since $k/2<1$, by Jensen's inequality, we have

$$\sum_{n=1}^{\infty} 2^{\alpha(1-k/2)n}\{\sum_{l=2^n}^{\infty} \nu_l^2\}^{k/2} = \sum_{n=1}^{\infty} 2^{\alpha(1-k/2)n}\{\sum_{j=n+1}^{\infty}\sum_{l=2^{j-1}}^{2^j-1} \nu_l^2\}^{k/2}$$

$$\leqslant A\sum_{n=1}^{\infty} 2^{\alpha(1-k/2)n}\sum_{j=n+1}^{\infty} 2^{-kj}\{\sum_{l=2^{j-1}}^{2^j-1} l^2\nu_l^2\}^{k/2} \leqslant A\sum_{n=1}^{\infty} 2^{\alpha(1-k/2)n}\sum_{j=n+1}^{\infty} 2^{-kj}\{\sum_{l=2^{j-1}}^{2^j-1} l^2\nu_l^2\}^{k/2}$$

$$\leqslant A\sum_{j=2}^{\infty} 2^{-kj}\{\sum_{l=2^{j-1}}^{2^j-1} l^2\nu_l^2\}^{k/2}\sum_{n=1}^{j-1} 2^{\alpha(1-k/2)n} \leqslant A\sum_{j=2}^{\infty} 2^{-(k+\alpha(k/2-1))j}\{\sum_{l=2^{j-1}}^{2^j-1} l^2\nu_l^2\}^{k/2}$$

$$\leqslant \sum_{j=1}^{\infty} 2^{-(k+\alpha(k/2-1))j}\{\sum_{l=1}^{2^j} l^2\nu_l^2\}^{k/2}.$$

Concerning the converse part, we proceed with the same method. Then we have by Jensen's inequality

$$\sum_{n=1}^{\infty} 2^{-(k+\alpha(k/2-1))n}\{\sum_{l=1}^{2^n} l^2\nu_l^2\}^{k/2} \leqslant \sum_{n=1}^{\infty} 2^{-(k+\alpha(k/2-1))n}\{\sum_{j=0}^{n-1}\sum_{l=2^j}^{2^{j+1}} l^2\nu_l^2\}^{k/2}$$

$$\leqslant A\sum_{n=1}^{\infty} 2^{-(k+\alpha(k/2-1))n}\{\sum_{j=0}^{n-1} 2^{2j}\sum_{l=2^j}^{2^{j+1}} \nu_l^2\}^{k/2} \leqslant A\sum_{n=1}^{\infty} 2^{-(k+\alpha(k/2-1))n}\sum_{j=0}^{n-1} 2^{kj}\{\sum_{l=2^j}^{2^{j+1}} \nu_l^2\}^{k/2}$$

$$\leqslant A\sum_{j=0}^{\infty} 2^{kj}\{\sum_{l=2^j}^{2^{j+1}} \nu_l^2\}^{k/2}\sum_{n=j+1}^{\infty} 2^{-(k+\alpha(k/2-1))n} \leqslant A\sum_{j=0}^{\infty} 2^{\alpha(1-k/2)j}\{\sum_{l=2^j}^{2^{j+1}} \nu_l^2\}^{k/2}$$

$$\leqslant A\sum_{j=0}^{\infty} 2^{\alpha(1-k/2)j}\{\sum_{l=2^j}^{\infty} \nu_l^2\}^{k/2}.$$

Thus the proof of the result (i) is completely proved.
The proof the result (ii) is proved similarly.

4 - Contraction theorems

We say, with Beurling [1], that f *is a contraction of* g if $|f(x)-f(y)| \leqslant A|g(x)-g(y)|$, where A is a constant.

As an extension of the theorems due to Beurling [1] and Boas [2], Kinukawa [4] proved the following theorem.

Theorem C. Let $f(x) \sim \sum f_n e^{inx}$ $g(x) \sim \sum g_n e^{inx}$.

Suppose that
$$|f(x)-f(y)| \leqslant A|g(x)-g(y)|$$
for any $x,\ y \in (0,\ 2\pi)$, or more generally,
$$\int_0^{2\pi} |f(x+t)-f(x)|^2 dx \leqslant A \int_0^{2\pi} |g(x+t)-g(x)|^2 dx$$
for any t, where A is an absolute constant, and suppose that there exists a positive sequence $\{\nu_n\}$ such that

$$(1)\quad |g_n| \leqslant \nu_n \qquad (2)\quad \sum_{n=1}^{\infty} n^{-3k/2}\{\sum_{l=1}^{n} l^2 \nu_l^2\}^{k/2} + \sum_{n=1}^{\infty} n^{-k/2}\{\sum_{l=n+1}^{\infty} \nu_l^2\}^{k/2} < \infty$$

then $\sum |f_n|^k < \infty$ where $0 < k \leqslant 2$.

Sunouchi [9]$_2$ proved that the convergence of two series $\sum_{n=1}^{\infty} n^{-k/2}\{\sum_{l=n}^{\infty} \nu_l^2\}^{k/2}$ and $\sum_{n=1}^{\infty} n^{-3k/2}\{\sum_{l=1}^{n} l^2 \nu_l^2\}^{k/2}$ is mutually equivalent, so the hypotheses of this theorem may be accordingly modified.

In this section, we generalize Theorem C in the following form by using the absolute Riesz summability with index k in place of the absolute convergence.

Theorem 3. *Let* $f(x) \sim \sum f_n e^{inx}$, $g(x) \sim \sum g_n e^{inx}$.

Suppose that $\int_0^{2\pi} |f(x+t)-f(x)|^2 dx \leqslant A \int_0^{2\pi} |g(x+t)-g(x)|^2 dx$ *for any* t,

and suppose that there exists a positive sequence ν_n *such that* $|g_n| \leqslant |\nu_n|$, *and*

$$(3)\qquad \sum_{n=2}^{\infty} n^{-1}(\log n)^{-1+\alpha(2-k)/2}\{\sum_{l=n}^{\infty} \nu_l^2\}^{k/2} < \infty .$$

Then the series $\sum f_n e^{inx}$ *is summable* $|R,\ \exp(\log n)^\alpha,\ 1|_k$ *almost everywhere, where* $\alpha > 0$ *and* $1 \leqslant k < 2$.

Theorem 4. *Let* $f(x) \sim \sum f_n e^{inx}$, $g(x) \sim \sum g_n e^{inx}$.

Suppose that $$\int_0^{2\pi} |f(x+t)-f(x)|^2 dx \leqslant A \int_0^{2\pi} |g(x+t)-g(x)|^2 dx$$

for any t, *and suppose that there exists a positive sequence* $\{\nu_n\}$ *such that* $|g_n| \leqslant \nu_n$, *and*

$$\sum_{n=2}^{\infty} n^{-1}(\log n)^{-1}(\log\log n)^{-k/2}\{\sum_{l=n}^{\infty} \nu_l^2\}^{k/2} < \infty . \tag{4}$$

Then the series $\sum f_n e^{inx}$ *is summable* $|R, (\log n)^{\alpha}, 1|_k$ *almost everywhere, where* $\alpha > 0$ *and* $1 \leqslant k < 2$.

Theorem 5. *Let* $f(x) \sim \sum f_n e^{inx}$, $g(x) \sim \sum g_n e^{inx}$.

Suppose that $$\int_0^{2\pi} |f(x+t)-f(x)|^2 dx \leqslant A \int_0^{2\pi} |g(x+t)-g(x)|^2 dx$$

for any t, *and suppose that there exists a positive sequence* $\{\nu_n\}$ *such that* $|g_n| \leqslant \nu_n$, *and*

$$\sum_{n=1}^{\infty} n^{-1-\alpha(k/2-1)}\{\sum_{l=n}^{\infty} \nu_l^2\}^{k/2} < \infty \quad \text{or} \tag{5}$$

$$\sum_{n=1}^{\infty} n^{-1-k-\alpha(k/2-1)}\{\sum_{l=1}^{n} l^2\nu_l^2\}^{k/2} < \infty . \tag{6}$$

Then the series $\sum f_n e^{inx}$ *is summable* $|R, \exp n^{\alpha}, 1|_k$ *almost everywhere, where* $0 < \alpha < 2/(2-k)$ *and* $1 \leqslant k < 2$.

Theorem 6. *Let* $f(x) \sim \sum f_n e^{inx}$, $g(x) \sim \sum g_n e^{inx}$.

Suppose that $$\int_0^{2\pi} |f(x+t)-f(x)|^2 dx \leqslant A \int_0^{2\pi} |g(x+t)-g(x)|^2 dx$$

for any t, *and suppose that there exists a positive sequence* $\{\nu_n\}$ *such that*

$|g_n| \leqslant \nu_n$, *and*

(7) $$\sum_{n=2}^{\infty} n^{-1}(\log n)^{-k/2}\{\sum_{l=n}^{\infty} \nu_l^2\}^{k/2} < \infty .$$

Then the series $\sum f_n e^{inx}$ *is summable* $|R,\ n^\alpha,\ 1|_k$ *almost everywhere, where* $\alpha > 0$ *and* $1 \leqslant k < 2$.

Theorem 7. *Let* $f(x) \sim \sum f_n e^{inx}, \qquad g(x) \sim \sum g_n e^{inx}.$

Suppose that $$\int_0^{2\pi} |f(x+t) - f(x)|^2 dx \leqslant A \int_0^{2\pi} |g(x+t) - g(x)|^2 dx$$

for any t, and suppose that there exists a positive sequence $\{\nu_n\}$ *such that* $|g_n| \leqslant \nu_n$, *and*

(8) $$\sum_{n=2}^{\infty} n^{-k/2}(\log n)^{\alpha(k/2-1)}\{\sum_{l=n}^{\infty} \nu_l^2\}^{k/2} < \infty$$ *or*

(9) $$\sum_{n=2}^{\infty} n^{-3k/2}(\log n)^{\alpha(k/2-1)}\{\sum_{l=1}^{n} l^2 \nu_l^2\}^{k/2} < \infty .$$

Then the series $\sum f_n e^{inx}$ *is summable* $|R,\ \exp n/(\log n)^\alpha,\ 1|_k$ *almost everywhere, where* $\alpha > 0$ *and* $1 \leqslant k < 2$.

Theorem 8. *Let* $f(x) \sim \sum f_n e^{inx}, \qquad g(x) \sim \sum g_n e^{inx}.$

Suppose that $$\int_0^{2\pi} |f(x+t) - f(x)|^2 dx \leqslant A \int_0^{2\pi} |g(x+t) - g(x)|^2 dx$$

for any t, and suppose that there exists a positive sequence $\{\nu_n\}$ *such that* $|g_n| \leqslant \nu_n$, *and*

(10) $$\sum_{n=2}^{\infty} n^{-1}(\log n)^{-k/2}\{\sum_{l=n}^{\infty} \nu_l^2\}^{k/2} < \infty .$$

Then the series $\sum f_n e^{inx}$ *is summable* $|R,\ n/(\log n)^\alpha,\ 1|_k$ *almost everywhere, where* $\alpha > 0$ *and* $1 \leqslant k < 2$.

The case $k=1$ of Theorems $3\sim 8$ is due to Okuyama $[7]_3$.

For the proofs of these theorems, we require the structure theorem due to Leindler $[5]_1$.

Theorem D. Let $0<\beta\leqslant 2$. Let $\lambda(x)$ $(x\geqslant 1)$ be a positive monotone function such that

$$\sum_{l=n}^{\infty} l^{-\beta}\lambda(l)^{-1}\leqslant An^{-\beta+1}\lambda(n)^{-1}.$$

Then the conditions

$$\int_0^1 t^{-2}\lambda(1/t)^{-1}\{\int_0^{2\pi}[f(x+t)-f(x-t)]^2\,dx\}^{\beta/2}\,dt<\infty \qquad \sum_{n=1}^{\infty}\lambda(n)^{-1}\{E_n^{(2)}(f)\}^{\beta}<\infty$$

are mutually equivalent.

Hence we prove only Theorem 3, because the other theorems can be shown similarly.

Proof of Theorem 3. By the hypotheses of Theorem 3 and Theorem D, we obtain

$$\int_0^1 t^{-1}(\log 1/t)^{-1+\alpha(2-k)/2}\{\int_0^{2\pi}[f(x+t)-f(x-t)]^2\,dx\}^{k/2}\,dt$$

$$\leqslant A\int_0^1 t^{-1}(\log 1/t)^{-1+\alpha(2-k)/2}\{\int_0^{2\pi}[g(x+t)-g(x-t)]^2\,dx\}^{k/2}\,dt<\infty .$$

Thus, by Theorem D, the series

$$\sum_{n=2}^{\infty} n^{-1}(\log n)^{-1+\alpha(2-k)/2}\{E_n^{(2)}(f)\}^{k}$$

converges. Therefore we see from Theorem 1(ii) that the series $\sum f_n e^{inx}$ is summable $|R,\ \exp(\log n)^{\alpha},\ 1|_k$ almost everywhere.

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Abstract

The purpose of this paper is to prove the generalization of the theorems due to Kinukawa [4] *and Okuyama* $[7]_2$ *on contraction of Fourier series by using the absolute Riesz summability with index k in place of the absolute convergence.*

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