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Medial and permutable near rings (**)

Introduction and preliminaries

This paper considers near-rings whose multiplicative semi-group satisfies one of the following identities:

1.	$abcd = acbd$	medial near-rings
2.	$abc = bac$	left permutable near-rings
3.	$abc = acb$	right permutable near-rings.

This terminology is used in semigroup and groupoid theory (see, for example [7]).

Near-rings with these identities have been studied by many Authors: medial near-rings in [10], left permutable near-rings in [5], [6], [8], [14], [15].

Recently in [2] Birkenmaier and Heatherly studied rings and near rings satisfying these identities and provided examples illustrating geometric applications of these structures. They answer a question submitted by Clay in $[4]_2$, showing that the MP-near-rings of $[10]_1$ arise naturally.

This paper is divided in four sections. In **1** we show that the near-rings with a left permutable idempotent element are special Φ-sums (see $[1]_2$ for the definition of Φ-sum); then mixed medial near-rings and left permutable near-rings with an idempotent are characterized as Φ-sums.

In **2** set $\mathcal{Z} = \{x \in N / xK = \{0\}$, for some essential N-subgroup $K\}$ is studied.

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In ring theory this set is defined via right ideals and it is called «singular ideal» [3].

In 3 the connection between $\mathcal{Z}$ and Q (the set of nilpotent elements of N) is shown as well as the links between the prime and the essential ideals for each near-ring class which satisfies the above mentioned identities. This enables us finally to classify the left permutable θ-near-rings in 4 and characterize them as special Φ-sums.

Throughout the paper, N stands for a left near-ring. In general we adhere to the notation and terminology used in [**11**]. In particular, a near-ring $N = N_0 + N_c$ with $N_0 \neq \{0\}$ and $N_c \neq \{0\}$ is called mixed near-ring. The multiplicative semigroup of N is denoted by $N^\bullet$; a subset $S \subseteq N$ is called ideal of $N^\bullet$ if $SN \subseteq S$ and $NS \subseteq S$. The set $r(x) = \{y \in N / xy = 0\}$ is the right annihilator of x, and $r(S) = \bigcap_{x \in S} r(x)$; $A_d(N) = \{x \in N / Nx = \{0\}\}$ ($A_s(N) = \{x \in N / xN = \{0\}\}$) denotes the right (left) annihilator of N and $A = A_d \cap A_s$. The set of the left divisors will be D and Q the set of nilpotent elements. An N-subgroup I is essential if $I \cap J \neq \{0\}$ for every N-subgroup J. An N-subgroup K including an essential N-subgroup H is called essential extension of N. An essential ideal is an ideal essential as N-subgroup. An N-subgroup K is called *left N-subgroup* if $NK \subseteq K$. A near-ring N is *strictly semiprime* if $K^2 = \{0\}$ implies $K = \{0\}$ for every N-subgroup K, N is *weakly semiprime* if $KH = \{0\}$ implies either $K = \{0\}$ or $H = \{0\}$ where K and H are principal N-subgroups. Finally we recall that if N is a medial near-ring and e is an idempotent of N, $r(e)$ is an ideal of N [**10**]$_3$; if N is a left permutable near-ring, $r(x)$ is an ideal of N for every $x \in N$.

1 - Φ-sums of mixed medial and left permutable near-rings

We observe that if N is a left or right permutable near-ring then N is a medial near-ring.

In fact if N is left permutable $(xy)zt = z(xy)t = xzyt$; if N is right permutable $xy(zt) = xzyt$.

Therefore we will looking principally at medial near-rings. Left and right permutability are independent, and generally mediality doesn't imply either left or right permutability.

Proposition 1. *Let N be a medial near-ring*:

(i) *If $N \neq D$ then N is a left permutable near-ring.*

(ii) *The set Q is a left ideal of $N^\bullet$ and an ideal of $N_0^\bullet$.*

(i) Let $N \neq D$ and $0 \neq x \in N \setminus D$, then $x(abc - bac) = 0$ for every a, b, $c \in N$ because N is a medial near-ring, so $abc = bac$ and N is a left permutable near-ring.

(ii) Let $q \in Q$ with $q^s = 0$, then $0 = n^s q^s = (nq)^s$ and this implies $nq \in Q$ for every $n \in N$, so Q is a left ideal of $N^\bullet$. Besides, if $m \in N_0$, then $0 = q^s m^s = (qm)^s$, thus Q is an ideal of $N_0^\bullet$.

We can note that a right permutable near-ring is always zero-symmetric: in fact $0n = 0(0n) = 0(n0) = (0n)0 = 0$.

In $[\mathbf{1}]_2$ we have shown that a class of near-rings can be constructed on every semidirect sum of additive groups A and B. We call the structure obtained in this way Φ-sum of A and B in particular

Theorem 1. *A near-ring N has a left permutable idempotent iff it is isomorphic to the Φ-sum of a near-ring A and a near-ring B with left identity e, when* $f_{0,e} = 0_A$.

If N is a near-ring with a left permutable idempotent e then $N = r(e) + eN$ ([**11**], p. 11) where $r(e)$ is an ideal, eN is a left N-subgroup and $r(e) \cap eN = \{0\}$. Thus the hypotheses of Theorem 1 of $[\mathbf{1}]_2$ are satisfied and N is isomorphic to the Φ-sum of $r(e)$ and eN. Obviously e is a left identity of eN and $f_{0,e} = 0_{r(e)}$, because $f_{0,e} = \gamma_{0+e/r(e)}$.

Conversely, let N be isomorphic to the Φ-sum of a near-ring A and a near-ring B with left identity e and $f_{0,e} = 0_A$. Then $\langle 0, e\rangle\langle 0, e\rangle = \langle f_{0,e}(0), \bar{f}_{0,e}(e)\rangle = \langle 0, e\rangle$, because $\bar{f}_{0,e} = \gamma_e$ (see Proposition 3 $[\mathbf{1}]_2$). So, $\langle 0, e\rangle$ is an idempotent of N. Moreover $\langle 0, e\rangle\langle a, b\rangle\langle a', b'\rangle = \langle f_{0,e} \circ f_{a,b}(a'), \bar{f}_{0,e} \circ \bar{f}_{a,b}(b')\rangle = \langle 0, \gamma_e \circ \bar{f}_{a,b}(b')\rangle = \langle 0, \bar{f}_{a,b}(b')\rangle$ and $\langle a, b\rangle\langle 0, e\rangle\langle a', b'\rangle = \langle f_{a,b} \circ f_{0,e}(a'), \bar{f}_{a,b} \circ \bar{f}_{0,e}(b')\rangle = \langle 0, \bar{f}_{a,b} \circ \gamma_e(b')\rangle = \langle 0, \bar{f}_{a,b}(b')\rangle$. Thus $\langle 0, e\rangle$ is a left permutable element.

Corollary 1. *A near-ring is a mixed medial near-ring iff it is isomorphic to the Φ-sum of a medial zero-symmetric near-ring A and a constant near-ring B where $f(A \times B) \subseteq End(A^+)$ and $\bar{f}(A \times B) \subseteq End(B^+)$ and both are right permutable subsets.*

Let N be a mixed medial near-ring, then 0 is a left permutable idempotent. In fact $0nm = 00nm = 0n0m = 0m = n0m$, and N is isomorphic to the Φ-sum of N_0 and N_c where N_0 is a medial near-ring. The remainder of the proof follows by Proposition 5 of [**1**]$_2$.

Corollary 2. *A near-ring N is a left permutable near-ring with an idempotent element iff it is isomorphic to the Φ-sum of a left permutable near-ring A and a left-permutable near-ring B with left identity e, where $f_{0,e} = 0_A$ and both $f(A \times B) \subseteq End(A^+)$, and $\bar{f}(A \times B) \subseteq End(B^+)$ are commutative subsets.*

Let N be a left permutable near-ring with idempotent element e. By Theorem 1 N is isomorphic to the Φ-sum of $r(e)$ and eN which are both left permutable near-rings. Moreover e is a left identity of eN. The remainder of the proof follows by Proposition 5 of [**1**]$_2$.

2 - The singular set $\mathcal{Z}$

Let $\mathcal{Z} = \{x \in N/xK = \{0\}$ for some K that is an essential N-subgroup of $N\}$.

Lemma 1. *Let K be an N subgroup of N and let M be an essential extention of K, then there is an essential N-subgroup L such that $aL \neq \{0\}$ and $aL \subseteq K$ for every $a \in N$.*

Let $a \in M$ and $L = \{r \in N/ar \in K\}$; obviously L is an N-subgroup of N and $aL \subseteq K$. Furthermore $aN \cap K \neq \{0\}$ because K is essential, so ar is a non zero element of K for some $r \in N$. Hence $r \in L$ and $aL \neq \{0\}$. Now, our goal is to show that L is essential: let $S \neq \{0\}$ be an N-subgroup of N. Obviously, if $aS = \{0\}$, then $S \subseteq L$, if $aS \neq \{0\}$, then $aS \cap K \neq \{0\}$ because K is essential, so, there is an $x \in S$ such that $ax \in K$ and therefore $S \cap L \neq \{0\}$.

Proposition 2. *The set $\mathcal{Z}$ is non empty iff N is a zero-symmetric near-ring.*

If N is a constant near-ring, then $xy = y$ for every x, $y \in N$, so $\mathcal{Z} = \phi$. If $N = N_0 + N_c$ and K is an N-subgroup, then $K_c = N_c$, so every N-subgroup is essential. Hence if $x \in \mathcal{Z}$ there is a K such that $xK = \{0\}$, but $N_c \subseteq xK$ and this is absurd. Finally, if N is a zero-symmetric near-ring then $\mathcal{Z}$ is a non empty set because obviously $0 \in \mathcal{Z}$.

In the following we will consider zero-symmetric near-rings. In this case the right annihilators, which generally are right ideals, are N-subgroups.

Proposition 3. *An element $x \in N$ belongs to $\mathcal{Z}$ iff $r(x)$ is an essential N-subgroup.*

If $x \in \mathcal{Z}$, there is an essential N-subgroup $K \subseteq N$ such that $xK = \{0\}$, so $r(x)$ is an essential extension of K. The converse is trivial.

Proposition 4. *The set $\mathcal{Z}$ is an ideal of the semigroup $N^{\bullet}$.*

If $x \in \mathcal{Z}$ then $nx \in \mathcal{Z}$ $\forall n \in N$ because $r(x) \subseteq r(nx)$. Moreover $r(x)$ is essential and by Lemma 1, there is an essential N-subgroup L such that $nL \subseteq r(x)$ $\forall n \in N$ ($x = 0$ implies $L = N$), thus $xnL = \{0\}$ and $xn \in \mathcal{Z}$.

Proposition 5. *If $\mathcal{Z} \neq \{0\}$, then N has nilpotent elements.*

Let N have no nilpotent elements and let $\mathcal{Z} \neq \{0\}$. If $0 \neq x \in \mathcal{Z}$, then $r(x) \cap xN \neq \{0\}$ because $r(x)$ is essential and $xN = \{0\}$ implies $x^2 = 0$, which is excluded. So $x^2\overline{n} = 0$ for some $\overline{n} \in N$. Because N has no nilpotent elements I.F.P. holds (see [13]). Thus $x\overline{n}x\overline{n} = 0$ and $x\overline{n} \in Q$, which is absurd.

Proposition 6. *If $\mathcal{Z} = \{0\}$ then $A_s(N) = \{0\}$.*

If $x \in A_s(N)$ then $xN = \{0\}$, so $x \in \mathcal{Z}$ and $x = 0$.

Theorem 2. *If N has right annihilator a.c.c. then $\mathcal{Z}$ is nilpotent.*

Let N have right annihilator a.c.c. Because $\mathcal{Z} \supseteq \mathcal{Z}^2 \supseteq \dots \supseteq \mathcal{Z}^n \supseteq \dots$ then $r(\mathcal{Z}) \subseteq r(\mathcal{Z}^2) \subseteq \dots$, so, there is a positive integer s such that $r(\mathcal{Z}^s) = r(\mathcal{Z}^{s+1})$. Now, our aim is to show that $\mathcal{Z}^{s+1} = \{0\}$. Assume $a \in \mathcal{Z}$ such that $\mathcal{Z}^s a \neq \{0\}$. If $b \in \mathcal{Z}$, then $r(b) \cap aN \neq \{0\}$, hence there is an element $\overline{n} \in N$ such that $a\overline{n} \neq 0$ and $ba\overline{n} = 0$, so $r(a) \subset r(ba)$. Since $ba \in \mathcal{Z}$ (by Proposition 3), if $\mathcal{Z}^s ba \neq \{0\}$, then $r(ba) \subset r(cba)$ $\forall c \in \mathcal{Z}$ and so on. Because right annihilator a.c.c. holds, there must be an element $\overline{a} \in \mathcal{Z}$ such that $\mathcal{Z}^s \overline{a} \neq \{0\}$ and $\mathcal{Z}^s b\overline{a} = \{0\}$ $\forall b \in \mathcal{Z}$, hence $\mathcal{Z}^{s+1}\overline{a} = \{0\}$. But given that $r(\mathcal{Z}^s) = r(\mathcal{Z}^{s+1})$, then $\mathcal{Z}^s\overline{a} = \{0\}$, which is excluded.

3 - Links between $\mathcal{Z}$ and Q

Now we turn our attention to medial and left or right permutable near-rings with $\mathcal{Z} \neq \phi$.

Proposition 7. *Let N be a medial near-ring with $\mathcal{Z} = \{0\}$, then:*

(i) I.F.P. *holds.* (ii) $abc = 0$ *implies* $bac = 0$. (iii) *N is weakly semiprime iff N is integral.*

(i) If $ab = 0$ then $anbN = abnN = \{0\}$, so $anb = 0$ $\forall n \in N$ by Proposition 6. (ii) If $abc = 0$, then $abcN = bacN = \{0\}$, so $bac = 0$ by Proposition 6. (iii) Assume $a \neq 0$, $b \neq 0$ and $ab = 0$. Then $aNbN = abN^2 = \{0\}$. Hence either $aN = \{0\}$ or $bN = \{0\}$. Thus $a = 0$ or $b = 0$ by Proposition 6.

Proposition 8. *Let N be a left permutable near-ring with $\mathcal{Z} = \{0\}$, then $ab = 0$ implies $ba = 0$.*

If $ab = 0$, then $abN = baN = 0$, so $ba = 0$.

Theorem 3. *Let N be a strictly semiprime medial near-ring, then:*

(i) $QN = \{0\}$. (ii) $Q \subseteq \mathcal{Z}$. (iii) *If right annihilator a.c.c. holds then* $Q = \mathcal{Z}$.

(i) Let q a nilpotent element of N and $q^s = 0$, then $q^s N^s = (qN)^s$ by the mediality of N and $qN = \{0\}$ $\forall q \in Q$ because N is strictly semiprime. (ii) This is obvious, by (i). (iii) This follows immediatly, by Theorem 2 and (ii).

Theorem 4. *Let N be a left permutable near-ring without left annihilator N-subgroups, then:*

(i) $Q \subseteq \mathcal{Z}$. (ii) *If right annihilator a.c.c. holds, then* $Q = \mathcal{Z}$.

(i) Let q be a nilpotent element of N and $q^s = 0$ and let K be a non trivial N-subgroup, then $qKq^{s-1} = Kq^s = \{0\}$. Assume $Kq^{s-1} \neq \{0\}$, then $r(q) \cap K \neq \{0\}$ and $q \in \mathcal{Z}$. If $Kq^{s-1} = \{0\}$, then $qKq^{s-2} = \{0\}$ and the previous condition arises again. Thus, after a finite number of steps we reach $Kq = \{0\}$, so $KqN = qKN = \{0\}$. Because $KN = \{0\}$ is impossible, it follows that $K \cap r(q) \supseteq KN \neq \{0\}$, so $q \in \mathcal{Z}$. (ii) This follows immediatly by Theorem 2 and (i).

Theorem 5. *Let N be a right permutable near-ring in which $ab = 0$ implies $ba = 0$ ($\forall a,\ b \in N$) then:*

(i) $Q \subseteq \mathcal{Z}$. (ii) *If right annihilator a.c.c. holds, then* $Q = \mathcal{Z}$.

(i) The proof is similar to Theorem 4 (i). (ii) This part follows immediatly by Theorem 2 and (i).

Examples. (a) Let's consider the near-ring $[Z_6,\ +,\ \cdot]$ where $[Z_6,\ +]$ is the cyclic group of order 6 and $[Z_6,\ \cdot]$ is defined as follows

	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	3	0	3	0	3
2	0	4	2	0	4	2
3	0	3	0	3	0	3
4	0	0	0	0	0	0
5	0	1	2	3	4	5

This near-ring satisfies the conditions of Theorem 3 and Theorem 4: in fact, $Q = \mathcal{Z} = \{0,\ 4\}$.

(b) Let's consider the near-ring $[Z_8,\ +,\ \cdot]$ where $[Z_8,\ +]$ is the cyclic group of order 8 and $[Z_8,\ \cdot]$ is defined as follows

	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0
2	0	4	0	4	0	4	0	4
3	0	6	4	2	0	6	4	2
4	0	0	0	0	0	0	0	0
5	0	4	0	4	0	4	0	4
6	0	0	0	0	0	0	0	0
7	0	6	4	2	0	6	4	2

This is a left permutable near-ring (hence a medial near-ring) that is not strictly semiprime ($\{0,\ 4\}$ is a nilpotent N-subgroup): therefore $\mathcal{Z} = N$ and $Q = \{1,\ 2,\ 4,\ 6\}$.

In [**13**] type 0, 1, 2 prime ideals and their respective radicals P_0, P_1, P_2 are defined. In the following a type 0 prime ideal will be called *prime* and a type 2 prime ideal will be called *completely prime,* as in [**9**] and [**11**]. In [**2**]$_2$ it has been shown that if N is a medial near-ring, then P is a type 1 prime ideal ($xNy \subseteq P$ implies $x \in P$ or $y \in P$) if and only if P is completely prime ($xy \in P$ implies $x \in P$ or $y \in P$).

Let us now take a look at a few links between prime ideals and essential ideals.

Proposition 9. *If N is a medial near-ring with $\mathcal{Z} = D$, then each type* 1 *prime ideal is an essential ideal.*

Let I be a type 1 ideal of N. If I is not essential, there is an N-subgroup $K \neq \{0\}$ such that $I \cap K = \{0\}$, so $K \subseteq D$ and $I \subseteq r(K)$. It is even true that $I = r(K)$. In fact if $s \in r(K) \setminus I$, then $ks = 0 \in I$ and $k \in I$, because I is completely prime. Hence I is an essential ideal, since $K \subseteq D = \mathcal{Z}$.

Proposition 10. *If N is a medial near-ring then each type* 1 *prime ideal which doesn't include $\mathcal{Z}$, is an essential ideal.*

Let I be a type 1 ideal of N which doesn't include z, then there is a $z \in \mathcal{Z} \setminus I$ where $r(z)$ is essential. Now $z\bar{z} = 0 \in I$ $\forall \bar{z} \in r(z)$ and $z \notin I$ so $\bar{z} \in I$ because I is completely prime, thus $r(\bar{z}) \subseteq I$ and I is essential.

However, there are medial near-rings with essential ideals which don't include $\mathcal{Z}$, but which are not prime ideals, as in the following example.

Example. (c) Let us consider the near-ring $[Z_8, +, \cdot]$ where $[Z_8, +]$ is the cyclic group of order 8 and $[Z_8, \cdot]$ is defined as follows

	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0
3	0	2	4	6	0	2	4	6
4	0	0	0	0	0	0	0	0
5	0	4	0	4	0	4	0	4
6	0	4	0	4	0	4	0	4
7	0	2	4	6	0	2	4	6

This is a medial near-ring in which $r(5) = \{0, 2, 4, 6\}$ is an essential ideal which doesn't include $\mathcal{Z}$ (because $\mathcal{Z} = N$), but which is not a type 1 prime ideal, in fact $3N5 = \{0, 4\} \subseteq r(5)$, but $3 \notin r(5)$ and $5 \notin r(5)$.

4 - θ-near-rings

Def. A. *A θ-near-ring is a near-ring N which satisfies the following conditions*: (1) *If $r(n)$ is an ideal, it is a type* 1 *prime ideal.* (2) $N = D$.

Theorem 6. *A left permutable near-ring N is a θ-near-ring iff Q is a type* 1 *prime ideal and $Q = A$.*

By Proposition 5, N has nilpotent elements because N is a zero-symmetric near-ring. Let Q' be the set of nilpotent elements, whose nilpotence index is 2. Assume $q \in Q'$, then $q^2 = 0 \in r(n)$, which is a completely prime ideal, so $q \in r(n)$ $\forall n \in N$. If $Q' = N$, then N is a zero-near-ring and the theorem is trivial, therefore there is an element $x \notin Q'$.

Let $y \notin r(x)$ with $y \neq x$ (such an element exists because if $r(x) = N \setminus \{x\}$, then $x(x+n) = 0$, so $x^2 + xn = x^2 = 0$, and this is excluded) and $\overline{y} \in r(y)$. Because N is left permutable, then $yN\overline{y} \subseteq r(x)$, where $r(x)$ is a type 1 prime ideal, hence $\overline{y} \in r(x)$ $\forall \overline{y} \in r(y)$, so $r(y) \subseteq r(x)$.

Since $y \notin r(x)$ and $r(y) \subseteq r(x)$, then $y \notin r(y)$, therefore $y \notin Q'$. For the same reason $x \notin r(y)$ and we can prove in the same way that $r(x) \subseteq r(y)$. So $r(x) = r(y)$.

Thus, $\forall x \in Q'$ $r(x)$ will be denoted by R.

If $x \in Q'$ then $x^2 = 0 \in R$ and $x \in R$ because R is completely prime, so $Q' \subseteq R$. Assume $p \in R \setminus Q'$, then $r(p) = R$ and $p^2 = 0$ but $p \notin Q'$, so $Q' = R$. Let z be a nilpotent element whose nilpotence index is greater than 2. But $z^s = 0 \in R$ implies that either z^{s-1} or z belong to R and, after a finite number of steps, $z \in Q'$ and $Q' = Q$.

Finally $qNq = \{0\} \subseteq r(n)$ implies $q \in r(n)$ $q \in Q$, $\forall n \in N$, so $NQ = \{0\}$. Moreover $nNn \subseteq r(q)$ implies $n \in r(q)$ $\forall n \in N$, $\forall q \in Q$, so $QN = \{0\}$, that is $Q = A$. Conversely, let $Q = A$ be a type 1 prime ideal of a left permutable near-ring N. Obviously condition (2) of Def. A holds. Assume now $xNy \subseteq r(n)$. If $n \in Q$, then $r(n) = N$ and therefore it is a type 1 prime. If $n \in N \setminus Q$, then $nx(ynx)y = 0$ and $nxy \in Q$, hence either $x \in Q$ or $y \in Q$ and, because $Q \subseteq r(n)$ $\forall n \in N$, $r(n)$ is a type 1 prime ideal.

Proposition 11. *A left-permutable θ-near-ring is a zero-near-ring iff* $\mathcal{Z} = N$.

If N is a zero-near-ring, it is obvious that $\mathcal{Z} = N$. Now let $\mathcal{Z} = N$ and $nN \neq \{0\}$. Then $r(n) \cap nN \ni i \neq 0$ hence $i = nh$ for some $h \notin r(n)$, so $n \in r(n)$ because $r(n)$ is completely prime, and therefore $n^2 = 0$. It follows that $nN = \{0\}$ and N is a zero-near-ring.

In the following $\mathcal{Z} = N$ *is excluded.*

Corollary 3. *If N is a left permutable θ-near-ring then* $\mathcal{Z} = Q = P_0 = P_1 = P_2$ *and N/Q is an integral near-ring.*

Given that N is a left permutable (hence medial) near-ring, $P_2 = P_1$ (see [7] for definition of P_0, P_1, P_2 and $[2]_2$ Propositions 2.7). Now, as Q is a type 1 prime ideal, $Q = P_1$. Let I be a proper prime ideal, then $Q \subseteq I$ because $NQ = \{0\} \subseteq I$ and $N \nsubseteq I$ and therefore $Q \subseteq P_0$. Given that $P_0 \subseteq Q$ (see [13]). $Q = P_0$. Finally, as $\mathcal{Z} \supseteq Q$, if $z \in \mathcal{Z} \setminus Q$, then $r(z) = Q$ and Q is an essential ideal, so $\forall n \in N$, $r(n)$ is an essential extension of Q and this implies $\mathcal{Z} = N$ but this is excluded, so $\mathcal{Z} = Q$. Finally N/Q is integral because Q is completely prime.

Theorem 7. *A near-ring N is a left permutable θ-near-ring with a non-zero idempotent element iff it is isomorphic to the Φ-sum of a zero-near-ring A and of an integral left permutable zero-symmetric near-ring B with a left identity e, when* $f_{a,b} = 0_A$ *and* $\bar{f}_{a,b} = \gamma_b$ $\forall \langle a,\ b \rangle \in A \times B$.

Let N be a left permutable θ-near-ring with a non-zero idempotent element e. Now $r(e) = Q$ is an ideal and eN is a left N-subgroup. Moreover $r(e) \cap eN = \{0\}$. So N is isomorphic to the Φ-sum of $r(e)$ and eN where $r(e) = Q$ is a zero-near-ring, eN is a left permutable zero-symmetric near-ring and e is a left identity of eN. Moreover eN is an integral near-ring because $enem = 0$ implies $enm = 0$, that is $nm \in r(e) = Q$. But Q is completely prime, so either $n \in Q$ or $m \in Q$ and either $en = 0$ or $em = 0$. Finally, now $f_{a,b} = \gamma_{q+en/Q} = 0_Q$ and $\bar{f}_{a,b} = \gamma_{q+en/eN} = \gamma_{en/eN}$ $\forall q \in Q$ $\forall en \in eN$.

Conversely, let N be isomorphic to the Φ-sum of a zero-near-ring A and of an integral left permutable zero-symmetric near-ring B with a left identity e. Moreover, $\forall \langle a,\ b \rangle \in A \times B$ let $f_{a,b} = 0_A$ and $\bar{f}_{a,b} = \gamma_b$. In these hypotheses N is a left permutable near-ring with an idempotent element (by Corollary 2) because $f_{0,e} = 0_A$, $f(A \times B) = \{0\}_A$ is a commutative subset of $\mathrm{End}(A^+)$ and

$\bar{f}(A \times B) = \{\gamma_b / b \in B\}$ is a commutative subset of End(B^+). Furthermore, easy calculation shows that Q is a type 1 ideal of N and $Q = A$. Thus, the hypotheses of Theorem 6 hold and N is a left permutable θ-near-ring.

In this way we have shown that the left permutable near-rings can be costructed by a semidirect sum of additive groups and a direct product of multiplicative semigroups of a zero-near-ring and of an integral zero-symmetric left permutable near-ring with a left identity.

References

[1] A. BENINI: $[\bullet]_1$ *Sums of near-rings*, Riv. Mat. Univ. Parma (4) **13** (1988), 135-141; $[\bullet]_2$ *Near-rings on certain groups*, Riv. Mat. Univ. Parma (4) **15** (1989), 149-158.

[2] G. BIRKENMEIER and H. HEATHERLY: $[\bullet]_1$ *Medial rings* (to appear); $[\bullet]_2$ *Medial near-rings*, Mt. Math. **107** (1989), 89-110.

[3] A. W. CHATTERS and C. R. HAJARNAVIS, *Rings with chain conditions*, Pitman Advanced Publ. Program Boston, 1980.

[4] J. R. CLAY: $[\bullet]_1$ *Generating balanced incomplete block designs from planar near-rings*, J. Algebra 22(1972), 319-331; $[\bullet]_2$ Math. Rev. 88a: 16070.

[5] J. R. CLAY and D. A. LAWVER, *Boolean near-rings*, Canad. Math. Bull. **12** (1969), 265-273.

[6] C. COTTI FERRERO, *Near-rings with E-permutable translations*, Near-rings and Near-fields edited by G. Betsch, 63-72 (North Holland Math. Studies **137**, Amsterdam, 1987).

[7] J. JEREK and T. KEPKA, *Permutable groupoids*, Czech. Math. 5 34 (1984), 396-410.

[8] S. LIGH, *The structure of a special class of near-rings*, J. Austral. Math. Soc. **13** (1972), 141-146.

[9] J. D. P. MELDRUM, *Near-rings and their links with groups* (Research Notes in Math. 134, Pitman, Marshfield, MA, 1985).

[10] S. PELLEGRINI MANARA: $[\bullet]_1$ *Sui quasi-anelli mediali in cui ogni elemento è potenza di se stesso*, Riv. Mat. Univ. Parma (4) **11**(1985), 223-228; $[\bullet]_2$ *On regular medial near-rings*, Boll. Un. Mat. Ital. Algebra e Geometria (VI) **IV** D, N. 1 (1985), 131-136; $[\bullet]_3$ *On medial near-rings*, Near-rings and near-fields edited by G. Betsch, 199-210. (North Holland Math. Studies 137, Amsterdam, 1987).

[11] G. PILZ, *Near-rings* (2nd ed.) North-Holland Math. Studies **23**, Amsterdam, 1983.

[12] D. RAMAKOTAIAH, *Radicals for near-rings*, Math. Z. **97**, 45-46.

[13] D. RAMAKOTAIAH and G. K. RAO, *On I.F.P. Near-rings*, J. Austral. Math. Soc. **27** (1979), 365-370.

[14] E. F. RATLIFF, *Some results on p-near-rings and related near-rings*, Ph. D. Diss. Univ. Oklahoma, 1971.

[15] N. V. SUBRANMANYAM, *Boolean semirings*, Math. Ann. **148**(1962), 395-401.

Sommario

Studiamo quasi-anelli soddisfacenti particolari identità polinomiali. Prendendo spunto dalla nota tecnica di J. R. Clay, caratterizziamo in termini di Φ-somme, i quasi-anelli mediali misti, i permutabili a sinistra con un elemento idempotente ed infine i θ-quasi-anelli.

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