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On critical q-near-fields (**)

Introduction

In [5] a group G is called critical if it is finite and does not belong to the variety generated by its proper factors. The fact that, for example, a locally finite variety is generated by its critical groups indicates the significance of such a structure. In this paper a near-ring is defined *critical* if it has proper factors and it doesn't belong to the variety generated by them.

By using the results obtained in $[8]_1$ we try to determine whether a near-ring whose proper factors are near-fields is critical, but we will not deal in the present paper with the construction of varieties of near-rings in more general cases. In this way we continue the line of thought put forward in $[8]_4$ (see also $[8]_{2,3}$).

We will prove that a q-near-field is critical iff it has only one ideal except when the q-near-field is an integral zero-symmetric near-ring. In this case we will give examples and describe the structure but we won't resolve the problem of criticality.

1 - General case

We denote by N a left near-ring and we call a near-ring *mixed* if $N = N_c + N_0$ where $N_c \neq \{0\}$ and $N_0 \neq \{0\}$. If $y \in N$ we say that $A(y) = \{x \in N / yx = 0\}$. For definitions and fundamental notations we refer to [9] without express recall.

Birkhoff in [1] proved that a class which is closed with respect to forming homomorphic images and subcartesian products of its members, is a variety. In

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the following, if H is a class of near-rings we denote by $V(H)$ the variety generated by H; in particular if $Q(N)$ is the set of homomorphic images of a near-ring N, we denote by $V(Q(N))$ the variety generated by $Q(N)$) (see [7]).

Def. A. A near-ring whose proper factors are near-fields is called *a q-near-field.*

In Theorem 1 of $[8]_2$ we prove that a q-near-field has at most two proper ideals, otherwise it is the sum of two constant near-fields (1).

Moreover as defined in [5] for groups:

Def. B. A near-ring N is called *critical* if it has proper factors but it doesn't belong to the variety generated by them.

Theorem 1. *Let N be a critical q-near-field. Then N has only one non-trivial proper ideal.*

Since N is critical, it must be subdirectly irreducible, otherwise it is isomorphic to a subnear-ring of a direct product of proper homomorphic images, hence lies in the variety generated by the proper homomorphic images. As a subdirectly irreducible near-ring, N possesses a unique minimal non-trivial ideal, I say (I is the intersection of all the non-trivial ideals of N). Then N/I is a near-field, since N is a q-near-field. This forces I to be a maximal ideal. Hence it is the unique proper non-trivial ideal.

2 - Constant q-near-fields

Proposition 1. *If N is a constant near-ring and if the additive group of N is cyclic of order p^α ($\alpha > 1$, p prime) then N is critical.*

The class $Q(N)$ is made up of constant near-rings whose additive group is cyclic of order p^β, $(0 < \beta < \alpha)$. Therefore $V(Q(N))$ contains near-rings whose elements are of order less than or equal to $p^{\alpha-1}$ and N is critical.

(1) A constant near-field is isomorphic to $M_c(Z_2)$ (see [9]).

Proposition 2. *A constant near-ring N is a q-near-field with only one ideal iff its additive group N^+ has only one subgroup whose index is* 2.

If N is a constant q-near-field and has only one ideal I, then N/I is isomorphic to the constant near-field $M_c(Z_2)$ and then I^+ has index 2 in N^+.

Conversely if N^+ has only one subgroup I^+ whose index is 2, it is normal in N^+ and therefore I is an ideal of N, and N/I is isomorphic to $M_c(Z_2)$.

Lemma 1. *The variety generated by the constant near-field $M_c(Z_2)$ contains all and only the near-rings which are direct sums of isomorphic copies of $M_c(Z_2)$.*

It will be sufficient to prove that the factors and sub-near-rings of direct sums of $M_c(Z_2)$ are still direct sums of isomorphic copies of $M_c(Z_2)$. This can by immediatly deduced from the fact that all the subgroups of a direct sum of cyclic groups of order 2 are still direct sums of cyclic groups of order 2 (see [3] Th. 2.2, p. 46).

Theorem 2. *A constant q-near-field N is critical iff it has only one non-trivial proper ideal.*

From Theorem 1 a critical q-near-field has only one ideal.

Conversely if N has only one ideal we can deduce from Lemma 1 that N doesn't belong to the variety generated by its unique factor. This is because this variety contains either near-rings without ideals or near-rings which have at least 2 ideals. Therefore N is critical.

3 - Mixed q-near-fields

Lemma 2. *If a mixed q-near-field $N=N_c+N_0$ has exactly 2 ideals, it is the direct sum of N_0 and N_c (which is isomorphic to $M_c(z_2)$).*

If N has 2 ideals I and J we can deduce from Lemma 1 of [2] that N is the direct sum of I and J where N/J is isomorphic to J and N/J is isomorphic to I. Besides, given that now the factors of N are near-fields, also its ideals must be near-fields and therefore if I is constant it is isomorphic to $M_c(Z_2)$ and isomorphic to N_c while J is isomorphic to N_0.

Theorem 3. *A mixed q-near-field N is critical iff it has only one non-trivial proper ideal.*

From Theorem 1, N has only one ideal. Conversely if N has only one ideal I, N/J is a near-field. Besides, if N/I is constant, N cannot belong to the variety generated by it because that variety contains only constant near-rings. If N/I is zero-symmetric, the variety it generates contains only zero-symmetric near-rings and consequently N cannot belong to it.

4 - Zero-symmetric q-near-fields

Lemma 3. *Let N be a non integral zero-symmetric q-near-field with only one ideal I. In this case this ideal coincides with the nil radical $\mathcal{N}$ of N and all the zero-divisors of N belong to I.*

If N is non integral and has only one ideal I, it must contain some nilpotent elements. If it didn't, it would be an I.F.P. near-ring(²) (see [**10**]). In an I.F.P. near-ring N, if $y \in N$ is a zero-divisor, $A(y)$ is a proper ideal of N and therefore $A(y) = I$. Now, if i is a zero-divisor and belongs to I, $A(i)$ would still be an ideal of N and therefore $A(i) = I$. But this would mean that $i^2 = 0$ and this is excluded, so N has nilpotent elements. Besides if $\mathcal{N}$ is the set of the nilpotent elements of N, $\mathcal{N} \subset I$ because N/I is a near-field. Moreover, as I is the unique ideal of N, it is prime and coincides with the prime radical $P_0(N)$(³). In [**10**] it has been proved that the elements of $P_0(N)$ are nilpotent and therefore $I = \mathcal{N} = P_0(N)$.

Theorem 4. *A non integral zero-symmetric q-near-field is critical iff it has only one non trivial proper ideal.*

If N is a zero-symmetric q-near-field, from Theorem 1 it has only one ideal.

Conversely if N is a q-near-field with only one proper ideal and zero-divisors, then, from Lemma 3 it has nilpotent elements which belong to I. Therefore $V(Q(N))$ doesn't contain near-rings with nilpotent elements and consequently N is critical.

(²) A near-ring is I.F.P. if $ab = 0$ implies $anb = 0$ for all a, b, $n \in N$.

(³) An ideal I is *prime* if $I_1 I_2 \subseteq I$ implies $I_1 \subseteq I$ or $I_2 \subseteq I$ where I_1 and I_2 are ideals of N.

In $[8]_1$ we described the planar q-near-fields. We can now see that they are critical because the ideals of a planar near-ring are contained in the annihilator and therefore a planar q-near-field has only one ideal.

Now we will take a look at integral zero-symmetric q-near-fields.

Proposition 3. *If N is an integral zero-symmetric q-near-field it satisfies the following conditions*:

(1) *It is infinite.*

(2) *It does not have the D.C.C.N.*

(3) *It has only one ideal I which is essential as an N-subgroup and I^+ contains an N-subgroup isomorphic to N^+.*

(4) *If $a \in N \setminus I$, $aN = N$ iff $aI = I$.*

(5) *If $a \in N \setminus I$ and aN is a proper N-subgroup of N, then $N = I + (aN \setminus I)$.*

(1) If N were finite, it would be strongly monogenic and therefore simple (see [9]).

(2) If N had the D.C.C.N. it would be 2-primitive on N and therefore simple (see [9]).

(3) From Theorem 1, it follows that N has only one ideal. So it is obvious that any ideal of an integral near-ring is essential as an N-subgroup. Besides, aN is isomorphic to N as an N-subgroup given that generally aN is isomorphic to $N/A(a)$. But in this case N is integral and thus $A(a) = \{0\}$. Finally, if $a \in I$ then $aN \subset I$, and the Proposition is proven.

(4) If $aN = N$, and if $aI \subset I$ there would exist an element $i \in I$ equal to an where a, $n \in N \setminus I$. This is impossible because N/I is an integral near-ring. Therefore if $aN = N$, $aI = I$.

Conversely if $aI = I$, given that N/I is a near-field, for any $n \in N$ there exists $n' \in N$ where $n + I = (a + I)(n' + I)$. Thus $n - an' = i$ for $i \in I$. As $aI = I$, there exists $i' \in I$ where $ai' = i$ and therefore $n = an' + ai'$, that is $n \in aN$.

(5) Because N/I is a near-field, for any $z \in N \setminus I$ there exists an $n \in N$ where $z + I = (a + I)(n + I)$, that is, $z - an \in I$, therefore any element $z \in N$ is expressible as an element of I and an element of $aN \setminus I$.

Examples of near-rings that satisfy the conditions of the Proposition 3 exist. An example of near-ring with only one ideal I where N-I is a field is given in [6] (p. 166 by Th. 8.6 and Cor. 8.7). In [4] there is an example of a near-ring that satisfies our Proposition 3 where N/I is a near-field.

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Summary

A near-ring is defined critical *if it has proper factors and it doesn't belong to the variety generated by them. We determine whether a near-ring whose proper factor are near-fields is critical. We prove that it is critical iff it has only one ideal except when it is an integral zero-symmetric near-ring. In this case we will give examples and describe the structure, but we don't solve the problem of criticality.*

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