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A generalization of some commutativity theorems for rings (II) (**)

1 - Introduction

Wedderburn's theorem, asserting that a finite division ring is necessarily commutative, has been generalized in several directions. A well known theorem of Jacobson states that if, for each x in a ring R, there exists an integer $n > 1$, depending upon x, such that $x^n = x$, then R is commutative. Herstein further generalized this result. He proved that, if for each x in a ring R there exists an integer $n > 1$, depending upon x, such that $(x^n - x)$ is central, then R is commutative.

Bell $[3]_2$ proved: «If R is a ring in which for any pair of elements x, and y in R, there exists an integer $n = n(x, y) \geqslant 1$ such that $xy = yx^n$, for all x and y in R, then R must be commutative». This result was later extended by Bell $[3]_3$ himself, and he proved that a ring R in which for every pair of elements x and y in R, there exist positive integers $m \geqslant 1$ and $n \geqslant 1$, satisfying $xy = y^m x^n$, is commutative. Recently, Quadri and Khan [**11**] generalized the above results.

The objective of this paper is to generalize the above mentioned results, and to extend the main theorems of [7] and $[9]_1$ for left (resp. right) s-unital rings.

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2 - Preliminaries

Throughout this paper, R represents an associative ring not necessarily with unity 1. Let $Z(R)$ denote the center of R, $C(R)$ the commutator ideal of R, N the set of all nilpotent elements of R, and N' the set of all zero divisors of R.

Def. 1. A ring R is called *left* (resp. *right*) *s-unital* if $x \in Rx$ (resp. $x \in xR$), for every x in R. Further, R is called *s-unital* if it is both left as well as right s-unital, that is $x \in Rx \cap xR$, for each x in R.

Def. 2. If R is s-unital ring (resp. left of right s-unital), then for any finite subset F of R, there exists an element e in R such that $ex = xe = x$ (resp. $ex = x$ or $xe = x$) for all x in F. Such an element e is called the *pseudo-identity* (resp. *pseudo left identity* or *pseudo right identity*) of F in R.

In preparation for the proof of our results, we first state the following well-known results.

Lemma 1 ($[8]_1$, Lemma 3). Let R be a ring such that $[x, [x, y]] = 0$, for all x and y in R. Then $[x^k, y] = kx^{k-1}[x, y]$ for any positive integer k.

Lemma 2 ([2], Lemma 2). Let R be a ring with unity 1, and let x and y be elements of R. If $kx^m[x, y] = 0$, and $k(x+1)^m[x, y] = 0$ for some integers $m \geq 1$ and $k \geq 1$, then necessarily $k[x, y] = 0$.

Lemma 3 ($[6]_1$, Theorem). Let f be a polynomial in n non-commuting indeterminates $x_1, x_2, \ldots, x_n$ with relatively prime integral coefficients. Then the following are equivalent:

(1) Every ring satisfying the polynomial identity $f = 0$ has nil commutator ideal.

(2) Every semi-prime ring satisfying $f = 0$ is commutative.

(3) For every prime p, the ring of 2×2 matrices over Z_p fails to satisfy $f = 0$.

Lemma 4 ($[11]_1$, Lemma 3). Let R be a ring with unity 1, and let k and m be natural numbers. If $(1 - y^k)x = 0$, then $(1 - y^{km})x = 0$, for all x and y in R.

Lemma 5 ($[12]_2$ Lemma). Let R be a left (resp. right) s-unital ring. If for each pair of elements x and y in R there exists a positive integer $k = k(x, y)$, and an element $e = e(x, y)$ of R such that $x^k e = x^k$ and $y^k e = y^k$ (resp. $ex^k = x^k$ and $ey^k = y^k$), then R is an s-unital ring.

The following theorem is due to Herstein.

Theorem H ([4], Theorem 18). Let R be a ring, and let $n > 1$ be a fixed integer. If $(x^n - x) \in Z(R)$, for each x in R, then R is commutative.

3 - Main result

The following theorem is the main result of this paper, which generalizes some commutativity conditions for rings.

Theorem. *Let $m > 1$, n, and k be non-negative integers, and let R be a left (resp. right) s-unital ring satisfying*

$$x^n [x, y] = [x, y^m] x^k \tag{1}$$

for all x and y in R. Then R is commutative.

First, we prove the following lemmas.

Lemma 6. *Let R be a left (resp. right) s-unital ring, and let $m > 1$, n and k be non-negative integers. If R satisfies* (1), *then R is as s-unital ring.*

Proof. Let R be a left (resp. right) s-unital ring, and let x and y be arbitrary elements of R. Then, we can find an element $e = e(x, y)$ of R such that $ex = x$ and $ey = y$ (resp. $xe = x$ and $ye = y$). From (1), we have $e^n [e, y] = [e, y^m] e^k$ (resp. $x^n [x, e] = [x, e^m] x^k$). Hence,

$$e^{n+1} y - e^n ye = ey^m e^k - y^m e^{k+1}$$

$$y = ye + y^m e^k - y^m e^{k+1} = y(e + y^{m-1} e^k - y^{m-1} e^{k+1}) \in yR$$

(resp. $x^{n+1} e - x^n ex = xe^m x^k - e^m x^{k+1}$, $x^{k+1} = e^m x^{k+1} \in Rx^{k+1}$) for $m > 1$.

Therefore, R is an s-unital ring by Lemma 5.

Lemma 7. *Let R be a ring with unity* 1, *and let $m > 1$, $n \geqslant 0$ and $k \geqslant 0$ be fixed integers. If R satisfies* (1), *then $N \subseteq Z(R)$. Further, $C(R) \subseteq Z(R)$.*

Proof. It is trivial to prove that for any natural number t, the polynomial identity (1) implies

$$x^{tn}[x, y] = [x, y^{m^t}]x^{tk} \tag{2}$$

for all x and y in R.

Now, let u be an element in N. Then by (2), we have

$$x^{tn}[x, u] = [x, u^{m^t}]x^{tk}$$

for every x in R, and any integer $t \geqslant 1$. But as u is nilpotent element, $u^{m^t} = 0$, for sufficiently large t, and $x^{tn}[x, u] = 0$, for all x in R and u in N.

Replace x by $(x+1)$ in the last polynomial identity to get

$$(x+1)^{tn}[x, u] = (x+1)^{tn}[(x+1), u] = 0 = x^{tn}[x, u]$$

for all x in R, and u in N. In view of Lemma 2, we obtain $[x, u] = 0$, for all x in R and u in N. Therefore, $u \in Z(R)$, and hence

$$N \subseteq Z(R). \tag{3}$$

Next, let

$$x = e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad y = e_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then x and y fail to satisfy the polynomial identity (1), for all non-negative integers n, k and $m > 1$. Hence, by Lemma 3, $C(R)$ is nil ideal, that is $C(R) \subseteq N$. Therefore, (3) gives

$$C(R) \subseteq N \subseteq Z(R). \tag{4}$$

Remark 1. In view of Lemma 7, it is guaranteed that the conclusion of Lemma 1 holds for any pair of elements x and y in a ring R with unity 1 which satisfies $x^n[x, y] = [x, y^m]x^k$, for fixed non-negative integers n, k, and $m > 1$.

Proof of the Theorem. Since R is a left (resp. right) s-unital ring which satisfies (1), for non negative integers n, k and $m > 1$, R is an s-unital ring by Lemma 5. Therefore, in view of Proposition 1 of [5], we may assume that R has 1.

Since R is isomorphic to a subdirect sum of subdirectly irreducible rings R_i, $(i \in I)$, each of which as a homomorphic image of R satisfies the hypothesis of the theorem placed on R, so we may assume that R is a subdirectly irreducible ring. Let S be the intersection of all its non-zero ideals. Then $S \neq (0)$.

Now, if $n = k = 0$, then $[x, y] = [x, y^m]$ for all x and y in R. Hence $[x, y^m - y] = 0$ for each x and y in R. Therefore, R is commutative by Theorem H. Let $n = k = 1$ in (1). Then $x[x, y] = [x, y^m]x$ for all x and y in R. Replacing x by $(x+1)$, we obtain $(x+1)[x, y] = [x, y^m](x+1)$, and hence $[x, y] = [x, y^m]$, that is $[x, y^m - y] = 0$ for all x and y in R and $m > 1$. Hence, R is commutative by Theorem H. Next, suppose that $n = 1$, and $k = 0$ in (1). Then $x[x, y] = [x, y^m]$ for all x and y in R. The usual argument of replacing x by $(x+1)$ in the last polynomial identity gives $[x, y] = 0$ for every x and y in R is commutative. If $n = 0$, and $k = 1$ in (1), we have $[x, y] = [x, y^m]x$, for all x,and y in R. Thus, R replace x by $(x+1)$ in the last identity to get $[x, y^m] = 0$. Hence $[x, y] = [x, y^m]x = 0$ for all x, and y in R. Therefore, R is commutative.

Next, suppose that $n > 1$, and $k > 1$. Let $q = 2^m - 2$. Then $q > 1$, for $m > 1$. Thus, by (1) we obtain

$$\begin{aligned} qx^n[x, y] &= (2^m - 2)x^n[x, y] = 2^m x^n[x, y] - 2x^n[x, y] \\ &= 2^m[x, y^m]x^k - 2x^n[x, y] = [x, (2y)^m]x^k - x^n[x, (2y)] \\ &= x^n[x, (2y)] - x^n[x, (2y)] = 0\,. \end{aligned}$$

Hence, $qx^n[x, y] = 0$, for all x and y in R. Replace x by $(x+1)$ in the last polynomial identity to have $q(x+1)^n[x, y] = 0 = qx^n[x, y]$, for all x and y in R. Therefore, Lemma 2 gives

$$q[x, y] = 0 \tag{5}$$

for all x and y in R. In view of (4) and Lemma 1, we have $[x^q, y] = qx^{q-1}[x, y] = 0$ for all x and y in R. Thus, (5) gives $[x^q, y] = 0$, for all x and y in R. Therefore,

$$x^q \in Z(R) \tag{6}$$

for all x and y in R and $q = (2^m - 2) > 1$, for $m > 1$.

Replace y by y^m in the polynomial identity (1) to get

$$x^n[x, y^m] = [x, (y^m)^m]x^k \tag{7}$$

for all x, and y in R.

Now, since by (4) commutators are central, then Lemma 1, and (1) gives

$$x^n[x, y^m] = [x, y^m]x^n = my^{m-1}[x, y]x^n$$

$$= my^{m-1}x^n[x, y] = my^{m-1}[x, y^m]x^k$$

$$[x, (y^m)^m]x^k = m(y^m)^{m-1}[x, y^m]x^k = my^{m^2-m}[x, y^m]x^k$$

$$= my^{m-1}y^{(m-1)^2}[x, y]x^k.$$

Thus, (7) gives

$$my^{m-1}(1 - y^{(m-1)^2})[x, y^m]x^k = 0 \tag{8}$$

for all x, and y in R.

Replace x by $(x+1)$ in (8) to get $my^{m-1}(1 - y^{(m-1)^2})[x, y^m](x+1)^k = 0$ for all x, and y in R. So, by Lemma 2, we obtain $my^{m-1}(1 - y^{(m-1)^2})[x, y^m] = 0$ for all x and y in R. Therefore, by (4) and Lemma 4, we have

$$my^{m-1}(1 - y^{q(m-1)^2})[x, y^m] = 0 \tag{9}$$

for all x and y in R.

Next, we claim that $N' \subseteq Z(R)$. Let $u \in N'$. Then by (6), we have

$$u^{q(m-1)^2} \in N' \cap Z(R) \qquad Su^{q(m-1)^2} = 0.$$

By using (4) and (9), we obtain

$$mu^{m-1}[x, u^m](1 - u^{q(m-1)^2}) = 0$$

for all $x \in R$, and $u \in N'$.

If $mu^{m-1}[x, u^m] \neq 0$, then $(1 - y^{q(m-1)^2}) \in N'$. Hence, $S(1 - u^{q(m-1)^2}) = 0$ for $u \in N'$. Thus, we have a contradiction as $S \neq (0)$. Therefore,

$$mu^{m-1}[x, u^m] = 0 \tag{10}$$

for all x and $y \in R$, and $u \in N'$.

Now, using (1) and (4) along with Lemma 1 repeatedly, we obtain

$$x^{2n}[x, u] = x^n(x^n[x, u]) = x^n[x, u^m]x^k = [x, u^{m^2}]x^{2k}$$

$$= mu^{m(m-1)}[x, u^m]x^{2k} = mu^{(m-1)}u^{(m-1)^2}[x, u^m]x^{2k}$$

$$= mu^{(m-1)}[x, u^m]u^{(m-1)^2}x^{2k}$$

for all $x \in R$, and $u \in N'$.

Thus, (10) implies that $x^{2n}[x, u] = 0$, for all $x \in R$ and $u \in N'$. Replace x by $(x+1)$ in the last polynomial identity to get $x^{2n}[x, y] = 0 = (x+1)^{2n}[x, u] = 0$ for all $x \in R$, and $u \in N'$. Hence, Lemma 2 yields $[x, u] = 0$, for all $x \in R$, and $u \in N'$, that is $u \in Z(R)$. Therefore,

$$N' \subseteq Z(R). \tag{11}$$

By (6), we have x^q, and x^{qm} are central for all $x \in R$ and $q = (2^m - 2) > 1$, for $m > 1$. Then (1) implies that

$$(x^q - x^{qm})x^n[x, y] = x^q(x^n[x, y]) - x^{qm}(x^n[x, y])$$

$$= x^n(x^q[x, y]) - x^{qm}([x, y^m]x^k) = x^n[x, x^q y] - [x, x^{qm}y^m]x^k$$

$$= x^n[x, x^q y] - [x, (x^q y)^m]x^k = x^n[x, x^q y] - x^n[x, x^q y].$$

Therefore, $(x^q - x^{qm})x^n[x, y] = 0$, for all x and y in R. Let $t = qm - q + 1$ and let $s = n + q - 1$, for $q = 2^m - 2$ and $m > 1$. Hence,

$$(x - x^t)x^s[x, y] = 0 \tag{12}$$

for all x and y in R.

Now, if $x^s[x, y] = 0$, then $(x+1)^s[x, y] = 0 = x^s[x, y]$, for all x and y in R. By Lemma 2, we obtain $[x, y] = 0$, for all x and y in R. Hence, R is commutative. But $x^s[x, y] \neq 0$ gives $(x - x^t) \in N' \subseteq Z(R)$, by (11). Thus, $[x - x^t, y] = 0$ for all x and y in R. Therefore, R is commutative by Theorem H.

As consequences of our main result, we derive the following corollaries.

Corollary 1. *Let R be a ring with unity 1, and let $m > 1$ and $k \geqslant 0$ be non-negative integers such that $[xy - y^m x^k, x] = 0$ for each x and y in R. Then R is commutative.*

Proof. By hypothesis we have $x[x, y] = [x, y^m]x^k$ for all x and y in R, where $m > 1$ and $k \geqslant 0$. Thus, R is commutative by the main Theorem.

Corollary 2. *Let R be a left s-unital ring such that there exist integers $m > 1$, and $k \geqslant 0$ satisfying the polynomial identity*

$$[xy - y^m x^k, x] = 0 \tag{13}$$

for all x and y in R. Then R is commutative.

Proof. We notice that (13) can be rewritten as follows: $x[x, y] = [x, y^m]x^k$ for all x and y in R, where $m > 1$ and $k \geqslant 0$. Therefore, R is commutative by Theorem.

Corollary 3. *Let R be an s-unital ring, and let $m > 1$ and $n \geqslant 0$ be fixed non-negative integers. If R satisfies the polynomial identity*

$$[x^n y - y^m x, x] = 0 \tag{14}$$

for all x and y in R, then R is commutative.

Proof. By (14), we have $x^n[x, y] = [x, y^m]x$ for all x and y in R, where $m > 1$ and $n \geqslant 0$. Hence, R is commutative by the main Theorem.

Remark 2. We would like to point out that Corollary 1 and Corollary 2, where proved in [**11**]$_2$ Theorem and [**11**]$_1$ Theorem, respectively, for $k \geqslant 1$. A similar comment applies to Corollary 3, which was obtained in [**9**]$_1$ Theorem for $n \geqslant 1$.

Corollary 4 ([7], Theorem). *Let m and n be fixed non-negative integers. Suppose that R satisfies the polynomial identity*

$$x^n[x, y] = [x, y^m] \tag{15}$$

for all x and y in R.

(a) *If R is a left s-unital, then R is commutative except for $m = 1$ and $n = 0$.*

(b) *If R is a right s-unital, then R is commutative except for $m = 1$ and $n = 0$, and also for $m = 0$, and $n \geqslant 1$.*

Proof. According to Lemma 6, and [5] (Proposition 1) it suffices to prove the theorem for R with 1.

(a) If $m = 0$, then $x^n [x, y] = 0$, for all x and y in R. Replace x by $(x+1)$, and apply Lemma 2, to get $(x+1)^n [x, y] = 0 = x^n [x, y]$, for all x and y in R, and hence $[x, y] = 0$, for all x and $y \in R$. Thus, R is commutative. Let $m = 1$. Then (15) becomes $[x, y] = x^{n+1} y - x^n yx$ for all x and $y \in R$. Hence, by $[6]_2$ (Theorem) R is commutative provided that $n \geqslant 1$. Now, for $m > 1$, commutativity of R follows from Theorem.

(b) Let $m = 1$ in (15). Then following the same argument as in the proof of (a), we can prove the commutativity of R. If $m = 0$, then $n = 0$, and hence the assertion is clear. In case $m > 1$, R is commutative by the main Theorem.

Remark 3. Let K be a field. Then, the non-commutative ring

$$R = \begin{pmatrix} K & 0 \\ K & 0 \end{pmatrix}$$

has a right identity element and satisfies the polynomial identity $x[x, y] = 0$, for all x and $y \in R$. Hence, in case $m = 0$ and $n > 0$, the main Theorem need not be true for right s-unital ring.

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Abstract

Let $m > 1$, and k be non-negative integers, and let R be an associative left (resp. right) s-unital ring satisfying

$$x^n[x, y] = [x, y^m]x^k$$

for all x and y in R. Then R is commutative. The result of this paper presents a generalization of some properties ensuring commutativity of certain ring.

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