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Boundary regularity for weighted quasi-minima (**)

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Introduction

In their paper [3] Di Benedetto and Trudinger established the Harnack inequality for all functions belonging to a De Giorgi class.

They also proved that the quasi-minima related to certain variational integrals belong to the previous classes. So the Harnack inequality holds for quasi-minima.

In this paper we deal with weighted quasi-minima.

In **1** we prove that the Di Benedetto and Trudinger result can be extended to the weighted quasi-minima provided that weighted De Giorgi classes are introduced.

In **2** we are concerned with boundary behaviour of weighted quasi-minima.

To this purpose we want underline the relevance of prove Harnack inequality, not only for Q-minima, but also and particularly for functions in De Giorgi classes. In fact, in **2**, following [12]$_1$, we use the Harnack inequality for functions in De Giorgi classes obtained in **1**, to prove a regularity criterion for Q-minima. The result that we obtain is

$$\textit{if}\quad \limsup_{r\to 0}\frac{w(B_r(x_0)-\Omega)}{w(B_r(x_0))}>0\,,\quad \textit{then } u(x) \textit{ is continuous at } x_0 .$$

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Applying again Harnack inequality for functions in De Giorgi classes we will state in a subsequent paper a Wiener-type criterion.

We observe that the Harnack inequality for weighted quasi-minima have been obtained by Modica [7] with weights in the A_p classes of Mouckenhoupt for any $p>1$ (for convenience we treat we treat only the case $p=2$).

But Modica doesn't treat with De Giorgi classes and his method is different of that of [3]. The method of [3] is followed by Scornazzani [10] who obtains the Harnack inequality for quasi-minima of non-uniformly degenerating functionals. The results of **1** are therefore a particular case of [10]. We choosed to expound that for sake of completeness and, moreover, our proof is indipendent. Precisely we are interested to functionals with not only second order terms and we take care of the best constants for which the stated estimates are valable.

1 - Let Ω be an open set. We consider functionals of the form

$$J(u,\ \Omega)=\int_{\Omega} f(x,\ u,\ \nabla u)\,dx \tag{1.1}$$

where $f(x,\ u,\ p)$ is a Caratheodory function, namely measurable in x for every $(z,\ p)$ and continuous in $(z,\ p)$ for almost all $x\in\Omega$. Moreover the function f satisfies the following conditions

$$|p|^2-b|z|^2-g(x)\leqslant\frac{f(x,\ z,\ p)}{w(x)}\leqslant\mu|p|^2+b|z|^2+g(x) \tag{1.2}$$

where $\mu\leqslant 1$, $b\in L^{\infty}_{\mathrm{loc}}(\Re^N)$, $g\in L^q(\Omega,\ w)$, $q>N$. The weight $w(x)$ belongs to the A_2 class of Muckenhoupt [8], that is

$$\sup_B\Big(\frac{1}{|B|}\int_B w\Big)\Big(\frac{1}{|B|}\int_B\frac{1}{w}\Big)\leqslant C \tag{1.3}$$

where B denote an arbitrary ball of $\Re^N$.

The spaces $H^1_0(.,\ w)$, $H^1(.,\ w)$, $L^p(.,\ w)$ are the usual weighted Sobolev spaces as in [9]. On this subject we recall an imbedding result due to Fabes, Kenig and Serapioni [4].

Proposition 1. There are positive constants c and $\bar{\delta}$ such that, for any ball

$B \subset \mathfrak{R}^N$ and for all $\varphi \in C_0^\infty(B)$,

$$\left(\frac{1}{w(B)} \int_B \varphi^s w dx\right)^{1/s} \leqslant C(\operatorname{diam} B)\left(\frac{1}{w(B)} \int_B |\nabla \varphi|^2 w dx\right)^{1/2}$$

where $2 \leqslant s \leqslant \frac{2N}{N-1} + \bar{\delta} = \overline{2}$.

The following property of A_2 weights is specially helpful.

Proposition 2 [1]. If $w \in A_2$, then there exists some positive constants $\delta_1 = 2$, $\delta_2 < 1$, c_1, c_2, such that

$$c_1\left(\frac{|E|}{|B|}\right)^{\delta_1} \leqslant \frac{w(E)}{w(B)} \leqslant c_2\left(\frac{|E|}{|B|}\right)^{\delta_2}$$

for any measurable set E of $\mathfrak{R}^N$ and any ball $B \supset E$. (Here we are using $w(E)$ to denote $\int_E w(x)\,dx$ and $|E|$ to denote the Lebesgue measure of E).

Using the terminology of Giaquinta and Giusti $[5]_2$ we give the following

Def. 1. We call a function u in $H^1_{\text{loc}}(\Omega, w)$ a *sub Q minimum* (*super Q-minimum*) *for J* if $Q \geqslant 1$ and

$$J(u, k) \leqslant QJ(u + \varphi, K) \tag{1.4}$$

for every $\varphi \leqslant 0 (\geqslant 0)$ in $H^1(\Omega, w)$ with $\operatorname{supp} \varphi = K \subset \Omega$.

A Q-minimum for J is both a sub and super Q-minimum.

Instead of proving directly the Harnack inequality for quasi minima, we prefer make a digression to provide the Harnack inequality for a more general class of functions. (This is also the way gone in [3]) This will be applied in 2.

Def. 2. *A weighted De Giorgi class* $DG_2^\pm(\Omega, w)$ is defined to consist of all functions u in $H^1_{\text{loc}}(\Omega, w)$ such that, for any ball $B_R = B_R(y) \subset \Omega$, $\sigma \in (0, 1)$, $k \geqslant 0$, the following inequality holds

$$\int_{B_{\sigma R}} |\nabla (u-k)^\pm|^2 w \leqslant \gamma\left\{\frac{1}{(1-\sigma)^2 R^2} \int_{B_R} |(u-k)^\pm|^2 w + (\chi^2 + (R^{-\alpha} k)^2 w(A^\pm_{k,R})^{\varepsilon + 2/\overline{2}}\right\} \tag{1.5}$$

where γ, χ, α, ε are non negative constants, $\varepsilon \leqslant 1 - 2/\overline{2}$, $\alpha = N\varepsilon/2$ and $A^\pm_{k,R} = B_R \cap [(u-k)^\pm \geqslant 0]$.

If u belongs contemporarily to the classes $DG_2^+(\Omega, w)$ and $DG_2^-(\Omega, w)$, then we will say that $u \in DG_2(\Omega, w)$.

The concepts of Q-minima and De Giorgi classes are linked together in the following

Proposition 3. *Let u in $H^1_{loc}(\Omega, w)$ be a sub (super) Q-minimum for J. Then $u \in DG_2^+(\Omega, w)$ ($DG_2^-(\Omega, w)$) with constants $2/\overline{2} + \varepsilon = 1 - 1/q$, $\chi^2 = \|g\|_{L^q_{loc}(\Omega, w)}$, γ depending on Q, μ, $\|b\|_{L^\infty_{loc}(\Omega, w)}$.*

Proof. By use of the well known hole-filling device and Lemma 1.1 of $[5]_1$ we easily obtain

$$\int_{B_{\sigma R}} |\nabla(u-k)^+|^2 w \leq \gamma\{\frac{1}{(1-\sigma)^2 R^2} \int_{B_R} |(u-k)^+|^2 w + (k^2 + \|g\|_q)\, w(A^+_{k,R})^{1-1/q}\}$$

if u is a sub Q-minimum for J, $B_R = B_R(y) \subset \Omega$, $\sigma \in (0, 1)$, $k \geq 0$. The case of a super Q-minimum is similar.

Remark. The Harnack inequality for functions in a weighted De Giorgi class will be proved in Theorem 3 as a direct consequence of the following two Theorems 1 and 2. The Corollary 1 and 2 translate the results of Theorems 1 and 2 for the particular case of sub and super Q-minima. Corollary 3 represents the Harnack inequality for the quasi-minima.

Theorem 1. *Let u in $DG_2^\pm(\Omega, w)$, $B_R = B_R(y) \subset \Omega$. Then, for any $\sigma \in (0, 1)$, $p > 0$,*

$$\sup_{B_{\sigma R}} u^\pm \leq C(1-\sigma)^{-2/p\varepsilon}\{(\fint_{B_R} (u^\pm)^2 w)^{1/p} + \chi R^\alpha\} \tag{1.6}$$

where $C = C(N, \gamma, \varepsilon, p)$. (Here we are using $\fint_{B_R} vw$ to denote $\frac{1}{w(B_R)} \int_{B_R} vw$).

Proof. Following [6], as adapted by [3] (Lemma 2.1) we can prove that, for u in $DG_2^\pm(\Omega, w)$, $B_R = B_R(y) \subset \Omega$ and any $\sigma \in (0, 1)$,

$$\sup_{B_{\sigma R}} u^\pm \leq \frac{c}{(1-\sigma)^{1/\varepsilon}}\{(\fint_{B_R} (u^\pm)\, w)^{1/2} + \chi R^\alpha\} \tag{1.7}$$

where $C = C(N, \gamma, \varepsilon)$.

We observe that (1.7) is obtained making use of Proposition 1 and 2. An interpolation argument allow us to conclude the Theorem 1.

Corollary 1. *Let u be a sub Q-minimum for J, $B_R = B_R(y) \subset \Omega$. Then, for every $\sigma \in (0, 1)$, $p > 0$, we have*

$$\sup_{B_{\sigma R}} u \leqslant C\big(\fint_{B_R} (u^+)^p w)^{1/p} + \chi R^{\alpha}\big) \tag{1.8}$$

where: $C = C(N, \mu, q, Q, \|g\|_q, \|b\|_{\infty,\mathrm{loc}}, \sigma)$ $\quad \chi^2 = \|g\|_q \quad \alpha = \dfrac{N}{2}(1 - 1/q - 2/\overline{2})$.

We state a preliminary Lemma, before proving Theorem 2.

Lemma 1. *Let $\theta \in (0, 1)$ fixed be and let $u \geqslant 0$, $u \in DG_2(\Omega, w)$, $B_4 = B_{4R}(y) \subset \Omega$. If, for a certain $\delta \in (0, 1)$, $w([u \geqslant \delta w(B_1)$, $B_1 = R_R(y)$, then there exists a positive integers s^* such that $w([u < \mu + 2^{-s^*}] \cap B_2) < \theta w(B_2)^d$ with*

$$s^* = s^*(N, \gamma, \theta, \delta) \qquad d = 1 + \delta_2(1/\overline{2} - 1/2 + 1/2N) \geqslant 1 - \delta_2/2N > 0$$

$$\mu = \inf u \qquad B_2 = B_{2_R}(y).$$

Proof. Taking a particular s^* to be fixed later, we may assume that $\mu < 2^{-s}$ for $s^* > s > 1$. By hypothesis, we have (see also Proposition 2)

$$w(B_2 - [u < \mu + 2^{-s}]) \geqslant C\delta w(B_2) \qquad \forall s^* > s > 1. \tag{1.9}$$

We observe now that $H^{1,1}(B) \subset H^{1,2}(B, w)$ for any ball $B \subset \Omega$. Applying a Lemma due to De Giorgi [2] (1) and Proposition 2, we have

$$2^{-s}|A_{s+1}|^{1-(1/N)} \leqslant \frac{\beta|B_2|}{|B_2 - [u < \mu + 2^{-s}]|} \int_{A_s - A_{s+1}} |\nabla u| \tag{1.10}$$

$$\leqslant C\beta\Big[\frac{w(B_2)}{w(B_2 - [u < \mu + 2^{-s}])}\Big]^{1/\delta_2} \int_{A_s - A_{s+1}} |\nabla u| \leqslant C\beta \int_{A_s - A_{s+1}} |\nabla u|$$

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(1) «Let $u \in W^{1,1}(B_r)$ and $l > k$. Then

$$(l - k)|[u < k] \cap B_r|^{1-(1/N)} \leqslant \frac{\beta r^N}{|B_r \setminus [u < l]|} \int_{\Delta_r} |\nabla u|$$

where β depends only on N and $\Delta_r = [k < u < l] \cap B_r$».

where: $B_2 = B_{2R}$, $A_s = A^-_{\bar{k}_s,2}$, $K_s = \mu + 2^{-s}$, $s = 1, 2, \ldots$, $\mu = \inf_{B_4} u$, and hence, using (1.9) and Proposition 2,

$$(1.11) \qquad 2^{-s} w(A_{s+1})^{1-(1/N)} \leqslant 2^{-s}\Big[\frac{w(B_2)}{|B_2|^{\delta_2}}\Big]^{1-(1/N)} |A_{s+1}|^{\delta_2(1-(1/N))}$$

$$\leqslant C\beta^{\delta_2}\Big[\frac{w(B_2)}{|B_2|^{\delta_2}}\Big]^{1-(1/N)} \Big(\int_{A_s - A_{s+1}} |\nabla u|\Big)^{\delta_2}.$$

On the other hand, from Def. 2, with $\chi = 0$ (replacing u by $u + \chi R^\alpha$)

$$\int_{B_2} |\nabla(u - k_s)^-|^2 w \leqslant C\{2^{-2s} w(A^-_{\bar{k}_s,4}) + (K_s)^2 w(A^-_{\bar{k}_s,4})^{2/\bar{2}+\varepsilon}\} \qquad \text{hence}$$

$$(1.12) \qquad \int_{A_s - A_{s+1}} |\nabla u| \leqslant C\Big(\int_{B_2} |\nabla(u - (\mu + 2^{-s}))^-|^2 w\Big)^{1/2}\Big(\int_{A_s - A_{s+1}} \frac{1}{w}\Big)^{1/2}$$

$$\leqslant C\Big(\int_{A_s - A_{s+1}} \frac{1}{w}\Big)^{1/2} w(A^-_{\bar{k}_s,4})^{1/\bar{2}}$$

provided $s < s^*$.

We observe that $w(A_{s+1}) \leqslant w(A^-_{\bar{k}_s,4})$ and then, from (1.11) and (1.12),

$$w(A_{s+1}) \leqslant C\Big(\int_{A_s - A_{s+1}} \frac{1}{w}\Big)^{\delta_2/2} w(B_4)^{\delta_2/\bar{2}+1/N}\Big(\frac{w(B_2)}{|B_2|^{\delta_2}}\Big)^{1-(1/N)}.$$

By summation from $s = 1$ to $s = s^* - 1$

$$w(A_{s^*}) \leqslant C\frac{|B_2|^{\delta_2} w(B_2)^{1+\delta_2/\bar{2}}}{w(B_2)^{\delta_2/2}(s_* - 2)|B_2|^{\delta_2(1-1/N)}} = Cw(B_2)^{1+\delta_2(1/\bar{2}-1/2)} |B_2|^{\delta_2/N} \leqslant Cw(B_2)^d.$$

The Lemma 1 is so proved.

Theorem 2. *Let u in $\mathrm{DG}^{\pm}_{\bar{2}}(\Omega, w)$ non negative, $B_R = B_R(y) \subset \Omega$. Then there exists $p > 0$, $p = p(N, \gamma, \varepsilon)$, such that for any σ, $\tau \in (0, 1)$, we have*

$$(1.13) \qquad \Big(\fint_{B_{\sigma R}} u^p w\Big)^{1/p} \leqslant C(\inf_{B_{\tau R}} u + \chi R^\alpha)$$

where $C = C(N, \gamma, \varepsilon, \sigma, \tau)$.

Proof. Following [3] (Proposition 3.1), we can prove that, if $u \geqslant 0$, $u \in \mathrm{DG}_2^-(\Omega, w)$, $B_{4R}(y) \subset \Omega$ and, for some $\delta \in (0, 1)$, $w([u \geqslant 1] \cap B_R) \geqslant \delta w(B_R)$, then we have

$$\inf_{B_R} u \geqslant \lambda - \chi R^{\alpha} \tag{1.14}$$

where λ is a positive constant, $\lambda = \lambda(N, \varepsilon, \gamma, \delta)$.

(1.14) is a consequence of the Lemma 1. To prove (1.14) we consider now the sequences

$$B_n = B_{\rho_n} \qquad \rho_n = 1 + 2^{-n} \qquad k_n = \mu + 2^{-s^*-1}(1 + 2^{-n}) \qquad n = 0, 1, 2, \dots .$$

By hypothesis we have $w(B_n - A^-_{k_n, \rho_{n+1}}) \geqslant C w(B_{n+1})$.

As before we apply Def. 2 over the balls B_{n+1} and B_n for the levels k_n, De Giorgi Lemma (cf. [1]) over the ball B_{n+1} for the levels $k_n > k_{n+1}$, and Proposition 2. We obtain

$$|A_{n+1}|^{1-1/N} \leqslant C\{w(A_n)^{1/2} + w(A_n)^{1/\bar{2}+\varepsilon/2}\}\Big(\int_{A_n} \frac{1}{w}\Big)^{1/2} \leqslant C|B_n|\{1 + w(A_n)^{1/\bar{2}+\varepsilon/2-1/2}\}.$$

We have $|A_{n+1}|^{1/N} \leqslant |A_n|^{1/N} \leqslant C w(A_n)^{1/2N}$ and then

$$|A_{n+1}| \leqslant C|B_n|\, w(A_n)^{1/2N}\{1 + w(A_n)^{1/\bar{2}+\varepsilon/2-1/2}\}.$$

So

$$w(A_{n+1}) \leqslant C \frac{w(B_{n+1})}{|B_{n+1}|^{\delta_2}} |A_{n+1}|^{\delta_2} \leqslant C w(B_n)\, w(A_n)^{\delta_2/2N}\{1 + w(A_n)^{1/\bar{2}+\varepsilon/2-1/2}\}^{\delta_2}. \tag{1.15}$$

We want to prove that, unless we extract a subsequence,

$$w(A_n) \to 0 \quad \text{as} \quad n \to 0. \tag{1.16}$$

We can suppose that $w(B_n - A_n) < w(B_n)$ $(n = 0, 1, 2, \dots)$ because otherwise $w(A_n) \equiv 0$ for every n sufficiently large, and (1.16) follows. If $w(B_n - A_n) \geqslant (1 - \frac{1}{2^n})\, w(B_n)$ for every n sufficiently large, then $w(A_n) \leqslant \frac{1}{2^n} w(B_n)$ and (1.16) follows. Hence, we will suppose there exists a sequence $\{n_j\}$ such that

$w(B_{n_j} - A_{n_j}) \leqslant (1 - \frac{1}{2^{n_j}}) w(B_{n_j})$ $(j = 0, 1, 2, \ldots)$ and then

$$w(A_{n_j}) \geqslant \frac{1}{2^{n_j}} w(B_{n_j}) \qquad j = 0, 1, 2, \ldots . \tag{1.17}$$

Substituting (1.17) in (1.15) and setting $Y_n = \dfrac{w(A_n)}{w(B_2)^d}$ we therefore have

$$Y_{n_j+1} \leqslant C b^{n_j} Y_{n_j}^{1+\eta} \qquad \eta \in (0,\ 1) \qquad b > 1$$

and we conclude, from Lemma 1 as in [3], [6], that $Y_{n_j} \to 0$ as $j \to +\infty$.

The proof of (1.14) and Theorem 2 follow as in [3] (cf. 303).

We observe that the Krylov and Safonov Lemma, as adapted by Trudinger [11] must be used with respect to the measure $w(x)\,dx$ instead of dx.

Corollary 2. *Let u be a non negative sub Q-minimum for J, $B_R = B_R(y) \subset \Omega$. Then there exists $p > 0$, $p = p(N, Q, \mu, q, \|g\|_q, \|b\|_{\infty,\mathrm{loc}})$ such that, for any σ, $\tau \in (0, 1)$ we have*

$$\Big(\fint_{B_{\sigma R}} u^p w \Big)^{1/p} \leqslant C\big(\inf_{B_{\tau R}} u + \chi R^\alpha\big) \tag{1.18}$$

where $C = C(N, \mu, q, Q, \|g\|_q, \|b\|_{\infty,\mathrm{loc}}, \sigma, \tau)$.

Theorem 3. *Let u in $\mathrm{DG}_2^{\pm}(\Omega, w)$ non negative, $B_R = B_R(y) \subset \Omega$. Then, for any $\sigma \in (0, 1)$,*

$$\sup_{B_{\sigma R}} u \leqslant C\big(\inf_{B_R} u + \chi R^\alpha\big) \tag{1.19}$$

where $C = C(N, \gamma, \varepsilon, \sigma)$.

Corollary 3. *Let u be a non negative Q-minimum for J, $B_R = B_R(y) \subset \Omega$. Then for any $\sigma \in (0, 1)$,*

$$\sup_{B_{\sigma R}} u \leqslant C\big(\inf_{B_{\sigma R}} u + \chi R^\alpha\big) \tag{1.20}$$

where $C = C(N, \mu, q, Q, \|g\|_q, \|b\|_{\infty,\mathrm{loc}}, \sigma)$.

2 - Boundary behaviour of quasi-minima

In this section we follow the outline of Ziemer [12]$_1$.

Let $\Omega \subset \mathfrak{R}^N$ an open bounded set and let $\beta \in H^1(\mathfrak{R}^N, w)$ such that $\beta/(\mathfrak{R}^N - \Omega)$ is continuous.

Def. 3. If $u \in H^1_{loc}(\Omega, w)$, $x_0 \in \partial\Omega$ and $L \in \mathfrak{R}$, we say that

$$u(x_0) \leqslant L \quad \textit{weakly} \tag{2.1}$$

if, for every $k > L$ there is an $r > 0$ such that $\eta(u-k)^+ \in H^1_0(\Omega, w)$ whenever $\eta \in C_0^\infty(B_r(x_0))$. The condition

$$u(x_0) \geqslant L \quad \textit{weakly} \tag{2.2}$$

is defined analogously and $u(x_0) = L$ if both (2.1) and (2.2) hold. If $u - \beta \in H^1_0(\Omega, w)$, then $u(x) = \beta(x)$ weakly for each $x \in \partial\Omega$.

Theorem 4. *Let $u \in H^1_{loc}(\Omega, w)$ be a sub Q-minimum for J such that $u - \beta \in H^1_0(\Omega, w)$. Suppose $x_0 \in \partial\Omega$ and $u(x_0) \leqslant L$ weakly. If*

$$\Lambda = \limsup_{\substack{x \to x_0 \\ x \in \Omega}} u(x) > L \tag{2.3}$$

then

$$\lim_{r \to 0} \fint_{B_r(x_0)} |u(x) - \Lambda| \, w(x) = 0 \,. \tag{2.4}$$

Proof. Let $\Lambda > k > L$ be, and $u_k = (u-k)^+$ on Ω, $u_k = 0$ otherwise. Let $r > 0$ so small that $\eta u_k \in H^1_0(\Omega, w)$ wheanever $\eta \in C_0^\infty(B_r(x_0))$. We define

$$\mu_k(r) = \sup\{u_k(x) : x \in B_r(x_0)\} \,.$$

We observe that

(2.5) there exists a positive constant $\Gamma = \Gamma(k)$ such that $\mu_k(r)$ is bounded above by Γ for $r \to 0$.

Then, for any $0<h<\mu_k(r)$, let $v(x)=\mu_k(r)-u_k(x)$ and $\varphi=\eta(v-h)^-$ where

$$\eta\in C_0^\infty(B_{2sr}(x_0)) \qquad 0\leqslant\eta\leqslant 1 \qquad \eta=1 \quad \text{on } B(x_0,\ 2tr) \qquad 0<t<s<1\,.$$

We observe that $v\geqslant 0$ and $\varphi\in H_0^1(\Omega,\ w)$. Following the method of [12]$_1$ (Theorem 3.1) one can proves that v is an element of the weighted De Giorgi class $DG_2^-(\Omega,\ w)$. Therefore, from Theorem 2, there exists $p>0$ such that

$$(2.6) \qquad \Big(\int_{B_{r/2}(x_0)} v^p w\Big)^{1/p}\leqslant C\inf_{B_{r/4}(x_0)} v+r^\alpha\leqslant C[(\mu_k(r)-\mu_k(r/4))+r^\alpha]\to 0$$

as $r\to 0$. Then $\int_{B_{r/2}(x_0)} vw\to 0$, that is $\int_{B_{r/2}(x_0)}(\mu_k(r)-u)\,w\to 0$ as $r\to 0$, where $\mu(r)=\sup\{u(x):x\in B_r(x_0)\}$ and $u(x)$ is prolonged by $\beta(x)$ out of Ω.

Then (2.4) follows.

Remarks. We observe also that $v=\mu_k$ in $B_r(x_0)-\Omega$, and then, from (2.5) and (2.6),

$$(2.7) \qquad \frac{w(B_r(x_0)-\Omega)}{w(B_r(x_0))}\to 0 \quad \text{as } r\to 0$$

Notice that also the analogous to Theorem 4 for u a super Q-minimum holds.

Theorem 5. *Let $u\in H_{loc}^1(\Omega,\ w)$ be a super Q-minimum for J such that $u-\beta\in H_0^1(\Omega,\ w)$. Suppose $x_0\in\partial\Omega$ and $u(x_0)\geqslant L$ weakly. If*

$$(2.8) \qquad \lambda=\liminf_{\substack{x\to x_0\\ x\in\Omega}} u(x)<L$$

then

$$(2.9) \qquad \lim_{r\to 0} \fint_{B_r(x_0)} |u(x)-\lambda|\,w(x)=0\,.$$

We suppose now that $u\in H_{loc}^1(\Omega,\ w)$ is a Q-minimum. For $\beta\in H^1(\Re^N,\ w)$ such that $(u-\beta)\in H_0^1(\Omega,\ w)$ and $x_0\in\partial\Omega$ we deduce from Theorems 4 and 5 that, if either

$$\limsup_{\substack{x\to x_0\\ x\in\Omega}} u(x)=\Lambda>\beta(x_0) \quad \text{or} \quad \liminf_{\substack{x\to x_0\\ x\in\Omega}} u(x)=\lambda<\beta(x_0)$$

then

$$\lim_{r\to 0} \fint_{B_r(x_0)} |u(x)-\Lambda|\, w(x) = 0 \qquad \text{or} \qquad \lim_{r\to 0} \fint_{B_r(x_0)} |u(x)-\lambda|\, w(x) = 0 .$$

It follows that it is impossible to have

$$\lambda = \liminf_{x\to x_0} u(x) < \beta(x_0) < \limsup_{x\to x_0} u(x) = \Lambda .$$

Then $u(x)$ is either upper or lower semicontinuous at x_0. Moreover, if

$$\limsup_{r\to 0} \frac{w(B_r(x_0)-\Omega)}{w(B_r(x_0))} > 0 \tag{2.10}$$

then (2.7) fails; therefore (2.3) and (2.8) fail too, that is $u(x)$ is continuous at x_0. We have so proved the following

Theorem 6. *Let $u \in H^1_{\mathrm{loc}}(\Omega, w)$ be a Q-minimum for J such that $u-\beta \in H^1_0(\Omega, w)$. Suppose $x_0 \in \partial\Omega$. If* (2.10) *holds, then $u(x)$ is continuous at x_0.*

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Abstract

We prove the Harnack inequality for functions belonging to weighted De Giorgi classes and apply this result to derive a regularity criterion at the boundary for weighted quasi-minima.

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