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GISÈLE FISCHER SERVI (*)

Finite axiomatizability in bimodal calculi (**)

Introduction

A *bimodal language* is determined by a set of propositional letters, the usual set of propositional connectives and two modal operators, say L_1 and L_2. By a *bimodal logic* we understand any system containing a standard version of classical propositional logic and a set of axiom schemes concerning L_1 and L_2 (1).

Since many of the results below apply to logics having at least two independent modal operators, Intensional and Pragmatic Logic (see [6]) can be considered here as cases in point. Epistemic logic also is basically bimodal («It is known that», «It is believed that») and becomes plurimodal when the person that knows or believes is brought into the formalization as an index for a modal operator (see [4]). By the same token, Dynamic Logic (or the logic of programs) can be viewed as a significant extension of a bimodal logic containing only either two program letters or one program letter and one program operator, say the iterative operator (see $[7]_2$). More standard but less well-known examples of bimodal logic are the Logic of viewpoints and the $S4+S5$ Fitting Logic combining time and logical necessity (see [3]). Another example of bimodal logic is the ($S4$, $*$)-C system, where $*$ stands for any classical normal logic. These logics have been introduced in $[2]_1$, $[2]_2$ and are instrumental for studying intuitionistic modal logic.

Now one can easily be led to think that the theory of simultaneous modal operators is a simple extension of the theory of single modal operators: it may seem that studying two modalities is very much like studying each one

(*) Indirizzo: Dipartimento di Filosofia, Facoltà di Lettere, Università, Parma, Italia.

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(1) The terminology is somewhat different from that used in $[2]_1$ and $[2]_3$, where a bimodal logic was defined as satisfying certain connecting axioms.

separately. That such is not the case is exemplified by the complexity of completeness and decidability proofs in Dynamic Logic (see $[7]_2$). Moreover, not all metatheoretical properties carry over from the monomodal to the plurimodal case. The purpose of this paper is to show, in fact, that such properties as having a finite number of distinct modalities and finite axiomatizability do not always hold for bimodal logics even when they hold for each monomodal logic when taken separately.

A common feature of the bimodal and plurimodal logics mentioned above is that the behavior of the two (or perhaps more) modal operators is described by the traditional modal systems (usually *S*4 or *S*5) (2). A discriminating property for these logics is the presence or absence of conditions expressing some relationship between the primitive modal concepts. These conditions, which we shall call *connecting axioms*, are of various nature and appear both in the Fitting logic and the (*S*4, $*$)-*C* systems. On the other hand, Pragmatic, Intensional, Dynamic logic and the Logic of viewpoints have no such connecting axioms.

Hence, in this paper we shall study systems in which at least one of the operators is *S*4 and compare results when some connecting conditions are added.

1 - Number of distinct modalities

First let us recall that a *modality* is any sequence of zero or more monadic operators. Hence in the language of bimodal logic, a modality is a sequence of $\neg$ (*negation*), L_1 (first *necessity operator*), L_2 (second *necessity operator*), M_1 (first *possibility operator*) and M_2 (second *possibility operator*). A modality is said to be *positive* if it contains no negation sign; a modality is in *standard form* if it is either positive or a negation of a positive one. In classical modal logic, L_1 and M_1 (as well as L_2 and M_2) are interdefinable, so that any modality can be expressed in standard form. Let us assume from now on that all modalities are expressed in standard form. Furthermore we call *length* of a modality the number of operators which occur in it and given a modality $\mathfrak{K}$, let $\mathfrak{K}^n$ denote the modality consisting of n occurrences of $\mathfrak{K}$ (for $n=0$, $\mathfrak{K}^n A$ is simply A). Last, two modalities $\mathfrak{M}$ and $\mathfrak{N}$ are said to be *equivalent* in a system *S* if, for every wff. A, the two formulae $\mathfrak{M}A$ and $\mathfrak{N}A$ are equivalent in *S*.

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(2) For the axioms of *S*4 and *S*5, refer to the standard versions presented in [1] (p. 4 and p. 10).

It is a well known fact that $S4$ and $S5$ have a finite number of non equivalent modalities. Yet the bimodal system which combines $S4$ with itself or with $S5$ without adding any connecting conditions has an infinite number of modalities. To see this let us define ($S4$, $*$) as the bimodal logic such that L_1 is an $S4$ modality and L_2 is of strength $*$, where $*$ is any normal monomodal calculus. In particular, we well be concerned with ($S4$, $S4$) and ($S4$, $S5$) (2). Clearly an ($S4$, $S4$) *model* is a quadruple $\mathcal{S} = (S, R_1, R_2, v)$ where S is a non-empty set, R_1 and R_2 are two reflexive and transitive relations on S and v is a truth function on formulas satisfying the usual modal conditions. In particular we have

(1) $v(L_1 A, s) = 1$ iff $v(A, s') = 1$ for all s' such that $sR_1 s'$

(2) $v(L_2 A, s) = 1$ iff $v(A, s') = 1$ for all s' such that $sR_2 s'$.

An ($S4$, $S5$) *model* is an ($S4$, $S4$) model with R_2 symmetrical. It is easy to see that the bimodal logic ($S4$, $S4$) – i.e. $S4$ axiom schemes and rules on L_1 and L_2 – is complete with respect to ($S4$, $S4$) models. Obviously the same holds for ($S4$, $S5$) logic with respect to ($S4$, $S5$) models.

Using these definitions we prove

Proposition 1. *The* ($S4$, $S4$) *logic and the* ($S4$, $S5$) *logic have an infinite number of non-equivalent modalities.*

Proof. We will prove that there is a modality $\mathfrak{K}$ such that for any atomic formula A,

(i) *if* $m \neq p$ *then* $\mathfrak{K}^m A \leftrightarrow \mathfrak{K}^p A$ *is not an* ($S4$, $S5$) *theorem.*

Clearly it is sufficient to show

(ii) $\mathfrak{K}^n A \rightarrow \mathfrak{K}^{n+1} A$ *is an* ($S4$, $S4$) *theorem,* but

(iii) $\mathfrak{K}^{n+1} A \rightarrow \mathfrak{K}^n A$ *is not an* ($S4$, $S5$) *theorem.*

For, suppose that (i) and (ii) hold but for some $m, p \in \mathrm{N}$, with $m < p$,

(iv) $\mathfrak{K}^m A \leftrightarrow \mathfrak{K}^p A$ *is an* ($S4$, $S5$) *theorem.*

Then $\mathfrak{K}^p A \to \mathfrak{K}^m A$ is also an $(S4, S5)$ theorem and by using (ii) and the transitivity of implication we can obtain the following $(S4, S5)$ theorems

(v) $\qquad \mathfrak{K}^p A \to \mathfrak{K}^m A,\ \mathfrak{K}^p A \to \mathfrak{K}^{m+1} A,\ \ldots,\ \mathfrak{K}^p A \to \mathfrak{K}^{p-1} A\,.$

The last element in the sequence (v) contradicts (iii) so (iv) cannot hold for any $m,\ p \in \mathbb{N}$.

Now define $\mathfrak{K}A$ to be $M_2 M_1 A$. It is obvious that in this case (ii) holds. To see that (iii) is also the case, consider the following $(S4, S5)$ infinite counter model: let $S = \{a_0, b_0, a_1, b_1, \ldots\}$. Let R_1 and R_2 be the reflexive closures of $\{(a_{n+1}, b_n)\colon n \geqslant 0\}$ and $\{(b_n, a_n)\colon n \geqslant 0\} \cup \{(a_n, b_n)\colon n \geqslant 0\}$ respectively

$$a_0 \overset{R_2}{\Leftrightarrow} b_0 \overset{R_1}{\leftarrow} a_1 \overset{R_2}{\Leftrightarrow} b_1 \overset{R_1}{\leftarrow} a_2 \overset{R_2}{\Leftrightarrow} b_2 \overset{R_1}{\leftarrow} \ldots$$

Diagram 1

Suppose now that $v(A, a_0) = 1$ but $v(A, b_0) = 0$ and for all $n \neq 0$, both $v(A, a_n) = 0$ and $v(A, b_n) = 0$. Then it is easy to check that $v(\mathfrak{K}\mathfrak{K}A, a_1) = 1$ but $v(\mathfrak{K}A, a_1) = 0$. For, $v(M_1 A, a_0) = 1$; hence $v(M_2 M_1 A, b_0) = 1$ and ultimately $v(M_1 M_2 M_1 A, a_1) = 1$. But R_2 is reflexive, so $v(\mathfrak{K}\mathfrak{K}A, a_1) = 1$. On the other hand, $v(M_1 A, a_1) = 0$ and thus $v(\mathfrak{K}A, a_1) = 0$. In fact an inductive argument shows that for all $n \neq 0$, $v(\mathfrak{K}^{n+1} A \to \mathfrak{K}^n A, a_n) = 0$.

It is not unreasonable to expect that if we add to the above bimodal systems some conditions connecting the two modal operators, then some combinations of operators might be reducible. We proceed to show that in the $(S4, *)$-$\boldsymbol{C}$ logics this is not always the case.

Let us recall that the $(S4, S4)$-$\boldsymbol{C}$ [$(S4, S5)$-$\boldsymbol{C}$ resp.] logic (see [2]$_2$) extends $(S4, S4)$ [$(S4, S5)$] with the following *connecting axioms*

(3) $$M_2 L_1 A \to L_1 M_2 A$$

(4) $\qquad L_1 L_2 A \to L_2 L_1 A \quad$ or, equivalently, $\quad M_2 M_1 A \to M_1 M_2 A\,.$

Remark 1. Conditions (3) and (4) are equivalent in $(S4, S5)$-$\boldsymbol{C}$, for the following wff.'s are theorems of the system

$$M_2 L_1 A \to M_2 L_1 M_2 A \qquad M_2 L_1 A \to M_2 L_1 L_2 M_2 A\,.$$

Then using (4) we have $M_2L_1A \to M_2L_2L_1M_2A$ and so $M_2L_1A \to L_2L_1M_2A$ which ultimately yields (3). Use the same type of ideas to prove that (3) implies (4) (see also $[2]_2$).

We first establish

Lemma 1. *Let both Q, U range over $\{L, M\}$. Then for any wff. A, the following are theorems of* $(S4, S5)$-C

(i) $Q_2L_1U_2A \leftrightarrow L_1U_2A$ (ii) $Q_2M_1U_2A \leftrightarrow M_1U_2A$

Proof. Put $Q = M$. Using (3), we have that $M_2L_1U_2A \to L_1M_2U_2A$ is an $(S4, S5)$-C theorem. But U_2 and Q_2 are of strength $S5$, so $Q_2L_1U_2A \to L_1U_2A$. The if part of (i) is immediate. Now let $Q = L$. Note that starting from (4), we have $L_1L_2U_2A \to L_2L_1U_2A$. Since L_2 $[M_2]$ is an $S5$ operator, we derive $L_1U_2A \to Q_2L_1U_2A$. The only if part of (i) is obvious. Passing to negations yields (ii).

Proposition 2. $(S4, S5)$-C *has a finite number of non equivalent positive modalities, viz. any positive modality can be reduced to an equivalent sequence having the form $R_1Q_2U_1$, where R_1 and U_1 range over the 7 irreducible positive modalities in $S4$ and Q_2 ranges over the 3 irreducible positive modalities in $S5$. Moreover, each modality can be reduced to one of length at most* 8.

Proof. Note first that the maximum length of irreducible modalities in $S4$ and $S5$ is 3 and 1 respectively, so that the maximum length of $R_1Q_2U_1$ is 7. Now, given any positive modality $\mathfrak{R}$, call $\mathfrak{R}'$ the modality obtained from $\mathfrak{R}$ by using all reduction laws of $S4$ $[S5]$ on combinations of L_1 and M_1 $[L_2$ and $M_2]$. We show by induction on the number of initial operators in $\mathfrak{R}'$ that Proposition 2 holds. For $\mathfrak{R}'$ of length 1 the result is obvious. Suppose that result holds for every modality of length m and let $\mathfrak{R}'$ be of length $m+1$, say $\mathfrak{R}' = Z\mathfrak{R}''$, with Z a single modal operator. By the inductive hypothesis, $\mathfrak{R}''$ is equivalent to $R_1Q_2U_1$ hence $\mathfrak{R}'$ is equivalent to $ZR_1Q_2U_1$. Suppose now that $Z = Z_1$ is either L_1 or M_1; then using the $S4$ reduction laws, ZR_1 can be substituted by an irreducible combination of L_1 and M_1, say T_1. Hence $\mathfrak{R}'$ is equivalent to $T_1Q_2S_1$ and has length $n \leqslant 7$.

Suppose on the other hand that $Z = Z_2$ is either L_2 or M_2 and that R_1 is some sequence $Y_1^1 \dots Y_1^k$ of L_1 and M_1 with $k \leqslant 3$. Now $Z_2R_1Q_2U_1$ is

$Z_2 Y_1^1 \dots Y_1^k Q_2 U_1$; starting from Y_1^k apply Lemma 1, $k-1$ times, to obtain

(i) $$Z_2 Y_1^1 Q_2 Y_1^2 \dots Q_2 Y_1^k Q_2 U_1 .$$

Then starting from $Z_2 Y_1^1 Q_2$, apply repeatedly Lemma 1 to modality (i) and obtain

(ii) $$Y_1^1 \dots Y_1^k Q_2 U_1 .$$

But (ii) is none other than $R_1 Q_2 U_1$.

Corollary 1. *The Fitting logic* $S4+S5$ *has a finite number of distinct modalities.*

Proof. Recall that the Fitting logic is a bimodal logic determined by an $S4$ operator L_1 and an $S5$ operator L_2 with the following connecting axiom

(5) $$L_2 A \to L_1 A .$$

But from (5) we can infer $L_2 A \to L_2 L_1 A$ and ultimately $L_1 L_2 A \to L_2 L_1 A$. So (3) as well as (4) (see Remark 1) hold in the Fitting logic.

Remark 2. *The property ascribed to* ($S4$, $S5$)-C *by* Proposition 2 *also applies to the system which differs from* ($S4$, $S5$)-C *by lacking the two necessitation rules, since both in the proof of* Proposition 2 *and in that of* Lemma 1, *only axiom schemes are used.*

Given any $*$, the connecting conditions of ($S4$, $*$)-C are not always sufficient to reduce modalities to a finite number. To see this we first prove

Lemma 2. *Let* $\mathfrak{K} = M_1 L_1 M_2 L_2$. *Then for any wff. A, we have that* $\mathfrak{K}^{n+1} A \to \mathfrak{K}^n A$ *is a theorem of* ($S4$, $S4$)-C.

Proof. We only need to prove that $\mathfrak{K}\mathfrak{K}A \to \mathfrak{K}A$ is an ($S4$, $S4$)-C theorem, so start with the obvious $L_2 \mathfrak{K}A \to M_1 L_1 M_2 L_2 A$; since M_2 is of strength $S4$, by (4) obtain $M_2 L_2 \mathfrak{K}A \to M_1 M_2 L_1 M_2 L_2 A$. Now apply (3) to this formula and get $M_2 L_2 \mathfrak{K}A \to M_1 L_1 M_2 M_2 L_2 A$, which in turn yields, first $M_2 L_2 \mathfrak{K}A \to \mathfrak{K}A$ and then its immediate consequence $L_1 M_2 L_2 \mathfrak{K}A \to \mathfrak{K}A$. Now apply M_1 on both sides of the last implication and the desired result follows from the characteristic $S4$ axiom on M_1.

Our aim is to prove that for some formula A,

(6) $$\mathfrak{K}^n A \to \mathfrak{K}^{n+1} A$$

is *not* a theorem of ($S4$, $S4$)-C, if $\mathfrak{K}$ is defined as in Lemma 2. To do this we use a semantic argument. So we first recall that in $[2]_2$ ($S4$, $S4$)-C was shown to be complete with respect to $S4$-double models, which were defined as follows: $\mathcal{S} = (S, R_1, R_2, v)$ is an $S4$-*double model* ($S4$-d.m.) if $\mathcal{S}$ is an ($S4$, $S4$) model and for every s, t, $u \in S$

(7) if sR_2t and tR_1u then there is $w \in S$ such that sR_1w and wR_2u

(8) if sR_2t and sR_1u then there is $w \in S$ with tR_1w and uR_2w.

Let us now construct an $S4$-d.m. in which (6) fails to be valid. Consider the infinite set

$$S = \{x_0, a_0, a_1, a_2, \ldots, b_0, b_1, b_2, \ldots, c_0, c_1, c_2, \ldots\}$$

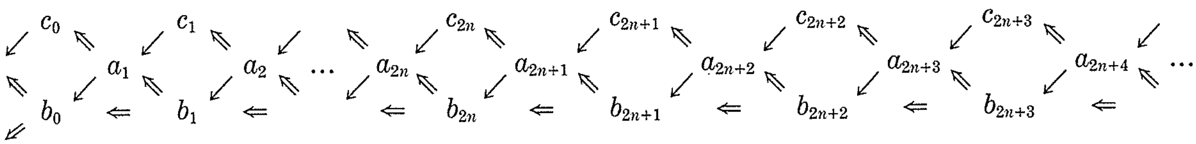
and for each $n > 0$ let

(9) $b_{n+1} R_2 b_n$ (10) $a_{n+1} R_1 b_n R_2 a_n$ (11) $a_{n+1} R_2 c_n R_1 a_n$

and let

(12) $$b_0 R_2 x_0 \qquad a_0 R_2 x_0 .$$

Furthermore let R_1 and R_2 be the smallest reflexive and transitive relations on S satisfying (9) to (12). The following diagram gives an account of the structure of (S, R_1, R_2) (omitting loops and transitive closures for the purpose of readability)



$\rightarrow$ for R_1
$\Rightarrow$ for R_2

Diagram 2

It is easy to check that conditions (7) and (8) are satisfied; so consider a truth valuation v such that for some atomic wff. A and for every $n \geqslant 0$, $v(A, x_0) = 0 = v(A, b_n)$ and $v(A, a_n) = 1 = v(A, c_n)$. Then $\mathcal{S} = (S, R_1, R_2, v)$ is an $S4$-d.m.

Lemma 3. *Let* $\mathfrak{K}$ *be as in Lemma 2. Then* $v(\mathfrak{K}A, c_0) = 0$.

Proof. Note that $v(M_2 L_2 A, a_0) = 0$, so

$$v(L_1 M_2 L_2 A, a_0) = 0 = v(L_1 M_2 L_2 A, c_0).$$

Hence $v(\mathfrak{K}A, c_0) = 0$.

In order to show that (6) is not valid in $\mathcal{S}$, we also prove

Lemma 4. *Let* $\mathfrak{K}$ *be as above. For every* $n \geqslant 0$ *the following are true*

$(13)_n$ $\quad v(L_1 M_2 L_2 \mathfrak{K}^n A, b_{2n+1}) = 1$ $\qquad (14)_n$ $\quad v(\mathfrak{K}^{n+1} A, a_{2n+2}) = 1$

$(15)_n$ $\quad v(\mathfrak{K}^{n+2} A, a_{2n+2}) = 0$ $\qquad (16)_n$ $\quad v(\mathfrak{K}^{n+1} A, c_{2n+2}) = 1$

$(17)_n$ $\quad v(\mathfrak{K}^{n+2} A, c_{2n+2}) = 0$

Proof. By induction on n. As the proof develops we shall frequently use, without explicit mention, conditions (9)-(12). In order to follow the steps of the proof, it is therefore advisable to refer to Diagram 2. Consider first $n = 0$. Then $(13)_0$ holds, since $v(L_2 A, c_0) = 1 = v(M_2 L_2 A, b_1)$, so $v(L_1 M_2 L_1 A, b_1) = 1$. But then

(i) $$v(\mathfrak{K}A, a_2) = 1$$

and $(14)_0$ is proved. Moreover by using Lemma 3 we obtain $v(L_2 \mathfrak{K}A, c_0) = v(L_2 \mathfrak{K}A, a_1) = v(L_2 \mathfrak{K}A, b_1) = 0$, therefore

(ii) $$v(M_2 L_2 \mathfrak{K}A, b_1) = 0.$$

Then we have

(iii) $$v(L_1 M_2 L_2 \mathfrak{K}A, b_1) = 0 = v(L_1 M_2 L_2 \mathfrak{K}A, a_2)$$

and conclude $v(\mathfrak{K}\mathfrak{K}A,\ a_2)=0$. So, $(15)_0$ is proved. To prove $(16)_0$, just note that $v(M_1\mathfrak{K}A,\ c_2)=1$ by (i), so $v(\mathfrak{K}A,\ c_2)=1$. Finally (ii) implies $v(L_1M_2L_2\mathfrak{K}A,\ c_2)=0$ which, together with (iii), yields $v(\mathfrak{K}\mathfrak{K}A,\ c_2)=0$.

Assume now $(13)_n$-$(17)_n$; we set to prove $(13)_{n+1}$-$(17)_{n+1}$. Because of $(16)_n$ we have

$$v(L_2\mathfrak{K}^{n+1}A,\ c_{2n+2})=1=v(M_2L_2\mathfrak{K}^{n+1}A,\ b_{2n+3})$$

so $v(L_1M_2L_2\mathfrak{K}^{n+1}A,\ b_{2n+3})=1$, i.e. $(13)_{n+1}$ is proved. Note that $(14)_{n+1}$, i.e. $v(\mathfrak{K}^{n+2}A,\ a_{2n+4})=1$, is an immediate consequence of $(13)_{n+1}$. Furthermore $(17)_n$ implies $v(L_2\mathfrak{K}^{n+2}A,\ c_{2n+2})=0=v(L_2\mathfrak{K}^{n+2}A,\ a_{2n+3})=v(L_2\mathfrak{K}^{n+2}A,\ b_{2n+3})$. So $v(M_2L_2\mathfrak{K}^{n+2}A,\ b_{2n+3})=0$, hence

$$\text{(iv)}\qquad v(L_1M_2L_2\mathfrak{K}^{n+2}A,\ b_{2n+3})=v(L_1M_2L_2\mathfrak{K}^{n+2}A,\ a_{2n+4})$$
$$=v(L_1M_2L_2\mathfrak{K}^{n+2}A,\ c_{2n+4})=0\,.$$

From (iv) we infer both $v(\mathfrak{K}^{n+3}A,\ a_{2n+4})=0$ and $v(\mathfrak{K}^{n+3}A,\ c_{2n+4})=0$, so that $(15)_{n+1}$ and $(17)_{n+1}$ are proved.

To obtain $(16)_{n+1}$, use $(14)_{n+1}$ to yield $v(M_1\mathfrak{K}^{n+2}A,\ c_{2n+4})=v(\mathfrak{K}^{n+2}A,\ c_{2n+4})=1$, concluding the proof of lemma.

We can now establish

Proposition 3. *$(S4,\ S4)$-C has an infinite number of non equivalent modalities.*

Proof. Because $(S4,\ S4)$-C is complete with respect to $S4$-d.m.'s Lemma 4 shows that there is a modality $\mathfrak{K}$ and a wff. A such that for every $n\geqslant 0$, $\mathfrak{K}^nA\to\mathfrak{K}^{n+1}A$ is not an $(S4,\ S4)$-C theorem. On the other hand, by Lemma 2, we have that for every $n>0$, $\mathfrak{K}^{n+1}A\to\mathfrak{K}^nA$ is a theorem of $(S4,\ S4)$-C. Reasoning as in Proposition 1, we have that if $p\neq m$, then $\mathfrak{K}^pA\leftrightarrow\mathfrak{K}^mA$ is not an $(S4,\ S4)$-C theorem and so $\mathfrak{K}^p$ is not equivalent to $\mathfrak{K}^m$.

A finite number of distinct modalities is often required for the successful use of the filtration method (see [6]). It is also closely connected with the property of finite axiomatizability. We recall that a propositional calculus S is said to be *finitely axiomatizable* (f.a.) if there is a finite set of schemas $\{A_1,\ \dots,\ A_n\}$ such that $\vdash_S A_i$ $(i=1, \ldots, n)$ and every theorem of S can be derived from $A_1, \ldots, A_n$ with $\boldsymbol{MP}$ only. For instance, modal calculi containing the necessitation rule are not necessarily finitely axiomatizable. It is well known that $S4$ and $S5$ are f.a., while $\boldsymbol{T}$ is not (see [5]). Since in the proofs of these facts the number of irreducible modalities of a specific system plays a crucial role, Lemmon, in [5], has been led to conjecture the following

Lemmon conjecture. *Any system having a finite number of irreducible modalities is finitely axiomatizable.*

To our knowledge this conjecture has not been either proved or disproved. The next result about $(S4,\ S5)$-$\boldsymbol{C}$ and the Fitting Logic provides other examples in favour of the Lemmon conjecture.

We first define the system $\boldsymbol{B}5$ as containing: the set X of the axioms of $(S4,\ S5)$-$\boldsymbol{C}$, the set $Y=\{\mathfrak{K}A\colon A\in X,\ \mathfrak{K}$ a finite sequence of alternate L_1 and $L_2\}$, and closed with respect to $\boldsymbol{MP}$.

Lemma 5. *The system* $\boldsymbol{B}5$ *is deductively equivalent to* $(S4,\ S5)$-$\boldsymbol{C}$.

Proof. Clearly we only need to show that

(i) *each element in* Y *is derivable in* $(S4,\ S5)$-$\boldsymbol{C}$

(ii) *one can derive in* $\boldsymbol{B}5$ *both necessitation rules.*

As for (i), if $A\in Y$, then it can be obtained by successive applications of both necessitation rules of $(S4,\ S5)$-$\boldsymbol{C}$. Suppose then that $A=L_1B$ with B a theorem of $\boldsymbol{B}5$. Let us proceed by induction on the length of proofs in $\boldsymbol{B}5$. If B is an axiom then result is obvious. So suppose that B is obtained by $\boldsymbol{MP}$ from the two $\boldsymbol{B}5$ theorems C and $C\to B$. From the inductive hypothesis we have that L_1C and $L_1(C\to B)$ are derivable in $\boldsymbol{B}5$, which using an $S4$ axiom provides $L_1C\to L_1B$. Hence, L_1B is a theorem of $\boldsymbol{B}5$. The case $A=L_2B$ is analogous.

Now consider the logic $\boldsymbol{C}5$ defined as the set of wff.'s which contains X, the set $Y'=\{L_1L_2L_1A\colon A\in X\}$ and which is closed with respect to $\boldsymbol{MP}$.

Proposition 4. *The system* $(S4,\ S5)$-$\boldsymbol{C}$ *and the Fitting Logic are finitely axiomatizable.*

Proof. Note that $\boldsymbol{C}5$ is equivalent to $\boldsymbol{B}5$, since by Remark 2 every element of Y is equivalent to an element of Y'. Therefore $\boldsymbol{C}5$ is equivalent to ($S4$, $S5$)-$\boldsymbol{C}$ and has the desired properties. To see that the Fitting Logic is also finitely axiomatizable, proceed as above and as in Corollary 1.

Here finite axiomatizability and a finite number of non equivalent modalities go together. Surprisingly, we can show that ($S4$, $S4$)-$\boldsymbol{C}$, although characterized by an infinite number of modalities, is also finitely axiomatizable. To do this, we first define the system $\boldsymbol{B}4$ determined by the set $V = \{A\colon A$ is an axiom of ($S4$, $S4$)-$\boldsymbol{C}\}$, the set $W = \{\mathfrak{K}A\colon A \in V$ and $\mathfrak{K}$ is a sequence of alternate L_1 and $L_2\}$ and $\boldsymbol{MP}$. For reasons analogous to those in Lemma 5, $\boldsymbol{B}4$ *is deductively equivalent to* ($S4$, $S4$)-$\boldsymbol{C}$. Now define another system $C4$ as having $\boldsymbol{MP}$ as the only inference rule and the following axioms:

(18) $L_1 A \to A$ (19) $L_2 A \to A$

(20) $L_1 L_2 B$ (21) $L_1 L_2 L_1 B$

where A is any wff. and B any axiom in ($S4$, $S4$)-$\boldsymbol{C}$. In order to show that $\boldsymbol{B}4$ and $\boldsymbol{C}4$ are deductively equivalent, we first prove

Lemma 6. *Suppose $\mathfrak{K}$ is a sequence of alternate L_1 and L_2 operators of length* > 2. *For any wff. A, either*

(i) $\mathfrak{K}A \leftrightarrow L_1 L_2 A$ or (ii) $\mathfrak{K}A \leftrightarrow L_1 L_2 L_1 A$

is derivable in $\boldsymbol{C}4$.

Proof. We proceed by induction on the length n of $\mathfrak{K}$. Let $n = 3$; then either

(iii) $\mathfrak{K}A = L_1 L_2 L_1 A$ or (iv) $\mathfrak{K}A = L_2 L_1 L_2 A$.

We only need to prove result for (iv). Using (4), we obtain the $\boldsymbol{C}4$ theorem $L_1 L_2 L_2 A \to L_2 L_1 L_2 A$ and by (18), (19) and (20) obtain (i).

Now let the length of $\mathfrak{K}$ be $n+1$ and suppose that Lemma 6 holds for any modality $\mathfrak{K}'$ made up of n alternate L_1 and L_2 operators. Then either

(v) $\mathfrak{K}A = \mathfrak{K}' L_1 A$ or (vi) $\mathfrak{K}A = \mathfrak{K}' L_2 A$.

We thus have that one of the following is a $C4$ theorem

(vii) $\mathfrak{K}A \leftrightarrow L_1 L_2 L_1 L_1 A$ (viii) $\mathfrak{K}A \leftrightarrow L_1 L_2 L_1 L_2 A$

(ix) $\mathfrak{K}A \leftrightarrow L_1 L_2 L_1 A$ (x) $\mathfrak{K}A \leftrightarrow L_1 L_2 L_2 A$.

Cases (vii) and (x) are equivalent to (ii) and (i) respectively (use (18)-(20)). As for (viii), by (4) and (18)-(20) we have first that $L_1 L_2 A \rightarrow L_1 L_2 L_1 L_2 A$ is a theorem, from which $L_1 L_2 L_1 A \rightarrow L_1 L_2 L_1 L_2 A$ is also such. The other implication in (viii) is obvious, thus concluding the proof.

Lemma 7. *For any wff. A, if A is derivable in* $C4$ *then so are* $L_1 A$ *and* $L_2 A$.

Proof. By induction on the length of proofs in $C4$. Suppose first that A is (18). Then $L_1 L_2 A$ is a theorem of $C4$, and so are $L_1 L_2 A \rightarrow L_2 A$ and $L_1 L_2 L_1 A \rightarrow L_1 A$. Hence L_1 and $L_2 A$ are $C4$ theorems. Similarly if A is (19).

If A is (20), A is equivalent to $L_1 A$; furthermore by Lemma 6, $L_2 A$ is equivalent to either (20) or (21). Hence both $L_1 A$ and $L_2 A$ are derivable in $C4$.

If A is (21), using again Lemma 4, we have that both $L_1 A$ and $L_2 A$ are equivalent to either (20) or (21).

If A is obtained by ***MP*** from B and $B \rightarrow A$, then by the inductive hypothesis

(i) $$L_1 B \quad \textit{and} \quad L_1(B \rightarrow A)$$

are derivable in $C4$. But

(ii) $$L_1(B \rightarrow A) \rightarrow (L_1 B \rightarrow L_1 A)$$

is an axiom of ($S4$, $S4$)-C, so $L_1 L_2$ (ii) is derivable in $C4$. By using (18) and (19) we conclude that (ii) is also a theorem of $C4$. Hence from (i) and (ii) we infer $L_1 A$. Analogously we prove that $L_2 A$ is derivable in $C4$, when A is obtained by ***MP*** as above.

The next proposition gives us an interesting counterexample for the converse of Lemmon's conjecture.

Lemma 8. *The system* $B4$ *is deductively equivalent to* ($S4$, $S4$)-C.

Proof. Using Lemma 7, proceed as in the analogous proof of Lemma 5.

Proposition 5. *The system (S4, S4)-C is finitely axiomatizable.*

Proof. We obtain result by showing that $\boldsymbol{B}4$ and $C4$ are deductively equivalent. Obviously $C4 \subseteq \boldsymbol{B}4$. Now let $A \in V$; using (18), (19) and (20) we derive A in $C4$. If $A \in W$, then $A = \mathfrak{K}\boldsymbol{B}$, with $\boldsymbol{B} \in V$. Suppose that the length of $\mathfrak{K}$ is greater than 2; then by Lemma 6, $\mathfrak{K}\boldsymbol{B}$ is derivable in $C4$. On the other hand if the length of $\mathfrak{K}$ is less than or equal to 2, then in all possible cases, by (18)-(21), $\mathfrak{K}\boldsymbol{B}$ is derivable in $C4$. Using Lemma 8, we can then conclude that $C4$ is deductively equivalent to ($S4$, $S4$)-$\boldsymbol{C}$ and has the required properties.

The next results indicate that the connecting axioms play a decisive role in obtaining finite axiomatizability; we show in fact that the logic ($S4$, $S4$) is *not* f.a., hereby providing another example in favour of the Lemmon conjecture. To do this, we shall make essential use of the following

Tarski theorem. *Let $\boldsymbol{S}$ be any system. Then a sufficient and necessary condition for $\boldsymbol{S}$ to* not *be finitely axiomatizable is that there be an infinity of systems $S_0, S_1, \ldots, S_n, \ldots$ such that for all n, $\boldsymbol{S}_n \subseteq \boldsymbol{S}_{n+1}$, $\boldsymbol{S}_n \neq \boldsymbol{S}$ and $\boldsymbol{S} = \bigcup_{n\in N} \boldsymbol{S}_n$.*

We begin by considering the system $\boldsymbol{D}4$ obtained from ($S4$, $S4$) by substituting both necessitation rules of that system with

(22) from A infer $L_1 L_2 A$

(23) from $A \to B$ infer $L_1 A \to L_1 B$.

Clearly,

Lemma 9. *The system $\boldsymbol{D}4$ is deductively equivalent to ($S4$, $S4$).*

Proof. That $\boldsymbol{D}4 \subseteq (S4, S4)$ is obvious. To show the converse, we suppose that A is a $\boldsymbol{D}4$ theorem; using (22) we infer $L_1 L_2 A$, so $L_2 A$ is derivable in $\boldsymbol{D}4$. On the other hand, by (23) the formula $L_1 L_2 A \to L_1 A$ is a theorem of $\boldsymbol{D}4$, so $L_1 A$ is also derivable.

Define now the system $\boldsymbol{D}$ to be as $\boldsymbol{D}4$ except for substituting (22) with

$$(24) \qquad \text{from} \quad A \to B \quad \text{infer} \quad L_1 L_2 A \to L_1 L_2 B$$

Then construct an infinite sequence of systems $\boldsymbol{D}_0, \boldsymbol{D}_1, \ldots, \boldsymbol{D}_n, \ldots$ where $\boldsymbol{D}_n$ has $\boldsymbol{MP}$ as the only inference rule and with the following set of axioms ($n \in \mathrm{N}$)

$$\mathcal{A}_n = \{A: \ \textit{is a theorem of } \boldsymbol{D}\} \cup \{(L_1 L_2)^n \top\}$$

$\top$ being a propositional tautology. Finally, set $\mathcal{A} = \bigcup_{n \in \mathrm{N}} \mathcal{A}_n$ and let $\boldsymbol{D}_\infty$ be the corresponding system. We show that $\boldsymbol{D}_\infty = \boldsymbol{D}4$ and that $\boldsymbol{D}_n \neq \boldsymbol{D}4$, for each n.

First let us prove

Lemma 10. *If A is a theorem of $\boldsymbol{D}4$, then for some $n \in \mathrm{N}$, $(L_1 L_2)^n \top \to A$ is a theorem of $\boldsymbol{D}$.*

Proof. By induction on the length of proofs in $\boldsymbol{D}4$. If A is in axiom, then for all $n \in \mathrm{N}$, $(L_1 L_2)^n \top \to A$ is a theorem in $\boldsymbol{D}$. Suppose that $A = L_1 L_2 B$ is obtained by (22) from a $\boldsymbol{D}4$ theorem B. Then by the inductive hypothesis there is $n \in \mathrm{N}$ such that $(L_1 L_2)^n \top \to B$ is a theorem of $\boldsymbol{D}$. By (24), we have $(L_1 L_2)^{n+1} \top \to A$ is a theorem of $\boldsymbol{D}$. Suppose now that $A = (L_1 B \to L_1 C)$ is obtained by (23) from the $\boldsymbol{D}4$ theorem $B \to C$; then by inductive hypothesis there is $n \in \mathrm{N}$ such that $(L_1 L_2)^n \top \to (B \to C)$ is a theorem of $\boldsymbol{D}$. But (23) is also a rule in $\boldsymbol{D}$, so by using $\boldsymbol{S}4$ axioms and rules on L_1 we conclude the $\boldsymbol{D}$ theorem $(L_1 L_2)^n \top \to (L_1 B \to L_1 C)$.

Lemma 11. *For each $n \in \mathrm{N}$, $D_n \subseteq D_{n+1}$; moreover $\boldsymbol{D}_\infty = \boldsymbol{D}4$.*

Proof. By using an axiom of $(\boldsymbol{S}4, \boldsymbol{S}4)$ we readily obtain $\boldsymbol{D}_n \subseteq \boldsymbol{D}_{n+1}$. Let A be derived in $\boldsymbol{D}4$. Then by Lemma 7, for some $n \in \mathrm{N}$, $(L_1 L_2)^n \top \to A$ is a theorem in $\boldsymbol{D}$ and thus an axiom of $\boldsymbol{D}_n$. By $\boldsymbol{MP}$, A is a theorem of $\boldsymbol{D}_n$ and hence one of $\boldsymbol{D}_\infty$. On the other hand, let A be an axiom of $\boldsymbol{D}_\infty$. If $A \in D$, then $A \in \boldsymbol{D}4$. If, for some n, $A = (L_1 L_2)^n \top$, then by (22) A is again a $\boldsymbol{D}4$ theorem.

Next we verify that for each $n \in \mathrm{N}$, $\boldsymbol{D}_n \neq \boldsymbol{D}_{n+1}$. To do this we define a sequence $\mathfrak{M}_1, \ldots, \mathfrak{M}_n$ of structures with $\mathfrak{M}_n = (M_n, R_1^n, R_2^n)$ where $M_n = \{b_0, a_1, b_1, a_2,$

$b_2, \ldots, a_n, b_n\}$ and

(25) $$b_i R_1^n a_i \qquad (i \geq 1)$$

(26) $$a_{i+1} R_2^n b_i \qquad (i \geq 0)$$

$$b_0 \Leftarrow a_1 \leftarrow \ldots \leftarrow b_{j-1} \Leftarrow a_j \leftarrow b_j \Leftarrow \ldots \leftarrow b_{n-1} \Leftarrow a_n \leftarrow b_n$$

Diagram 3

Furthermore, let R_1^n, R_2^n be the smallest reflexive and transitive relations on $M_n - \{b_0\}$ and M_n respectively, satisfying both (25) and (26). Now let v be a truth function on $\mathfrak{M}_n$ such that

(27) $v(L_1 A, s)$ satisfies (1) for all $s \in M_n - \{b_0\}$

(28) $$v(L_1 A, b_0) = 0$$

(29) $v(L_2 A, s)$ satisfies (2) for all $s \in M_n$.

Let A be a wff. and $n \in \mathrm{N}$. We say that $\mathfrak{M}_n$ *is nice* for A if $v(A, a_n) = 1$, for every truth function v satisfying (27)-(29). We say that A is *valid* in $\mathfrak{M}_n$ if $v(A, s) = 1$, for every such truth function v and every $s \in M_n$.

Lemma 12. *If A is a theorem of $\boldsymbol{D}$ and $n \in \mathrm{N}$, then A is valid in $\mathfrak{M}_n$.*

Proof. We only show that (24) is valid in $\mathfrak{M}_n$ ($n \in \mathrm{N}$). So take a structure $\mathfrak{M}_n$ defined as above and suppose that $v(B \to C, s) = 1$, for all $s \in M_n$. Suppose furthermore $v(L_1 L_2 B, b_j) = 1$, where $b_j \in M_n$ and $j \neq 0$. Then $v(L_2 B, b_j) = 1 = v(L_2 B, a_j)$ and so $v(B, b_j) = v(B, a_j) = v(B, b_{j-1}) = 1$. By the inductive hypothesis we infer

(i) $v(C, b_j) = 1$ (ii) $v(C, b_{j-1}) = 1$ (iii) $v(C, a_j) = 1$.

Using (ii) and (iii) we first conclude

(iv) $$v(L_2 C, a_j) = 1$$

and hence $v(L_1 L_2 C,\ a_j) = 1$. Furthermore by (i), $v(L_2 C,\ b_j) = 1$ which with (iv) yields $v(L_1 L_2 C,\ b_j) = 1$. We conclude the proof by noting that $v(L_1 L_2 B \rightarrow L_1 L_2 C,\ b_0) = 1$.

We now prove that, for every n, the system $\boldsymbol{D}_n$ can be characterized by $\mathfrak{M}_n$.

Proposition 6. *For every wff. A, if A is a theorem of $\boldsymbol{D}_n$ then $\mathfrak{M}_n$ is nice for A.*

Proof. It is sufficient to prove that $\mathfrak{M}_n$ is nice for the axioms of $\boldsymbol{D}_n$ and niceness is preserved by ***MP***. Then suppose first that A is a theorem of $\boldsymbol{D}$; by Lemma 12, A is valid in $\mathfrak{M}_n$ and so $v(A,\ a_n) = 1$ for every v. Now let $A = (L_1 L_2)^n \top$. For $0 < j < n$, it is easy to see, by induction on j, that $v((L_1 L_2)^j \top,\ b_j) = 1 = v((L_1 L_2)^j \top,\ a_{j+1})$, for every v on M_n. So

$$v((L_1 L_2)^{n-1} \top,\ b_{n-1}) = 1 = v((L_1 L_2)^{n-1} \top,\ a_n).$$

By (26), we can infer $v(L_2(L_1 L_2)^{n-1} \top,\ a_n) = 1$ and so $v((L_1 L_2)^n \top,\ a_n) = 1$.

To show that the system $\boldsymbol{D}_n$ is distinct from $\boldsymbol{D}_{n+1}$, we establish

Proposition 7. *For all $n \in \mathrm{N}$, $\mathfrak{M}_n$ is not nice for $(L_1 L_2)^{n+1} \top$.*

Proof. It is immediate to check, by induction on j, that, for each j with $0 < j \leqslant n$,

$$\text{(i)} \qquad v((L_1 L_2)^j \top,\ b_{j-1}) = 0$$

since (28) yields (i) for $j = 1$ and the inductive step is obvious. From (i) and (26) we infer $v(L_2(L_1 L_2)^n \top,\ a_n) = 0$ and ultimately $v((L_1 L_2)^{n+1} \top,\ a_n) = 0$.

Corollary 2. *For each $n \in \mathrm{N}$, $\boldsymbol{D}_n \neq \boldsymbol{D}_\infty$.*

Proof. By Lemma 11, Proposition 6 and Proposition 7.

Corollary 3. *The system ($\boldsymbol{S}4$, $\boldsymbol{S}4$) is not finitely axiomatizable.*

Proof. By Lemma 9, Lemma 11, Corollary 2 and Tarski's theorem.

A similar argument shows that

Corollary 4. *The system (S4, S5) is not finitely axiomatizable.*

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Sunto

Sistemi che contengono due o più modalità distinte (calcoli bimodali e calcoli plurimodali) compaiono negli ambiti più diversi: in Teoria della computazione, in Linguistica e in Logica matematica vera e propria. Una metateoria concernente la classe delle logiche (bi- e) pluri-modali non è stata sufficientemente sviluppata; il presente lavoro avanza in questa direzione, misurandosi rispetto ad una congettura di Lemmon che riguarda il numero di modalità non equivalenti di un sistema e la sua finita assiomatizzabilità.

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