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Solutions of Mathieu's equation (**)

1 - Introduction

The Mathieu equation, first introduced by Mathieu [4] in connection with the vibrations of an elliptic membrane, has attracted the attention of many authors on account of its occurrence in many branches of applied mathematics. This equation is also of more general interest, in that it is the simplest linear differential equation which is not reducible to hypergeometric form.

The canonical form of Mathieu's equation is

$$y'' + (a - 2q \cos 2z)\, y = 0 \tag{1.1}$$

and many discussions of its solutions have appeared in the literature. However, it seems that no explicit expressions for such solutions have so far been put on record.

In this study, solutions of Mathieu's equation are constructed as convergent power series of a parameter by the intermediate use of inhomogeneous hypergeometric functions. The principle of this method is not new, but the use of properties of inhomogeneous hypergeometric functions has not previously occurred in this context.

For a detailed study of inhomogenous functions, see Babister [2]. The method has been applied by Exton $[3]_2$ in the treatment of a hitherto intractable extension of the Bessel-Clifford equation. Extensive bibliographies

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on Mathieu functions and related topics can be found in McLachlan [5], Arscott [1], Meixner, Schäfke and Wolf [6] as well as elsewhere.

2 - An auxiliary differential equation

Before proceeding directly to Mathieu's equation, we consider the differential equation

$$x(1-x)\,y'' + [c-(a+b+1)\,x]\,y' - aby = k^2 x^p y \tag{2.1}$$

where, for the present purposes, p and c are real, such that $0 \leqslant p \leqslant 1$ and $0 < c < 1$ and $1 < c < 2$. The quantities a, b, k and x may be any numbers, real or complex, provided that $\mathrm{Re}(c-a-b) > 0$ and that $|x| \leqslant 1$.

If we replace y by $x^{1-c}y$ in (2.1), this equation becomes

$$x(1-x)\,y'' + [2-c-(a+b-2c+3)\,x]\,y' - (a+1-c)(b+1-c)\,y = k^2 x^p y \tag{2.2}$$

and for $p=1$, replacing x by $1-x$ in (2.1) gives

$$x(1-x)\,y'' + [a+b-c+1-(a+b+1)\,x]\,y' - (ab+k^2)\,y = -k^2 xy\,. \tag{2.3}$$

Hence, if $y(a,\, b;\, c;\, p,\, k;\, x)$ is a solution of (2.1), then so also is

$$x^{1-c}y(a+1-c,\ b+1-c;\ 2-c;\ p,\ k;\ x)\,. \tag{2.4}$$

If $p=1$, the functions

$$y\Big(\frac{a+b+\sqrt{(a-b)^2-4k^2}}{2},\ \frac{a+b-\sqrt{(a-b)^2-4k^2}}{2};\ a+b-c+1;\ 1,\ ik;\ 1-x\Big) \tag{2.5}$$

$$(1-x)^{c-a-b}\,y\Big(c-\frac{a+b+\sqrt{(a-b)^2-4k^2}}{2},\ c-\frac{a+b-\sqrt{(a-b)^2-4k^2}}{2};\ c-a-b+1;\ 1,\ ik;\ 1-x\Big) \tag{2.6}$$

also satisfy (2.1).

3 - The solution of (2.1)

Consider an initially tentative solution of (2.1) in the form

$$y(a,\ b;\ c;\ p,\ k;\ x) = \sum_{r=0}^{\infty} k^{2r} y_r(x)\,. \tag{3.1}$$

If this expression is substituted into (2.1), then on equating the coefficients of successive powers of k to zero, we have

$$x(1-x)\,y_0'' + [c-(a+b+1)\,x]\,y_0' - aby_0 = 0 \tag{3.2}$$

$$x(1-x)\,y_r'' + [c-(a+b+1)\,x]\,y_r' - aby_r = x^p\,y_{r-1} \qquad (r=1,2,3,\ldots). \tag{3.3}$$

A suitable form of y_0 is clearly the hypergeometric function

$$y_0 = {}_2F_1(a,\ b;\ c;\ x) \qquad \text{where} \tag{3.4}$$

$$ {}_AF_B\left(\begin{matrix} a_1,\ a_2,\ldots,a_A\,; \\ b_1,\ b_2,\ldots,b_B\,;\end{matrix} x\right) = \sum_{m=0}^{\infty} \frac{(a_1,\ m)(a_2,\ m)\ldots(a_A,\ m)\,x^m}{(b_1,\ m)(b_2,\ m)\ldots(b_B,\ m)\,m!} \tag{3.5}$$

and, as usual,

$$(a,\ m) = a(a+1)(a+2)\ldots(a+m-1) = \Gamma(a+m)/\Gamma(a) \qquad (a,0)=1\,. \tag{3.6}$$

(See $[3]_1$, Chapter I, for example).

If $A \leqslant B$, the series (3.5) converges for all finite values of x; if $A = B+1$, (3.5) converges for $|x|<1$ and also for $|x|=1$ if $\mathrm{Re}(b_1+b_2+\ldots+b_B-a_1-a_2-\ldots-a_A)>0$. When $A > B+1$, the series (3.5) does not converge at all, except in the trivial case when $x=0$. When at least one of the parameters $a_1,\ a_2,\ldots,a_A$ is a non-positive integer, then the series in question terminates, when the matter of convergence does not arise. Any exceptional values of the parameters when the series (3.5) does not make sense are tacitly excluded.

From (3.3) and (3.4), the function y_1 is determined by the inhomogeneous hypergeometric equation

$$x(1-x)\,y_1'' + [c-(a+b+1)\,x]\,y_1' - aby_1 = x^p\,{}_2F_1(a,\ b;\ c;\ x) \tag{3.7}$$

$$= \sum_{m=0}^{\infty} \frac{(a,\ m)(b,\ m)}{(c,\ m)\,m!}\,x^{p+m}\,.$$

This gives the result

$$y_1 = \sum_{m=0}^{\infty} \frac{(a, m)(b, m)}{(c, m)\, m!} f_{p+1+m}(a, b; c; x) \tag{3.8}$$

where the inhomegeneous hypergeometric function $f_{p+1+m}(a, b; c; x)$ is given by

$$f_{p+1+m}(a, b; c; x) \tag{3.9}$$

$$= \frac{x^{p+1+m}}{(p+1+m)(p+c+m)} {}_3F_2\left(\begin{matrix} a+1+p+m, & b+1+p+m, & 1; \\ c+1+p+m, & 2+p+m; & \end{matrix} x\right).$$

(See [2], page 201).

After a little re-arrangement, it is found that

$$y_1 = \frac{x^{p+1}}{(p+1)(p+c)} {}_4F_3\left(\begin{matrix} a, \ b, \ c+p, \ 1+p; \\ a+1+p, \ b+1+p, \ c; \end{matrix} 1\right) \tag{3.10}$$

$$\times {}_3F_2\left(\begin{matrix} a+1+p, & b+1+p, & 1; \\ c+1+p, & 2+p; & \end{matrix} x\right).$$

If this process is repeated successively, we have eventually

$$y_0 = G_0(a, b; c; p, x) \tag{3.11}$$

$$y_r = \frac{1}{(p+1)^{2r} r! (\frac{p+c}{p+1}, r)} \left[\prod_{s=1}^{r} F_s(a, b; c, p)\right] G_r(a, b; c; p, x) \quad \text{where} \tag{3.12}$$

$$F_r(a, b; c; p) \tag{3.13}$$

$$= {}_5F_4\left(\begin{matrix} a+(r-1)(1+p), \ b+(r-1)(1+p), \ c-1+r(1+p), \ r(1+p), \ 1; \\ a+r(1+p), \ b+r(1+p), \ c+(r-1)(1+p), \ 1+(r-1)(1+p); \end{matrix} 1\right)$$

$$r = 1, 2, 3, \ldots$$

$$G_r(a, b; c; p, x) = x^{rp+r} {}_3F_2\left(\begin{matrix} a+r(1+p), b+r(1+p), \ 1; \\ c+r(1+p), \ 1+r(1+p); \end{matrix} x\right) \tag{3.14}$$

$$r = 0, 1, 2, 3, \ldots .$$

4 - The convergence of the series (3.1)

From the properties of the generalised hypergeometric function ${}_AF_B$ listed in 3, the series (3.13) and (3.14) converge absolutely and uniformly when the restrictions on the parameters associated with (2.1) are taken into account. We now consider the ratio of the $(r+1)^{\text{th}}$ and the r^{th} terms of the series (3.1), namely

$$(4.1)\quad R_r = \frac{k^2}{(p+1)^2(r+1)\left(\dfrac{p+c}{p+1}+r\right)} F_{r+1}(a,\ b;\ c;\ p) \frac{G_{r+1}(a,\ b;\ c;\ p,\ x)}{G_r(a,\ b;\ c;\ p,\ x)}.$$

Since the series representation of $F_r(a,\ b;\ c;\ p)$ converges for all values of r, this function is bounded. Also, by the inspection of (3.14),

$$(4.2)\qquad \lim_{r\to\infty} \frac{G_{r+1}(a,\ b;\ c;\ p;\ x)}{G_r(a,\ b;\ c;\ p,\ x)} = x^{p+1}.$$

Hence, $\lim_{r\to\infty} R_r = 0$ and the series (3.1) converges absolutely. Since (3.14) converges uniformly and absolutely, (3.1) consists of an absolutely convergent series of series which are uniformly convergent. Thus (3.1) converges uniformly also.

5 - Solutions of Mathieu's equation

In Mathieu's equation (1.1), put $x = \cos^2 z$ and $x = \cos^2 2z$. We then have, respectively, the differential equations

$$(5.1)\qquad x(1-x)\,y'' + (1/2 - x)\,y' + (a/4 + q/2 - qx)\,y = 0$$

$$(5.2)\qquad x(1-x)\,y'' + (1/2 - x)\,y' + (a/16 - qx^{1/2}/8)\,y = 0\,.$$

Both of these equations are special cases of (2.1) and from (3.1), (2.4), (2.5) and (2.6), the following solutions of (5.1) may be written down:

$$(5.3)\qquad y\left(\frac{\sqrt{a+2q}}{2},\ -\frac{\sqrt{a+2q}}{2};\ 1/2;\ 1,\ \sqrt{q};\ \cos^2 z\right)$$

$$(5.4)\qquad \cos z\, y\left(\frac{1+\sqrt{a+2q}}{2},\ -\frac{1-\sqrt{a+2q}}{2};\ 3/2;\ 1,\ \sqrt{q};\ \cos^2 z\right)$$

$$y\left(\frac{\sqrt{a-2q}}{2}, -\frac{\sqrt{a-2q}}{2}; 1/2; 1, i\sqrt{q}; \sin^2 z\right) \tag{5.5}$$

$$\sin z y\left(\frac{1+\sqrt{a-2q}}{2}, -\frac{1-\sqrt{a-2q}}{2}; 3/2; 1, i\sqrt{q}; \sin^2 z\right). \tag{5.6}$$

Similarly, using (3.1) and (2.4), we have the two following solutions of (5.2):

$$y\left(\frac{\sqrt{a}}{4}, -\frac{\sqrt{a}}{4}; 1/2; 1/2, \sqrt{q/8}; \cos^2 2z\right) \tag{5.7}$$

$$\cos 2z y\left(\frac{2+\sqrt{a}}{4}, -\frac{2-\sqrt{a}}{4}; 3/2; 1/2, \sqrt{q/8}; \cos^2 2z\right). \tag{5.8}$$

The six expressions (5.3) to (5.8) are all solutions of Mathieu's equation which converge if the absolute value of each respective argument does not exceed unity. This clearly applies if z is real. The solutions (5.5) and (5.6) may respectively be identified with constant multiples of $ce_\nu(q; z)$ and $se_\nu(q; z)$, where ν is the corresponding characteristic exponent. The determination of the normalisation constants and the associated characteristic exponent is to be undertaken in a further study.

6 - Numerical implementation

The expression (5.5) and (5.6), or (5.3) and (5.4) when the sign of q is reversed, lend themselves quite readily to the computation of the functions which they represent. This is facilitated by observing that, for $p = 1$, $F_r(a, b; c; 1)$ and $G_r(a, b; c, 1, x)$ possess the two following recurrence relations ($r = 1, 2, 3, \ldots$)

$$F_r(a, b; c; 1) = \frac{(a+2r, 2)(b+2r, 2)(c+2r, -2)(2r-1)}{(a+2r-2, 2)(b+2r-2, 2)(c+2r)(2r+1)} \tag{6.1}$$

$$\times[F_{r-1}(a, b; c; 1) - 1 - \frac{(a+2r-2)(b+2r-2)(c+2r-1)2r}{(a+2r)(b+2r)(c+2r-2)(2r-1)}]$$

$$G_r(a, b; c; l, x) = \frac{(c+2r-2, 2)(2r-1)}{(a+2r-2, 2)(b+2r-2, 2)} \tag{6.2}$$

$$\times[G_{r-1}(a, b; c; 1, x) - x^{2r-2} - \frac{(a+2r-2)(b+2r-2)}{(c+2r-2)(2r-1)} x^{2r-1}].$$

Both of these results are simple consequences of the fact that each series representing $F_{r+1}(a, b; c; 1)$ or $G_{r+1}(a, b; c; 1, x)$ is the same as that representing $F_r(a, b; c; 1)$ or $G_r(a, b; c; 1, x)$ respectively, each with the first two terms subtracted. The portion of the calculation which converges most slowly is, in the case of (5.5), the computation of

$$F_1\left(\frac{\sqrt{a-2q}}{2}, -\frac{\sqrt{a-2q}}{2}; 1/2; 1\right) \tag{6.3}$$

$$= {}_4F_3\left[\begin{matrix} \frac{\sqrt{a-2q}}{2}, -\frac{\sqrt{a-2q}}{2}, 3/2, 2; \\ 2+\frac{\sqrt{a-2q}}{2}, 2-\frac{\sqrt{a-2q}}{2}, 1/2; \end{matrix} 1\right].$$

This only needs to be carried out for each value of $a-2q$.

Furthermore, the form of $G_0\left(\frac{\sqrt{a-2q}}{2}, -\frac{\sqrt{a-2q}}{2}; 1/2; 1, \sin^2 z\right)$, namely

$$ {}_2F_1\left(\frac{\sqrt{a-2q}}{2}, -\frac{\sqrt{a-2q}}{2}, 1/2; \sin^2 z\right) = \cos(z\sqrt{a-2q}) \tag{6.4}$$

is most convenient in this context. The situation where (5.3), (5.4) and (5.6) are concerned is similar. If $|q| \leqslant 15$, accuracy of at least six decimal places is easily effected using a small computer.

References

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Summary

By utilising series in terms of powers of a parameter, explicit solutions of Mathieu's equation are deduced. This is achieved by the use of auxiliary inhomogeneous hypergeometric functions. A mean of the numerical implementation of the results is indicated.

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