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Submanifolds of codimension r of a H-structure manifold (**)

1 - Preliminaries

Let V_n be an n-dimensional differentiable manifold of class C^∞. Suppose that there exists on V_n a tensor field $F \neq 0$ of type (1, 1) satisfying

$$F^2 = a^2 I \tag{1.1}$$

where a is any non-zero complex number. Suppose further that V_n admits a hermite metric G satisfying

$$G(FX^*,\ FY^*) + a^2 G(X^*,\ Y^*) = 0 \tag{1.2}$$

for arbitrary vector fields X^* and Y^* on V_n. Thus, in view of the equations (1.1) and (1.2) V_n will be said to possess an H-structure.

If $'F(X^*, Y^*)$ is the tensor field of type (0, 2) given by

$$'F(X^*,\ Y^*) = G(FX^*,\ Y^*) \tag{1.3}$$

the following results can be proved easily

$$\begin{gathered} 'F(FX^*,\ Y^*) = -\,'F(X^*,\ FY^*) = a^2 G(X^*,\ Y^*) \\ 'F(FX^*,\ FY^*) + a^2\,'F(X^*,\ Y^*)=0 \qquad 'F(X^*,\ Y^*) + 'F(Y^*,\ X^*)=0\,. \end{gathered} \tag{1.4}$$

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Let $\tilde{D}$ be the Riemannian connection on V_n; then

$$\tilde{D}_{X^*} Y^* - \tilde{D}_{Y^*} X^* = [X^*, Y^*] \qquad \tilde{D}_{X^*} G = 0 . \tag{1.5}$$

Let $\tilde{N}$ be the Nijenhuis tensor formed with F; then

$$\tilde{N}(X^*, Y^*) = [FX^*, FY^*] - F[FX^*, Y^*] - F[X^*, FY^*] + F^2[X^*, Y^*] . \tag{1.6}$$

An H-structure manifold V_n will be called a *K-manifold* if the structure tensor F is parallel i.e.

$$(\tilde{D}_{X^*} F)(Y^*) = 0 . \tag{1.7}$$

A submanifold V_{n-r} of codimension r of the H-structure manifold V_n will be said to possess a *generalised para r-contact structure* if there exists a tensor field f of type (1, 1), r C^∞ contravariant vector fields $\underset{x}{U}$, r C^∞ 1-forms $\overset{x}{u}$ (r some finite integer) satisfying

$$f^2 = a^2 I - \sum_{x=1}^{r} \overset{x}{u} \otimes \underset{x}{U}. \tag{1.8}$$

Also

$$\overset{y}{u} \circ f + \sum_{x=1}^{r} \theta_x^y \overset{x}{u} = 0 \qquad f\underset{x}{U} + \sum_{y=1}^{r} \theta_x^y \underset{y}{U} = 0$$

(1.9)

$$\overset{x}{u}(\underset{y}{U}) + \sum_{z=1}^{r} \theta_z^x \theta_y^z = a^2 \delta_y^x$$

where $x, y = 1, 2, \ldots, r$, δ_y^x denotes the Kronecker delta and θ_y^x are scalar fields.

If in addition, the submanifold V_{n-r} admits a Riemannian metric g satisfying

$$g(fX, fY) + a^2 g(X, Y) + \sum_{x=1}^{r} \overset{x}{u}(X) \overset{x}{u}(Y) = 0 \tag{1.10}$$

we say that V_{n-r} admits a *generalised para r-contact metric structure.*

2 - Submanifolds of codimension r

Let V_{n-r} be the subamnifold of codimension r of a H-structure manifold V_n. If B denotes the differential of the immersion i: $V_{n-r} \to V_n$, a vector field X in the tangent space of V_{n-r} determines a vector field BX in that of V_n. Let $\underset{x}{N}$, $x = 1, 2, \ldots, r$ be r mutually orthogonal fields of unit normal vectors defined on V_{n-r}. Thus we have

$$(2.1) \qquad G(BX,\ BY)=g(X,\ Y) \qquad G(BX,\ \underset{x}{N})=0 \qquad G(\underset{x}{N},\ \underset{y}{N})=\delta^x_y.$$

The vector fields FBX and $F\underset{x}{N}$ can be expressed by

$$(2.2) \qquad FBX = BfX - \sum_{x=1}^{r} \overset{x}{u}(X)\underset{x}{N} \qquad F\underset{x}{N} = -B\underset{x}{U} + \sum_{y=1}^{r} \theta^y_x \underset{y}{N}$$

where f is a (1, 1) tensor field, $\overset{x}{u}$ 1-forms and $\underset{x}{U}$ vector fields on the submanifold V_{n-r}, $(x = 1, 2, \ldots, r)$.

Operating by F on both the sides of $(2.2)_1$ and making use of equations (1.1) and (2.2), we obtain

$$a^2 BX = Bf^2 X - \sum_{y=1}^{r} \overset{y}{u}(fX)\underset{y}{N} - \sum_{x=1}^{r} \overset{x}{u}(X)\{-B\underset{x}{U} + \sum_{y=1}^{r} \theta^y_x \underset{y}{N}\}.$$

Comparison of tangential and normal vectors gives

$$(2.3) \qquad f^2 = a^2 I - \sum_{x=1}^{r} \overset{x}{u} \otimes \underset{x}{U} \qquad \overset{y}{u} \circ f + \sum_{x=1}^{r} \theta^y_x \overset{x}{u} = 0.$$

Multiplying both the sides of the equation $(2.2)_2$ by F and using again equations (1.1) and (2.2), we get

$$a^2 \underset{x}{N} = -\{Bf\underset{x}{U} - \sum_{z=1}^{r} \overset{z}{u}(\underset{x}{U})\underset{z}{N}\} + \sum_{y=1}^{r} \theta^y_x \{-B\underset{y}{U} + \sum_{z=1}^{r} \theta^z_y \underset{z}{N}\}.$$

Comparison of tangential and normal vectors gives

$$(2.4) \qquad f\underset{x}{U} + \sum_{y=1}^{r} \theta^y_x \underset{y}{U} = 0 \qquad \overset{z}{u}(\underset{x}{U}) + \sum_{y=1}^{r} \theta^z_y \theta^y_x = a^2 \delta^z_x.$$

Further, in view of the equations (1.1), (2.1) and (2.2), if g is the induced metric on V_{n-r}, then we have

$$(2.5) \qquad g(fX, fY) + a^2 g(X,\ Y) + \sum_{x=1}^{r} \overset{x}{u}(X)\overset{x}{u}(Y) = 0.$$

In view of the equations (2.3), (2.4) and (2.5), we have

Theorem 2.1. *The submanifold V_{n-r} of codimension r of an H-structure manifold V_n admits a generalised para r-contact metric structure.*

Suppose further that $\tilde{D}$ is the Riemannian connection on V_n and D the induced connection on the submanifold V_{n-r}. Then the equations of Gauss and Weingarten can be expressed as

$$\tilde{D}_{BX}BY = BD_X Y + \sum_{x=1}^{r} \overset{x}{h}(X, Y) \underset{x}{N} \tag{2.6}$$

$$\tilde{D}_{BX} \underset{x}{N} = -B\overset{x}{H}(X) + \sum_{y=1}^{r} \theta_x^y \underset{y}{N} \tag{2.7}$$

where $\overset{x}{h}(X; Y)$ are second fundamental forms, and

$$\overset{x}{h}(X, Y) = g(\overset{x}{H}(X), Y). \tag{2.8}$$

Suppose that the enveloping manifold V_n is a K-manifold. Hence we have $(\tilde{D}_{BX}F)(BY) = 0$ or equivalently $\tilde{D}_{BX}FBY = F\tilde{D}_{BX}BY$.

In view of the equations (2.2), (2.6) and (2.7) the last equation takes the form

$$D_{BX}\{BfY - \sum_{x=1}^{r} \overset{x}{u}(Y) \underset{x}{N}\} = F\{BD_X Y + \sum_{x=1}^{r} \overset{x}{h}(X, Y) \underset{x}{N}\}$$

or equivalently

$$BD_XfY + \sum_{x=1}^{r} \overset{x}{h}(X, fY) \underset{x}{N} - \sum_{x=1}^{r} \overset{x}{u}(Y)\{-B\overset{x}{H}(X) + \sum_{y=1}^{r} \theta_x^y \underset{y}{N}\}$$

$$= BfD_X Y - \sum_{x=1}^{r} \overset{x}{u}(D_X Y) \underset{x}{N} + \sum_{x=1}^{r} \overset{x}{h}(X, Y)\{-B\underset{x}{U} + \sum_{y=1}^{r} \theta_x^y \underset{y}{N}\}.$$

The comparison of the tangential vectors in both the sides gives

$$D_XfY + \sum_{x=1}^{x} \overset{x}{u}(Y) \overset{x}{H}(X) = fD_X Y - \sum_{x=1}^{r} \overset{x}{h}(X, Y) \underset{x}{U}$$

or equivalently

$$(D_Xf)(Y) + \sum_{x=1}^{r} \{\overset{x}{u}(Y) \overset{x}{H}(X) + \overset{x}{h}(X, Y) \underset{x}{U}\} = 0. \tag{2.9}$$

If $N(X, Y)$ is the Nijenhuis tensor for the submanifold V_{n-r}, we can write

$$N(X, Y) = (D_{fX}f)(Y) - (D_{fY}f)(X) + f(D_Y f)(X) - f(D_X f)(Y). \tag{2.10}$$

A necessary and sufficient condition that the submanifold V_{n-r} be totally geodesic is that $\overset{x}{h}(X, Y) = 0$ $(x = 1, 2, \ldots, r)$. Thus in view of the equations (2.8) and (2.9), it follows that $D_X f = 0$. Hence from (2.10), we have $N(X, Y) = 0$.

But V_{n-r} is said to be *integrable* id and only if $N(X, Y) = 0$. Thus we have

Theorem 2.2. *A totally geodesie submanifold V_{n-r} with a generalised para r-contact structure of an H-structure manifold is integrable.*

3 - Curvature tensors

Suppose that W, X, Y, Z are arbitrary vector fields on an open set A in the neighbourhood of a point of the submanifold V_{n-r}. If $'\tilde{L}$ and $'L$ are the Riemann-Christoffel curvature tensors of V_n and V_{n-r} respectively, we have

$$'\tilde{L}(BW, BX, BY, BZ) = {}'L(W, X, Y, Z) + \sum_{x=1}^{r} \{\overset{x}{h}(X, Z)\overset{x}{h}(W, Y) - \overset{x}{h}(X, Y)\overset{x}{h}(W, Z)\}. \tag{3.1}$$

If the manifold V_n admits constant holomorphic sectional curvature C, we have

$$\begin{aligned} &'\tilde{L}(BW, BX, BY, BZ) \\ &= \frac{C}{4}[G(BW, BZ)G(BX, BY) - G(BX, BZ)G(BW, BY) \\ &+ {}'F(BX, BZ)\,{}'F(BW, BY) - {}'F(BX, BY)\,{}'F(BW, BZ) \\ &+ 2\,{}'F(BW, BX)\,{}'F(BY, BZ)]. \end{aligned} \tag{3.2}$$

From equations (1.3) and (2.2) it can be proved that

$$'F(BX, BY) = f(X, Y) \overset{\text{def}}{=} g(fX, Y). \tag{3.3}$$

Hence, in view of the equations (2.1), (3.1) and (3.3), the equation (3.2) takes the form

$$(3.4) \qquad 'L(W, X, Y, Z)$$

$$= \frac{C}{4}[g(W, Z)g(X, Y) - g(X, Z)g(W, Y) + 'f(X, Z)'f(W, Y)$$

$$-'f(X, Y)'f(W, Z) + 2'f(W, X)'f(Y, Z)]$$

$$+ \sum_{x=1}^{r} \{\overset{x}{h}(X, Y)\overset{x}{h}(W, Z) - \overset{x}{h}(X, Z)\overset{x}{h}(W, Y)\}.$$

Thus we have

Theorem 3.1. *Let V_n be an H-structure manifold of constant holomorphic sectional curvature C. Then the curvature tensor of the submanifold V_{n-r} satisfies the equation* (3.4).

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Abstract

In this paper, we consider an H-structure manifold and show that its submanifold of codimension r admits a generalised para r-contact structure. Conditions for the integrability of such a structure are studied. A result connecting the curvature tensors of the manifold and of its submanifolds is established.

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