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On weakly left duo rings (**)

Introduction

A ring is called *left duo* if every left ideal is two-sided. As a generalization of left duo rings, Yao [**16**] called a ring R *weakly left duo* in case for every $r \in R$ there is a natural number $n(r)$ such that $Rr^{n(r)}$ is a two-sided ideal of R. A local ring with nil radical is weakly left duo, but not necessarily left duo. Recently, Yue Chi Ming [**17**] studied weakly duo rings in connection with strong regular rings.

Bass [**1**] proved that if R is a left perfect ring, then R has no infinite set of orthogonal idempotents and every non-zero left R-module has a maximal submodule. The converse is false, as shown by Cozzens [**4**] and Koifman [**10**]. However, the converse is true for commutative rings (see Hamsher [**8**], Renault [**14**], or [**10**]), and more general it is true for left duo rings (see Chandran [**3**]). In this paper, we generalize the above result to weakly left duo rings, and we give an example of a perfect ring that is weakly left duo but not left duo, so our generalization is non-trivial.

Recall that a ring R is a *left* (*right*) *V-ring* if every simple left (right) R-module is injective. A well-known result of Kaplansky states that a commutative ring is (von Neumann) regular if and only if it is a V-ring. In the noncommutative case neither the necessary nor the sufficient part holds (see [**4**], [**5**]). Brown ([**2**], Theorem 4.8) and Chandran ([**3**], Theorem 1) proved that a left duo ring is regular if and only if it is a left V-ring. This result has been extended to weakly left duo rings [**17**]. Using this we prove that the group ring $R[G]$ over a weakly duo ring R is regular if and only if each left (right) ideal of $R[G]$ is idempotent.

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Examples in [4] show that the above results for weakly duo rings can not be extended to a larger class of rings, namely those rings with each idempotent central.

1 – Throughout, R represents an associative ring with identity and R-modules are unital. Let J denote the Jacobson radical of R. Using an idea of [8] (Lemma 2), we first generalize this result to weakly left duo rings.

Lemma 1. *Let R be a weakly left duo ring and let every non-zero left R-module have a maximal submodule. Then every element of R which is not a right zero divisor is a unit.*

Proof. Let x be an element of R which is not a right zero divisor. Let $A = \bigoplus_{i=1}^{\infty} Ry_i$ where $Ry_i \cong R/Rx^i$, that is, $L_R(y_i) = Rx^i$. Let $B = \sum_{i=1}^{\infty} R(xy_{i+1} - y_i)$. Then $A/B = \sum_{i=1}^{\infty} R\bar{y}_i$ where $\bar{y}_i = y_i + B$. Suppose $A \neq B$, then A/B has a maximal submodule M and $\bar{y}_i \notin M$ for some i. It follows that $\bar{y}_k \notin M$ for all $k \geq i$. Since R is weakly left duo, $x^j R \subseteq Rx^j$ for some j. Let $n = ij$, then $\bar{y}_n \notin M$ and $x^n R \subseteq Rx^n$. Now M is a maximal submodule of A/B, so $r\bar{y}_n + m = \bar{y}_{2n}$, for some $r \in R$ and $m \in M$. Then

$$\bar{y}_n = x^n \bar{y}_{2n} = x^n(r\bar{y}_n + m) = (x^n r)\bar{y}_n + x^n m$$

$$= (r' x^n)\bar{y}_n + x^n m \quad (\text{since } x^n R \subseteq Rx^n) \quad = x^n m \in M \quad (\text{since } x^n \bar{y}_n = 0)$$

which is a contradiction. Hence $A = B$. So there exists elements $r_1, r_2, \ldots, r_n \in R$ with

$$y_1 = \sum_{i=1}^{n} r_i(xy_{i+1} - y_i) = -r_1 y_1 + \sum_{i=2}^{n} (r_{i-1}x - r_i)\, y_i + r_n x y_{n+1}.$$

Since the y_i are independent

$$y_1 = -r_1 y_1, \; r_{i-1}x - r_i \in L_R(y_i) = Rx^i \qquad (i = 2, \ldots, n) \qquad r_n x \in L_R(y_{n+1}) = Rx^{n+1}.$$

Since $r_n x \in Rx^{n+1}$, and x is not a right zero divisor, $r_n \in Rx^n$. Suppose that $r_k \in Rx^k$, $2 \leq k \leq n$. Since $r_{k-1}x - r_k \in Rx^k$, $r_{k-1}x \in Rx^k$. As x is not a right zero divisor, $r_{k-1} \in Rx^{k-1}$. By introduction we have $r_1 \in Rx$. Then $y_1 = -r_1 y_1 = 0$ so $R = Rx$. Left $yx = 1$, then $(xy - 1)x = 0$. Since x is not a right zero divisor, $xy = 1$.

A ring without non-zero nilpotent elments is called *reduced*. Yue Chi Ming ([17], Proposition 3) proved that if R is a weakly left duo ring and $J = 0$, then R is reduced. We need this result to prove the following lemma.

LEMMA 2. *Let R be a weakly left duo ring and $J = 0$. Then:* (1) *$L_R(a)$ is an ideal for every $a \in R$;* (2) *$Ra \cap L_R(a) = 0$ for every $a \in R$;* (3); *if every non-zero left R module has a maximal submodule, then R is a regular ring.*

PROOF. We know that R is reduced.

(1) Suppose $x \in L_R(a)$, then $xa = 0$. Since $(ax)^2 = 0$, $ax = 0$. Let $y \in R$. Then $(xya)^2 = 0$, so $xya = 0$. Hence $xy \in L_R(a)$.

(2) Let $x \in Ra \cap L_R(a)$. Then $xa = 0$ and $x = ya$ for some $y \in R$. By (2), $xy \in L_R(a)$ and then $x^2 = xya = 0$. Thus $x = 0$.

(3) Let $0 \neq a \in R$. Since $L_R(a)$ is an ideal by (1), we can form the quotient ring $\bar{R} = R/L_R(a)$. Moreover $\bar{R}$ is weakly left duo and every non-zero left $\bar{R}$-module has a maximal submodule. Let $\bar{r} = r + L_R(a)$ be any element in $\bar{R}$. If $\bar{r} \cdot \bar{a} = 0$, then $ra \in Ra \cap L_R(a) = 0$ by (2). So $r \in L_R(a)$, that is, $\bar{r} = 0$. Thus $\bar{a}$ is not a right zero divisor. Then by Lemma 1, $\bar{a}$ is a unit in $\bar{R}$, that is, $\bar{R}\bar{a} = \bar{R}$. Thus $Ra + L_R(a) = R$. Since $Ra \cap L_R(a) = 0$, Ra is a direct summand of R and so R is regular.

Now we are in a position to prove Bass' converse for perfect rings.

THEOREM 3. *If R is a weakly left duo ring, then R is a left perfect ring if and only if R has no infinite set of orthogonal idempotents and every non-zero left R-module has a maximal submodule.*

PROOF. ($\Rightarrow$). Bass [1].

($\Leftarrow$). Let $J = J(R)$ and $S = R/J$. Since J is left T-nilpotent, it suffices to show that S is semisimple. Now J is nil, so by ([9], pp. 54-55), countable sets of orthogonal idempotents in $S = R/J$ lift orthogonally to R. Thus S can not have infinite set of orthogonal idempotents. Let $s_1, \ldots, s_n$ be a maximal set of orthogonal idempotents in S with $1 = s_1 + \ldots + s_n$. Every non-zero left S-module also has a maximal submodule, so S is regular by Lemma 2. Since S is also weakly left duo, S is normal and then each idempotent s_i is central. A ring is called *normal* if every idempotent is central. Each weakly left duo ring is normal by [**16**] (Lemma 4). Thus we have a ring decomposition $S = \bigoplus_{i=1}^{\infty} Ss_i$ where each Ss_i is regular and does not contain non-trivial idempotents. It follows that each Ss_i is a division ring and so S is semisimple.

We note that Cozzens' example ([4], p. 76) is a normal ring, so the converse of the above result is not true for normal rings.

Theorem 3 has been proved in commutative case by several authors (see [8], [10], [14]), and Chandran ([3], Theorem 3) proved it for left duo rings. To show that our generalization is non-trivial, we give a perfect ring which is weakly left duo but not left duo.

Example. Let F be a field and

$$R=\{\begin{bmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{bmatrix} | 0,\ a,\ b,\ c,\ d \in F\}.$$

Since R is a local semiprimary ring, it is weakly left duo. But R is not left duo, since

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}=\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\notin R\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

2 – A ring R is called *fully left (right) idempotent* if $I^2 = I$ for each left (right) ideal I of R. It is known that a left (right) *V-ring* is fully left (right) idempotent ([12], Corollary 2.2), and it is easy to see that regular rings are both fully left and fully right idempotent.

Lemma 3. *Let R be a weakly left duo ring. If R is fully Left (or right) idempotent, then R is reduced.*

Proof. Let $r \in R$ and $r^2 = 0$. Since $Rr = RrRr$, by [**16**] (Lemma 2), $r = \sum_{i=1}^{n} x_i r$ for some nilpotent elements $x_i \in RrR$. Then $(1 - x_1)r = \sum_{i=2}^{n} x_i r$. Let $y = (1 - x_1)^{-1}$ and we have $r = \sum_{i=2}^{n} yx_i r$, where each yx_i is still nilpotent by [**16**] (Lemma 2).

Using the induction we have $r = zr$ for some nilpotent element z, and it follows that $r = zr = z^2 r = \ldots = 0$. Hence R is reduced. Similarly if R is fully right idempotent then R is reduced, too.

Recall that a ring R is called *strongly regular* if for each $r \in R$ there exists $x \in R$ such that $r = xr^2$. It is well-known that a ring is strongly regular if and only if it is regular and duo (see [15], Chapter 1, § 12).

The equivalence of (1) through (5) of the following result has been proved in [17] (Proposition 7).

Proposition 1 (Yue Chi Ming [17]). *Let R be a weakly left fyo ring. The following are equivalent*: (1) *R is a strongly regular ring*; (2) *R is a left V-ring*; (3) *R is a right V-ring*; (4) *R is fully left idempotent*; (5) *R is fully right idempotent*; (6) *Each factor ring of R is reduced.*

Proof. (1) $\Rightarrow$ (2), (3). Since R is regular and duo, the results follow from [2] (Theorem 4.8) or [3] (Theorem 1].

(2) $\Rightarrow$ (4) and (3) $\Rightarrow$ (5) ([12] Corollary 2.2).

(4) $\Rightarrow$ (6). Let I be a ideal of R. The, R/I is also fully left idempotent by Ramamurthi ([13] Proposition 5), and clearly R/I is weakly left duo. Thus R/I is reduced by Lemma 3.

(5) $\Rightarrow$ (6). Similar to (4) $\Rightarrow$ (6).

(6) $\Rightarrow$ (1). Let $r \in R$. There is a natural number $n = n(r)$ such that Rr^n is a two-sided ideal of R, and then Rr^{2n} is also a two-sided ideal. Now $r + Rr^{2n}$ is a nilpotent element of the reduced ring R/Rr^{2n} and then $r = xr^2$ for some $x \in R$.

Corollary 1. *Let R be a weakly left duo ring. The following are equivalent*: (1) *Every non-zero left R-module has a maximal submodule*; (2) *J is left T-nilpotent and $S = R/J$ is a regular ring.*

Proof. It is known (see [8], Lemma 1) that (1) holds if and only if J is left T-nilpotent and every non-zero left S-module has a maximal submodule. Thus by Lemma 2(3) we have (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (1). Since S is also weakly left duo, S is a left V-ring by Proposition 6. It follows from [12] (theorem 2.1(2)] that every non-zero left S-module has a maximal submodule, and then (1) follows.

Since Cozzens ([4], p. 76) has constructed a normal right V-ring that is not regular, Proposition 1 can not be extended to normal rings.

Let R be a ring, G be a group, and $R[G]$ the group ring. It is well-known (see [**11**], p. 155, for example) that the group ring $R[G]$ is regular if and only if: (1) R is regular; (2) G is locally finite; (3) the order of each element of G is a unit in R. Thus by [**12**], (Lemma 2.3(c) and Lemma 6.5), $R[G]$ is regular if and only if R is regular and $R[G]$ is fully left (or right) idempotent.

There is a normal ring R and a group G such that $R[G]$ is a left V-ring but not regular [7] (p. 112). This can not happen if we strengthen the condition of R to be a weakly left duo ring.

Proposition 2. *Let R be a weakly left duo ring and G a group. The following are equivalent*: (1) $R[G]$ *is regular*; (2) $R[G]$ *is fully left idempotent*; (3) $R[G]$ *is fully right idempotent.*

Proof. We only verify $(2) \Rightarrow (1)$.

Let I be a left ideal of R. Then $I[G]$ is a left ideal of the fully left idempotent ring $R[G]$. We have $I[G] = I[G]^2 = I^2[G]$. It follows that $I = I^2$, and then R is fully left idempotent. Thus R is regular by Proposition 1, and then $R[G]$ is regular.

Corollary 2. *Let R be a weakly left duo ring and G a group. If $R[G]$ is a left (or right) V-ring, then $R[G]$ is regular.*

The converse of Corollary 2 is not true: Farkas and Snider [**6**] produced examples of regular rings $F[G]$ over field F such that $F[G]$ is neither left nor right V-ring (see [7], p. 109, Example 3).

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Abstract

Weakly left duo rings are studied in connection with perfect rings, von Neumann regular rings, and V-rings.

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