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Constraint and scaling methods to derive shell theory from three-dimensional elasticity (**)

Dedicated to TRISTANO MANACORDA on his seventieth birthday

1 - Introduction

I here wish to show how the linear theory of thin shells can be obtained as an exact consequence of three-dimensional elasticity in two alternative ways: by the method of internal constraints and by the scaling method. I shall first sketch the two methods, and then discuss briefly how they lend significance to each other.

The method of internal constraints to derive the classical plate equation of Germain-Lagrange has been proposed in $[14]_1$. As is well-known, Germain-Lagrange equation describes the equilibrium of a thin elastic plate of maximal response simmetry when it is acted upon by transverse loadings. Slightly more general versions of the constraint method have been shown to work in the case of in-plane loadings [11] and in the case of single or multilayered plates of arbitrary material symmetry subject to arbitrary loadings, in both statical and dynamical situations [12].

The scaling method has been introduced in 1979 by Ciarlet & Destuynder [4]; a major achievement of it is Ciarlet's «justification» of von Kármán plate equations $[3]_1$. The method has been later applied to a number of linear and nonlinear models of plates, shallow shells and rods by Ciarlet himself and others (vid. $[3]_2$ for an updated review and an inclusive list of references).

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The constraint and the scaling methods have been contrasted and put in reciprocal perspective by Ciarlet & Podio-Guidugli [7]. Of course, both methods have precursors and antecedents: to quote just the oldest ones, germs of the idea of a constrained kinematics are found in a famous paper by Kirchhoff [**10**], published in 1850; expansions in a thickness parameter appear in the papers of Cauchy [2] and Poisson [**16**] on the subject, published in 1828 and 1829, respectively. Deductions of the equations of shell dynamics from the three-dimensional balance laws are given by Truesdell & Toupin in Section 213 of [17], and Naghdi [**13**], where inclusive reference lists are also found.

2 - Preliminaries

Although a more general setting is easy to assemble [**14**]$_2$, for the purpose of this presentation I shall let $\mathcal{S}$ be a compact oriented regular surface in a three-dimensional Euclidean space $\mathcal{E}$ with associated translation space $\mathcal{V}$, and assume that $\mathcal{S}$ has a global parametrization

$$\boldsymbol{x} = \boldsymbol{x}(z^1,\ z^2)\,(^1)\,. \tag{2.1}$$

Let $\boldsymbol{n}(\boldsymbol{x})$ be the unit normal at a typical point $\boldsymbol{x} \in \mathcal{S}$. For a *shell-like region, modelled on* $\mathcal{S}$ and having *thickness* 2ε, I shall mean a region $\mathcal{G}(\varepsilon) \subset \mathcal{E}$ such that the mapping

$$\boldsymbol{p} = \boldsymbol{p}(z^1,\ z^2,\ \zeta) := \boldsymbol{x}(z^1,\ z^2) + \zeta \boldsymbol{n}(\boldsymbol{x}(z^1,\ z^2)) \tag{2.2}$$

be a global *orthogonal* parametrization of $\mathcal{G}(\varepsilon)$ itself, with $\zeta \in [-\varepsilon,\ \varepsilon]$ the oriented distance from a point $\boldsymbol{p} \in \mathcal{G}(\varepsilon)$ to the foot $\boldsymbol{x} \in \mathcal{S}$ of the unique normal line passing through $\boldsymbol{p}$, and with the vectors

$$\boldsymbol{e}_\alpha := \boldsymbol{x}_{,\alpha} \quad (\alpha = 1,\ 2)\,(^2) \tag{2.3}$$

of the covariant basis at a point $\boldsymbol{x} \in \mathcal{S}$ tangent to the lines of curvature of $\mathcal{S}$. The geometry of such a shell-like region is well-understood, and I shall summarize it here only to introduce my notation.

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(1) My geometric terminology is after Do Carmo [**9**].

(2) Here $(\cdot)_{,\alpha}$ denotes differentiation with respect to z^α.

At a point $p \in \mathscr{G}(\varepsilon)$ the covariant basis is

$$g_\alpha := p_{,\alpha} = e_\alpha + \zeta n_{,\alpha} \qquad g_3 := n \tag{2.4}$$

the contravariant basis is

$$g^\alpha := \mathrm{D}_p z^\alpha \,(^3) \qquad g^3 := n \tag{2.5}$$

the metric tensor is

$$G := g_\alpha \otimes g^\alpha + n \otimes n = g^\alpha \otimes g_\alpha + n \otimes n . \tag{2.6}$$

Consistently, at a point $x \in \mathscr{S}$,

$$e^\alpha := \mathrm{D}_x z^\alpha \qquad e^3 := n \tag{2.7}$$

is the contravariant basis, and

$$I := e_\alpha \otimes e^\alpha + n \otimes n = e^\alpha \otimes e_\alpha + n \otimes n \tag{2.8}$$

the metric tensor. For $x \in \mathscr{S}$ and $\zeta \in [-\varepsilon, \varepsilon]$ fixed, the shifters

$$A := g_\alpha \otimes e^\alpha + n \otimes n \qquad (B := g^\alpha \otimes e_\alpha + n \otimes n) \tag{2.9}$$

transforms the covariant (contravariant) basis at x into the corresponding basis at $p = x + \zeta n(x)$; in particular, either one of the surface shifters

$$^sA := g_\alpha \otimes e^\alpha \qquad ^sB := g^\alpha \otimes e_\alpha \tag{2.10}$$

transforms linearly $\mathscr{T}_x$, the tangent space to $\mathscr{S}_x$, into $\mathscr{T}_p$, the tangent space to $\mathscr{S}_p$. I shall denote by

$$^sG := g_\alpha \otimes g^\alpha = G - n \otimes n \tag{2.11}$$

the restriction to $\mathscr{T}_p$ of the linear transformation $G(p)$ of $\mathscr{V}$; similarly, I shall

(3) Here $\mathrm{D}_p z^\alpha$ denotes the gradient of the scalar field $z^\alpha = z^\alpha(p)$ on the surface patch $\mathscr{S}_p$ through p obtained by parallel transport along the normal of the patch $\mathscr{S}_x$ of $\mathscr{S}$.

write

$$ {}^{s}\boldsymbol{I} := \boldsymbol{e}_\alpha \otimes \boldsymbol{e}^\alpha \qquad {}^{s}\boldsymbol{L} := -\boldsymbol{n}_{,\alpha} \otimes \boldsymbol{e}^\alpha \tag{2.12} $$

for the surface identity ${}^{s}\boldsymbol{I}$ and the Weingarten tensor ${}^{s}\boldsymbol{L}$, so that, in particular,

$$ {}^{s}\boldsymbol{A} = {}^{s}\boldsymbol{I} - \zeta\, {}^{s}\boldsymbol{L}\,. \tag{2.13} $$

Furthermore, I shall record here the transformation laws

$$ \boldsymbol{n}\mathrm{d}s_p := \boldsymbol{g}_1 \times \boldsymbol{g}_2 \,\mathrm{d}z^1\,\mathrm{d}z^1 = \alpha \boldsymbol{n}\mathrm{d}s_x \qquad \alpha := \det \boldsymbol{A} \tag{2.14} $$

for oriented area elements,

$$ \mathrm{d}v_p := (\boldsymbol{n}\mathrm{d}\zeta)\cdot(\boldsymbol{n}\mathrm{d}s_p) = \alpha\,\mathrm{d}\zeta\,\mathrm{d}s_x \tag{2.15} $$

for volume elements (4). From (2.15) a basic integration formula follows: for $\mathscr{P}$ a shell-like part of $\mathscr{S}(\varepsilon)$ modelled on the portion $\mathcal{S}_{\mathscr{P}}$ of $\mathcal{S}$, and for $\Psi = \Psi(\boldsymbol{p})$ a field over $\mathscr{S}(\varepsilon)$,

$$ \int_{\mathscr{P}} \Psi \mathrm{d}v_p = \int_{\mathcal{S}_{\mathscr{P}}} {}^{s}\Psi \mathrm{d}s_x \qquad {}^{s}\Psi := \int_{-\varepsilon}^{\varepsilon} \Psi \alpha \mathrm{d}\zeta\,. \tag{2.16} $$

Finally, I shall define the gradient of a scalar field $v = v(\boldsymbol{p})$ and a vector field

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(4) Notice that, under my present hypotheses, $\boldsymbol{g}_\alpha(\boldsymbol{p})$ and $\boldsymbol{e}_\alpha(\boldsymbol{x})$ are parallel and Weingarten tensor is diagonal, so that, in particular,

$$ {}^{s}\boldsymbol{A} = (1 + \chi_1\zeta)\,\boldsymbol{e}_1 \otimes \boldsymbol{e}^1 + (1 + \chi_2\zeta)\,\boldsymbol{e}_2 \otimes \boldsymbol{e}^2 $$

where χ_1, χ_2 denote the principal curvatures of $\mathcal{S}$. As

$$ \boldsymbol{B}\boldsymbol{A}^T = \boldsymbol{G} \qquad \boldsymbol{A}^T\boldsymbol{B} = \boldsymbol{I} $$

the cofactor $\boldsymbol{A}^*$ of $\boldsymbol{A}$ can be expressed as

$$ \boldsymbol{A}^* = \alpha \boldsymbol{B} \qquad \alpha := \det \boldsymbol{A} = (1 + k_1\zeta)(1 + \varkappa_2\zeta)\,. $$

Then,

$$ \boldsymbol{g}_1 \times \boldsymbol{g}_2 \,\mathrm{d}z^1\,\mathrm{d}z^2 = \boldsymbol{A}^*(\boldsymbol{e}_1 \times \boldsymbol{e}_2)\,\mathrm{d}z^1\,\mathrm{d}z^2 = \alpha \boldsymbol{A}^* \boldsymbol{n}\mathrm{d}s_x $$

(cf. (2.14)); moreover, $\alpha \neq 0$ because, for $\mathscr{S}(\varepsilon)$ to be a shell-like region modelled on $\mathcal{S}$, ε must be (greater than 0 and) less than or equal to $\min\{|\chi_1|^{-1}, |\chi_2|^{-1}\}$ over $\mathcal{S}$.

$v = v(p)$ over $\mathcal{G}(\varepsilon)$ as, respectively,

$$\nabla v := v_{,i}\, g^i = \mathrm{D}_p v + v_{,\zeta}\, n \qquad \mathrm{D}_p v := v_{,\alpha}\, g^\alpha \tag{2.17}$$

(5)

$$\nabla \boldsymbol{v} := \boldsymbol{v}_{,i} \otimes g^i = \mathrm{D}_p \boldsymbol{v} + \boldsymbol{v}_{,\zeta} \otimes n \qquad \mathrm{D}_p \boldsymbol{v} := \boldsymbol{v}_{,\alpha} \otimes g^\alpha . \tag{2.18}$$

Likewise, for $w = w(x)$ and $\boldsymbol{w} = \boldsymbol{w}(x)$, respectively, a scalar and a vector field over $\mathcal{S}$, I shall write

$$\mathrm{D}_x w := w_{,\alpha}\, e^\alpha \qquad \mathrm{D}_x \boldsymbol{w} := \boldsymbol{w}_{,\alpha} \otimes e^\alpha . \tag{2.19}$$

Notice that, for $w(p) \equiv w(x)$ and $\boldsymbol{w}(p) \equiv \boldsymbol{w}(x)$ for all $p \in \mathcal{G}(\varepsilon)$ and $x \in \mathcal{S}$ such that $p = x + \zeta n(x)$, one has, respectively,

$$\nabla w = \mathrm{D}_p w = {}^s\boldsymbol{B}(\mathrm{D}_x w) \qquad \nabla \boldsymbol{w} = (\mathrm{D}_p \boldsymbol{w}) \qquad (\mathrm{D}_p \boldsymbol{w})\, {}^s\boldsymbol{A} = \mathrm{D}_x \boldsymbol{w} . \tag{2.20}$$

3 - The method of internal constraints

Central to the method is the stipulation as *exact* mathematical restrictions, prevailing in all possible motions, of the traditional verbal hypotheses underlying plate and shell theory, namely, that material fibers orthogonal to the middle surface before loading remain approximately orthogonal to it after loading, and suffer negligible stretching. Within the framework of linear strain analysis, those restrictions take the form

$$\boldsymbol{E}(\boldsymbol{u})\boldsymbol{n} = \boldsymbol{0} \quad \text{in } \mathcal{G}(\varepsilon) \qquad 2\boldsymbol{E}(\boldsymbol{u}) := \nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T . \tag{3.1}$$

It is possible to show (cf. e.g. [13], $[14]_2$) that the general solution of the partial differential system (3.1) has the representation

$$(3.2)_1 \qquad \boldsymbol{u}(z^1, z^2, \varepsilon) = \boldsymbol{d}(z^1, z^2, \zeta) - \zeta \mathrm{D}_x w(z^1, z^2) + w(z^1, z^2)\, \boldsymbol{n}(z^1, z^2)$$

parametrized by two fields defined over $\mathcal{S}$, the scalar field w and the *tangent*

(5) Cf. the definition of $\mathrm{D}_p z^\alpha$ in footnote (3).

vector field $\hat{\boldsymbol{u}}$

$$(3.2)_2 \qquad \hat{\boldsymbol{u}}(z^1, z^2)\cdot\boldsymbol{n}(z^1, z^2) = 0 \qquad \boldsymbol{d}(z^1, z^2, \zeta) = {}^s\boldsymbol{A}(z^1, z^2, \zeta)\,\hat{\boldsymbol{u}}(z^1, z^2).$$

For $\boldsymbol{p} \in \mathscr{S}(\varepsilon)$ fixed, the collection of all admissible strain tensors, i.e., the subspace $\mathcal{D}_p$ of all symmetric tensors satisfying $(3.1)_1$ at $\boldsymbol{p}$, is such that

$$(3.3) \qquad \mathcal{D}_p \equiv \mathcal{D}_x \qquad \text{for} \quad \boldsymbol{p} = \boldsymbol{x} + \zeta\boldsymbol{n}(\boldsymbol{x}) \quad \text{and for all } \zeta \in [-\varepsilon, \varepsilon].$$

As customary when internal frictionless constraints on possible motions prevail, I shall split the stress field $\boldsymbol{S}$ into a *reactive* part $\boldsymbol{S}^{(R)}$ and an *active* part $\boldsymbol{S}^{(A)}$ such that

$$(3.4)_1 \qquad \boldsymbol{S} = \boldsymbol{S}^{(R)} + \boldsymbol{S}^{(A)}$$

$$(3.4)_2 \qquad \boldsymbol{S}^{(R)}(\boldsymbol{p}) \in (\mathcal{D}_p)^{\perp}\,(^6) \qquad \boldsymbol{S}^{(A)}(\boldsymbol{p}) \in \mathcal{D}_p$$

$$(3.4)_3 \qquad \boldsymbol{S}^{(A)}(\boldsymbol{p}) = \tilde{\mathbb{C}}(\boldsymbol{p})[\boldsymbol{E}(\boldsymbol{u}(\boldsymbol{p}))]$$

with $\tilde{\mathbb{C}}(\boldsymbol{p})$ a linear, symmetric transformation of $\mathcal{D}_p$. Moreover, I shall assume that the form of $\tilde{\mathbb{C}}(\boldsymbol{p})$ reflect the *maximal material symmetry* compatible with the constraints, i.e. (cf. [15]), that

$$(3.4)_4 \qquad \tilde{\mathbb{C}}(\boldsymbol{p}) := 2\tilde{\mu}\,\mathbb{1}(\boldsymbol{p}) + \tilde{\lambda}\,{}^s\boldsymbol{G}(\boldsymbol{p}) \otimes {}^s\boldsymbol{G}(\boldsymbol{p})$$

where $\tilde{\mu}$, $\tilde{\lambda}$ are two constant material moduli and $\mathbb{1}(\boldsymbol{p})$ is the identical transformation of $\mathcal{D}_p$. In words, (3.4) prescribe that $\mathscr{S}(\varepsilon)$ is made of a *transversely isotropic* linearly elastic material such as to comply with the internal kinematical constraints expressed by (3.1).

The stored-energy functional associated with the elastic state $\{\boldsymbol{u}, \boldsymbol{E}, \boldsymbol{S}\}$ in $\mathscr{S}(\varepsilon)$ is

$$(3.5) \qquad \tilde{\Sigma} := \int_{\mathscr{S}(\varepsilon)} \tilde{\sigma}\,dv_p \qquad \tilde{\sigma}(\boldsymbol{p}) := \frac{1}{2}\boldsymbol{E}(\boldsymbol{u}(\boldsymbol{p}))\cdot\tilde{\mathbb{C}}(\boldsymbol{p})[\boldsymbol{E}(\boldsymbol{u}(\boldsymbol{p}))]$$

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(6) I.e., $\boldsymbol{S}^{(R)}$ may be represented as

$$\boldsymbol{S}^{(R)} = \sum_{i=1}^{3} S^{(R)}_{3i}\,\frac{1}{2}(\boldsymbol{n}\otimes\boldsymbol{g}^i + \boldsymbol{g}^i\otimes\boldsymbol{n}).$$

with the use of (2.16) and (3.2) one has

$$\tilde{\Sigma}(\hat{\boldsymbol{u}},\ w) = \int_{\mathscr{S}} {}^{s}\tilde{\sigma} ds_x . \tag{3.6}$$

The total energy functional for thin elastic shells is obtained by addition of an appropriate loading potential to functional (3.6); field and boundary equations for the unknown fields $\hat{\boldsymbol{u}}$ and w on $\mathscr{S}$ then follow along completely standard variational procedures.

To sum up, the method of internal constraints is based on the *geometrical* assumption that $\mathscr{S}(\varepsilon)$ is a shell-like region; the *kinematical* assumption that possible strain fields obey (3.1); and the *constitutive* assumption that the material comprising $\mathscr{S}(\varepsilon)$ is transversely isotropic: all together those assumptions integrate an explicit declaration of *thinness* for a shell-like body occupying $\mathscr{S}(\varepsilon)$ in one of its natural reference placements; that such a body respond elastically is a contingent fact, but linearity is a key feature.

4 - The scaling method

In the succinct and simplified version I shall give here the method consists of two steps: first, both the data and the solution are rescaled; secondly, the energy functional is required to stay bounded above under rescaling (7).

As to the data, let a shell-like region $\mathscr{S}(\varepsilon)$ be made of a linearly elastic *unconstrained isotropic* material described by the constitutive law

$$\boldsymbol{S} = \mathbb{C}[\boldsymbol{E}(\boldsymbol{u})] \qquad \mathbb{C} := 2\mu\mathbb{I} + \lambda \boldsymbol{G} \otimes \boldsymbol{G} \qquad \mu > 0 \qquad 2\mu + 3\lambda > 0 \tag{4.1}$$

where μ, λ are the Lamé moduli and $\mathbb{I}$ is the identical transformation of the space of symmetric tensors. The Lamé moduli are assumed to be such that

$$\mu(\varepsilon) = \varepsilon^{-3}\bar{\mu} \qquad \lambda(\varepsilon) = \varepsilon^{-3}\bar{\lambda}\ (8); \tag{4.2}$$

the domain $\mathscr{S}(\varepsilon)$ is mapped one-to-one onto

$$\bar{\mathscr{S}} := \{\bar{\boldsymbol{p}} \in \mathscr{E} \,|\, \bar{\boldsymbol{p}} = \bar{\boldsymbol{x}} + \bar{\zeta}\boldsymbol{n}(\bar{\boldsymbol{x}}) \qquad \bar{\boldsymbol{x}} \in \mathscr{S},\ \bar{\zeta} \in [-1,\ 1]\} \tag{4.3}$$

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(7) For a full presentation of the scaling method as applied to plates vid. e.g. $[3]_2$.

(8) As in the former section, for brevity I here omit to spell out the rescaling of loading data.

a shell-like region of *fixed* thickness modelled on a copy of $\mathcal{S}$ and parametrized by

(4.4) $$\bar{z}^{\alpha} = z^{\alpha} \quad (\alpha = 1,\ 2) \qquad \bar{\zeta} = \varepsilon^{-1}\zeta .$$

It follows from (4.3) and (4.4) that

(4.5) $$\boldsymbol{p}(\varepsilon) = \boldsymbol{x} + \varepsilon(\bar{\boldsymbol{p}} - \bar{\boldsymbol{x}})$$

and thus

(4.6) $$\boldsymbol{g}_{\alpha}(\varepsilon) = \boldsymbol{e}_{\alpha} + \varepsilon(\bar{\boldsymbol{g}}_{\alpha} - \bar{\boldsymbol{e}}_{\alpha}) \qquad \boldsymbol{g}_3(\varepsilon) = \bar{\boldsymbol{g}}_3 = \boldsymbol{n}\,;$$

the dependence on ε of contravariant basis, metric tensor etc. are equally easy to derive.

As to the solution, notice preliminarly that it follows from (2.18), $(3.1)_2$ that

(4.7) $$2E_{ij} := 2\boldsymbol{E}\cdot\boldsymbol{g}_i \otimes \boldsymbol{g}_j = u_{i,j} + u_{j,i} - \boldsymbol{u}\cdot(\boldsymbol{g}_{i,j} + \boldsymbol{g}_{j,i})\,.$$

For simplicity I shall restrict myself here to considering such situations that

(4.8) $$u_3\chi_{\alpha} \ll u_{\alpha,\alpha} \qquad u_{\alpha}\chi_{\alpha} \ll (u_{\alpha,3} + u_{3,\alpha}) \qquad (\alpha = 1,\ 2 \text{ not summed})\,(^9).$$

Consequently, (4.7) yield

(4.9) $$2E_{\alpha\beta} = u_{\alpha,\beta} + u_{\beta,\alpha} - u_{\gamma}\boldsymbol{g}^{\gamma}\cdot(\boldsymbol{g}_{\alpha,\beta} + \boldsymbol{g}_{\beta,\alpha})$$
$$2E_{\alpha 3} = u_{\alpha,3} + u_{3,\alpha} \qquad E_{33} = u_{3,3}\,.$$

I shall then rescale the displacement field as follows

(4.10) $$\boldsymbol{u}(z^1,\ z^1,\ \zeta;\ \varepsilon) = \varepsilon\bar{u}_{\alpha}(\bar{z}^1,\ \bar{z}^1,\ \bar{\zeta})\,\boldsymbol{g}^{\alpha}(\varepsilon) + \bar{u}_3(\bar{z}^1,\ \bar{z}^1,\ \bar{\zeta})\,\boldsymbol{n}\,.$$

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(9) In an extended presentation of the scaling method one might replace assumptions (4.8) by taking the position vector in $\mathcal{G}(\varepsilon)$ as follows (cf. [6])

$$\boldsymbol{p}(\varepsilon) = \boldsymbol{x}(\varepsilon) + \varepsilon\boldsymbol{n}(\boldsymbol{x}(\varepsilon)) \qquad \boldsymbol{x}(\varepsilon) = z^{\alpha}\boldsymbol{i}_{\alpha} + \varepsilon z^3(z^1,\ z^2)\,\boldsymbol{i}_3$$

where coordinates z^k and basis vectors $\boldsymbol{i}_k$ are cartesian orthogonal.

Passing now to rendering the dependence on ε of the energy functional explicit, I begin to recall the expression of the stored energy density associated with (4.1)

$$2\sigma(\boldsymbol{E}) = 2\mu\boldsymbol{E}\cdot\boldsymbol{E} + \lambda(\boldsymbol{E}\cdot\boldsymbol{G})^2 . \tag{4.11}$$

Notice now that, in view of (4.6) and the like, (4.9) yield

$$E_{\alpha\beta}(\varepsilon) = \varepsilon\bar{E}_{\alpha\beta} + O(\varepsilon^2) \qquad E_{\alpha3}(\varepsilon) = \bar{E}_{\alpha3} \qquad E_{33}(\varepsilon) = \varepsilon^{-1}\bar{E}_{33} \tag{4.12}$$

where

$$\begin{aligned} 2\bar{E}_{\alpha\beta} &:= \bar{u}_{\alpha,\beta} + \bar{u}_{\beta,\alpha} - \bar{u}_\gamma\bar{\boldsymbol{e}}^\gamma\cdot(\bar{\boldsymbol{e}}_{\alpha,\beta} + \bar{\boldsymbol{e}}_{\beta,\alpha}) \\ 2\bar{E}_{\alpha3} &:= \bar{u}_{\alpha,3} + \bar{u}_{3,\alpha} \qquad \bar{E}_{33} := \bar{u}_{3,3} . \end{aligned} \tag{4.13}$$

Moreover,

$$\alpha\,\mathrm{d}\zeta = \varepsilon\,\mathrm{d}\bar{\zeta} + O(\varepsilon^2) . \tag{4.14}$$

With (4.11)-(4.14) it follows from (2.16) that

$${}^s\sigma(\varepsilon) = \varepsilon^{-4}\int_{-1}^{1}\sigma_3(\bar{\boldsymbol{E}})\,\mathrm{d}\bar{\zeta} + \varepsilon^{-2}\int_{-1}^{1}\sigma_2(\bar{\boldsymbol{E}})\,\mathrm{d}\bar{\zeta} + \int_{-1}^{1}\sigma_1(\bar{\boldsymbol{E}})\,\mathrm{d}\bar{\zeta} + O(\varepsilon) \tag{4.15}$$

where

$$2\sigma_1(\bar{\boldsymbol{E}}) := 2\bar{\mu}[(\bar{I}^{11}\bar{E}_{11})^2 + (\bar{I}^{22}\bar{E}_{22})^2 + 2\bar{I}^{11}\bar{I}^{22}(\bar{E}_{12})^2] + \bar{\lambda}(\bar{I}^{11}\bar{E}_{11} + \bar{I}^{22}\bar{E}_{22})^2 \tag{4.16$_1$}$$

$$2\sigma_2(\bar{\boldsymbol{E}}) := 4\bar{\mu}[\bar{I}^{11}(\bar{E}_{13})^2 + \bar{I}^{22}(\bar{E}_{23})^2] + 2\bar{\lambda}(\bar{I}^{11}\bar{E}_{11} + \bar{I}^{22}\bar{E}_{22})\bar{E}_{33} \tag{4.16$_2$}$$

$$2\sigma_3(\bar{\boldsymbol{E}}) := (2\bar{\mu} + \bar{\lambda})(\bar{E}_{33})^2 . \tag{4.16$_3$}$$

I now introduce the crucial assumption that the energy functional stay bounded above under rescaling, i.e., I require that

$$\lim_{\varepsilon\to 0}{}^s\sigma(\varepsilon) < +\infty . \tag{4.17}$$

As $(4.1)_3$ prevail, (4.17) implies that

$$\bar{E}_{33} = \bar{E}_{23} = \bar{E}_{13} = 0\,; \tag{4.18}$$

but, in view of (4.7), (4.8) and (4.12), (4.18) is equivalent to (3.1): the rescaling method yields precisely the kinematical restrictions that are the starting point of the constraint method.

5 - Conclusions

A simplified version of the rescaling method has been shown to be sufficient to «justify», in the terminology of Ciarlet and coworkers, the classical Kirchhoff-Love representation of the displacement field in a shell-like region. Actually, in its full strength the scaling method has an even broader scope: the method serves to legitimate well-known linear and nonlinear equations for plates and shells (Germaine-Lagrange; von Kármán; Marguerre-von Kármán) as special consequences of the general equations of three-dimensional elasticity, consequences that are arrived at by means of rather sophisticated techniques of asymptotic analysis (10). One may well wish to have a mechanical rationale for the various scaling choices made for data and solution, and their somewhat miraculous effects. Grounds have been given here for asserting that, at least for linear problems in the mechanics of structures derived from three-dimensional linear elasticity, such a rationale is provided by the constraint method.

I believe that one may safely conclude that the scaling method offers an *analytical legitimation* of the method of internal constraints, whereas the latter lends *mechanical significance* to the former.

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(10) The idea of scaling works also for *rod* problems and, with care, for modelling *junctions* between elastic bodies of different «dimensions», such as blocks, plates, rods etc. [5] [1]. Conceivably, the method of internal constraints works as well in the same circumstances (vid. e.g. [8] for an application of the constraint method to derive rod equations).

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