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Rizza's conjecture concerning the bisectional curvature (**)

1 - Introduction

Let M be a C^∞-differentiable manifold of dimension $n \geqslant 3$ and denote by T_x the tangent space to M at the point $x \in M$.

In his paper [3], G. B. Rizza obtains a useful formula for the bisectional curvature χ_{pq} with respect to the oriented planes p and q of T_x, in terms of the sectional curvature K_r of some convenient planes r of T_x. For two orthonormal bases X_1, X_2 and X_3, X_4 of p and q, respectively, this formula is

$$\begin{aligned}\tfrac{3}{2}\chi_{pq} = & \sum_{\sigma_2} sK_{S_{i3}S_{j4}} \cos^2 \tfrac{1}{2}X_i X_3 \cos^2 \tfrac{1}{2}X_j X_4 \sin^2 S_{i3} S_{j4} \\ & - \sum_{\sigma_2} sK_{S_{i3}D_{j4}} \cos^2 \tfrac{1}{2}X_i X_3 \sin^2 \tfrac{1}{2}X_j X_4 \sin^2 S_{i3} D_{j4} \\ & - \sum_{\sigma_2} sK_{D_{i3}S_{j4}} \sin^2 \tfrac{1}{2}X_i X_3 \cos^2 \tfrac{1}{2}X_j X_4 \sin^2 D_{i3} S_{j4} \\ & + \sum_{\sigma_2} sK_{D_{i3}D_{j4}} \sin^2 \tfrac{1}{2}X_i X_3 \sin^2 \tfrac{1}{2}X_j X_4 \sin^2 D_{i3} D_{j4}\end{aligned} \tag{1}$$

where $S_{ij} = X_i + X_j$, $D_{ij} = X_i - X_j$, σ_2 is the group of the permutations (i, j) of $(1, 2)$ and $s = \text{sign}(i, j)$.

As a consequence of the formula (1), it is proved in [3] that, *if* $|K_r| \leqslant C$ *for any plane* r *of* T_x, *then* $|\chi_{pq}| \leqslant \frac{4}{3}C$ *for all planes* p *and* q *of* T_x. Moreover, for

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some couples of planes we have $|\chi_{pq}| \leqslant C$, which suggests to G. B. Rizza the following

Conjecture. *Let C be the maximum of $|K_r|$ as r varies in T_x. Then $|\chi_{pq}| \leqslant C$ for any couple p, q of oriented planes of T_x.*

Remark. If $n = 3$ then the planes p, q have in common a line and by Corollary 2 of [3], the Rizza's conjecture has an affirmative answer.

Our purpose, in this paper, is to give a negative answer to the Rizza's conjecture, in dimension greater that 3. In fact, we prove that if $K_r \in [-\alpha C,\ C]$ or $K_r \in [-C,\ \alpha C]$ and $\alpha \in [-1,\ \frac{1}{2}]$ then $|\chi_{pq}| \leqslant C$. For $\alpha \in (\frac{1}{2},\ 1]$ this is not generally true and we present some counter-examples (Theorem 2 and Proposition 1).

2 - A theorem

In this section we prove the following

Theorem 1. *Let C be the maximum of $|K_r|$ at the point $x \in M$, when the plane r varies in T_x. If $K_r \in [-\alpha C,\ C]$ or $K_r \in [-C,\ \alpha C]$ and $\alpha \in [-1,\ \frac{1}{2}]$ then $|\chi_{pq}| \leqslant C$ for all oriented planes p, q of T_x.*

Proof. For the oriented planes p, q, we choose orthonormal bases X_1, X_2 and X_3, X_4, satisfying the conditions

$$X_1 \cdot X_3 = 0 \qquad X_2 \cdot X_4 = 0$$

where $X \cdot Y$ denotes the inner product of the vectors X and Y. An elementary calculation proves the existence of such bases.

From the formula (1) we obtain by simple computations

$$\begin{aligned}
\tfrac{3}{2}\chi_{pq} = {} & \tfrac{1}{16}(K_{S_{13}S_{24}} + K_{D_{13}D_{24}})[4 - (X_1 \cdot X_4 + X_2 \cdot X_3)^2] \\
& - \tfrac{1}{16}(K_{S_{13}D_{24}} + K_{D_{13}S_{24}})[4 - (X_1 \cdot X_4 - X_2 \cdot X_3)^2] \\
& - \tfrac{1}{4}K_{S_{23}S_{14}}(1 + X_2 \cdot X_3)(1 + X_1 \cdot X_4) + \tfrac{1}{4}K_{S_{23}D_{14}}(1 + X_2 \cdot X_3)(1 - X_1 \cdot X_4) \\
& + \tfrac{1}{4}K_{D_{23}S_{14}}(1 - X_2 \cdot X_3)(1 + X_1 \cdot X_4) - \tfrac{1}{4}K_{D_{23}D_{14}}(1 - X_2 \cdot X_3)(1 - X_1 \cdot X_4)
\end{aligned} \tag{2}$$

and two cases must be analyzed.

Case 1. $\alpha \in [-1, 0]$.

In this case M has positive (or negative) sectional curvature at x, for any plane of T_x. Suppose $0 \leqslant K_r \leqslant C$. Then from (2) we deduce

$$\tfrac{3}{2}\chi_{pq} \leqslant \frac{C}{8}[4-(X_1\cdot X_4+X_2\cdot X_3)^2]+\frac{C}{2}[1-(X_2\cdot X_3)(X_1\cdot X_4)].$$

But $$|X_1\cdot X_4+X_2\cdot X_3| \leqslant 2 \qquad |(X_1\cdot X_4)(X_2\cdot X_3)| \leqslant 1$$

and then $\chi_{pq} \leqslant C$.

Now, renouncing to the terms preceded by the sign plus in (2) and proceding as above, we obtain $\chi_{pq} \geqslant -C$, which, together with $\chi_{pq} \leqslant C$, gives our result.

If $-C \leqslant K_r \leqslant 0$ we use the same argument.

Case 2. $\alpha \in (0, \frac{1}{2}]$.

We know that $K_r \in [-\alpha C, C]$ or $K_r \in [-C, \alpha C]$ and because these two possibilities are analogous, we analyze only the first one. From (2) we obtain

$$\tfrac{3}{2}\chi_{pq} \leqslant \frac{C}{8}[4-(X_1\cdot X_4+X_2\cdot X_3)^2]+\frac{\alpha C}{8}[4-(X_1\cdot X_4-X_2\cdot X_3)^2]$$

$$+\frac{\alpha C}{4}(1+X_2\cdot X_3)(1+X_1\cdot X_4)+\frac{C}{4}(1+X_2\cdot X_3)(1-X_1\cdot X_4)$$

$$+\frac{\alpha C}{4}(1-X_2\cdot X_3)(1-X_1\cdot X_4)+\frac{C}{4}(1-X_2\cdot X_3)(1+X_1\cdot X_4)$$

and

$$\tfrac{3}{2}\chi_{pq} \geqslant -\frac{\alpha C}{8}[4-(X_1\cdot X_4+X_2\cdot X_3)^2]-\frac{C}{4}[4-(X_1\cdot X_4-X_2\cdot X_3)^2]$$

$$-\frac{C}{4}(1+X_2\cdot X_3)(1+X_1\cdot X_4)-\frac{\alpha C}{4}(1+X_2\cdot X_3)(1-X_1\cdot X_4)$$

$$-\frac{C}{4}(1-X_2\cdot X_3)(1-X_1\cdot X_4)-\frac{\alpha C}{4}(1-X_2\cdot X_3)(1+X_1\cdot X_4).$$

Introducing the notations $U=X_1\cdot X_4$, $V=X_2\cdot X_3$, from the precedent inequalities we deduce immediately that if

$$\frac{C}{8}[8(1+\alpha)-(1+\alpha)U^2-6(1-\alpha)UV-(1+\alpha)V^2] \leqslant \frac{3C}{2}$$

(3)

$$-\frac{C}{8}[8(1+\alpha)-(1+\alpha)U^2+6(1-\alpha)UV-(1+\alpha)V^2] \geqslant -\frac{3C}{2}$$

for all $U, V \in [-1, 1]$, then $|\chi_{pq}| \leqslant C$.

By simple computations, (3) becomes

$$f_1(U, V) \equiv 4(2\alpha-1)-(1+\alpha)U^2-6(1-\alpha)UV-(1+\alpha)V^2 \leqslant 0$$

(3)′

$$f_2(U, V) \equiv 4(2\alpha-1)-(1+\alpha)U^2+6(1-\alpha)UV-(1+\alpha)V^2 \leqslant 0.$$

But we have $f_2(U, V) = f_1(U, -V)$ therefore we study only the extremum of f_1. Since

$$\Delta = \frac{\partial^2 f_1}{\partial U^2}\frac{\partial^2 f_1}{\partial V^2} - \left(\frac{\partial^2 f_1}{\partial U \partial V}\right)^2 = -16(2\alpha^2-5\alpha+2) < 0$$

for $\alpha \in [0, \frac{1}{2})$, the function f_1 attains its extremum on the boundary of the set $[-1, 1] \times [-1, 1]$.

Using this remark, we shall prove that $f_1(U, V) \leqslant 0$ for all $U, V \in [-1, 1]$.

First, from the expression of f_1 it follows that if $UV \geqslant 0$ then $f_1(U, V) \leqslant 0$. For $UV < 0$ we have $f_1(U, 1) < 0$ if $U < 0$ and $f_1(U, -1) < 0$ if $U > 0$. As f_1 is symmetric with respect to U and V, we have proved that for $U, V \in [-1, 1]$ (3)′ is valid in the case $\alpha \in [\frac{1}{2})$.

If $\alpha = \frac{1}{2}$, (3)′ is obviously true and the proof is complete.

The precedent proof supplies some information on the case $\alpha \in (\frac{1}{2}, 1]$. Thus, if $\alpha \in (\frac{1}{2}, 1]$ then

$$\Delta > 0 \qquad \frac{\partial^2 f_1}{\partial U^2} = -2(1+\alpha) < 0$$

which prove that for $U = 0$, $V = 0$ the function f_1 has a maximum on $[-1, 1] \times [-1, 1]$. But $f_1(0, 0) = 4(2\alpha - 1) > 0$ and then (3)′ is not satisfied for all $U, V \in [-1, 1]$.

This argument suggests that *examples of manifolds with the property*

(P) $$|K_r| \leqslant C \qquad and \qquad |\chi_{pq}| > C$$

for some p, q *must be sought among the manifolds whose sectional curvature varies in* $[-\alpha C,\ C]$ *or in* $[-C,\ \alpha C]$ *with* $\alpha \in (\frac{1}{2},\ 1]$.

3 - Examples

In this section we present examples of manifolds with the property (P).

Let M be a Sasakian manifold of dimension $n = 2m + 1 \geqslant 5$. This means that on M is given an almost contact metric structure $(F,\ \xi,\ \eta,\ \cdot)$, the tensor fields F, ξ, η satisfying the conditions

(4) $$F^2 = -I + \eta \otimes \xi \qquad \eta(\xi) = 1 \qquad FX \cdot FY = X \cdot Y - \eta(X)\,\eta(Y)$$

for any vector fields X and Y on M (I is the identity transformation).

As it is well-known, M is Sasakian if and only if the almost contact metric structure has the property

(5) $$(\nabla_X F)\,Y = (X \cdot Y)\,\xi - \eta(Y)\,X$$

where ∇ denotes the Riemannian connection of the metric $\cdot$ (see for instance [**1**], p. 73).

Theorem 2. *Let* M *be a Sasakian manifold and* $\alpha \in (\frac{1}{2},\ 1]$. *If for any plane* r *of* T_x *we have*

(a) $$K_r \in [-\alpha C,\ C] \quad and \quad C \in [1,\ \frac{2}{2-\alpha})$$

or

(b) $$K_r \in [-C,\ \alpha C] \quad and \quad C \in [\frac{1}{\alpha},\ \frac{2}{\alpha})$$

then there exist two oriented planes p *and* q, *of* T_x, *such that* $|\chi_{pq}| > C$.

Proof. For any unit vectors X, $Y \in T_x$ we have

(6) $$\begin{aligned} X &= a\xi + bT_1 \quad \text{where} \quad a^2 + b^2 = 1, \quad T_1 \perp \xi \quad \text{and} \quad \|T_1\| = 1 \\ Y &= c\xi + dT_2 \quad \text{where} \quad c^2 + d^2 = 1, \quad T_2 \perp \xi \quad \text{and} \quad \|T_2\| = 1\,. \end{aligned}$$

Moreover, X and Y are orthogonal if and only if

$$ac + bdT_1 \cdot T_2 = 0. \tag{7}$$

Using the properties of the Riemann-Christoffel tensor $\mathcal{R}$ we have

$$\mathcal{R}(X, Y, X, Y) = a^2 d^2 \mathcal{R}(\xi, T_2, \xi, T_2) + 2abcd\mathcal{R}(T_1, \xi, \xi, T_2)$$
$$+2abd^2 \mathcal{R}(\xi, T_2, T_1, T_2) + 2b^2 cd\mathcal{R}(T_1, \xi, T_1, T_2)$$
$$+b^2 c^2 \mathcal{R}(T_1, \xi, T_1, \xi) + b^2 d^2 \mathcal{R}(T_1, T_2, T_1, T_2).$$

But on a Sasakian manifold the following equality holds

$$R(U, V)\xi = \eta(V)U - \eta(U)V \tag{8}$$

for all $U, V \in T_x$ ([1], p. 75) and then

$$\mathcal{R}(\xi, T_2, \xi, T_2) = \mathcal{R}(T_1, \xi, T_1, \xi) = 1$$
$$\mathcal{R}(T_1, \xi, \xi, T_2) = -T_1 \cdot T_2$$
$$\mathcal{R}(\xi, T_2, T_1, T_2) = \mathcal{R}(T_1, \xi, T_1, T_2) = 0.$$

Now, using (6) and (7) we obtain

$$K_{XY} = 2 - b^2 - d^2 + b^2 d^2 [1 - (T_1 \cdot T_2)^2] K_{T_1 T_2}. \tag{9}$$

Suppose $K_{XY} = -\alpha C$. If $b^2 d^2 [1 - (T_1 \cdot T_2)^2] = 0$ then from (9) follows $K_{XY} \geqslant 0$, which contradicts the hypothesis $K_{XY} < 0$. Therefore $b^2 d^2 [1 - (T_1 \cdot T_2)^2] \neq 0$ and from (9) we obtain

$$K_{T_1 T_2} = -\frac{2 - b^2 - d^2 + \alpha C}{b^2 d^2 [1 - (T_1 \cdot T_2)^2]} \leqslant -(2 - b^2 - d^2 + \alpha C) \leqslant -\alpha C. \tag{10}$$

But $-\alpha C$ is the minimum of the sectional curvature and then by (10) it follows $b^2 = d^2 = 1$ and $T_1 \cdot T_2 = 0$.

In this way we have proved that there exists two orthonormal vectors T_1, T_2, with the properties

$$K_{T_1 T_2} = -\alpha C \qquad T_1 \perp \xi \qquad T_2 \perp \xi.$$

Let $\{e_1, e_2, \dots, e_m, e_{1^*} = Fe_1, e_{2^*} = Fe_2, \dots, e_{m^*} = Fe_m, \xi\}$ be an adapted base of T_x. By a theorem of E. Moskal (see [4], p. 39-43 or [1], lemma, p. 93) we have

$$\mathcal{R}(e_i,\ e_{i^*},\ e_j,\ e_{j^*}) = 2 - K_{e_i e_j} - K_{e_{i^*} e_{j^*}} \qquad \text{for } i \neq j. \tag{11}$$

By the above argument, (11) becomes

$$\mathcal{R}(T_1,\ FT_1,\ T_2,\ FT_2) \geqslant 2 + \alpha C - C > C \tag{12$_a$}$$

in the case (a), and

$$\mathcal{R}(T_1,\ FT_1,\ T_2,\ FT_2) \geqslant 2 + C - \alpha C > C \tag{12$_b$}$$

in the case (b).

Now, from the general formula (see for instance [3])

$$\chi_{pq} = \mathcal{R}(X_1,\ X_2,\ X_3,\ X_4) \begin{vmatrix} X_1 \cdot X_1 & X_1 \cdot X_2 \\ X_2 \cdot X_1 & X_2 \cdot X_2 \end{vmatrix}^{-\frac{1}{2}} \begin{vmatrix} X_3 \cdot X_3 & X_3 \cdot X_4 \\ X_4 \cdot X_3 & X_4 \cdot X_4 \end{vmatrix}^{-\frac{1}{2}}$$

we deduce that the bisectional curvature of the planes p, q, spanned by the vectors T_1, FT_1 and T_2, FT_2 respectively, is $\chi_{pq} = \mathcal{R}(T_1,\ FT_1,\ T_2,\ FT_2)$ and from $(12)_a$, $(12)_b$ we deduce the result.

Finally, we present some Sasakian manifolds which satisfy the hypotheses of Theorem 2.

Lemma. 1. *Let M be a Sasakian manifold and $x \in M$. If X, $Y \in T_x$ are orthonormal, then*

$$-1 \leqslant 3(X \cdot FY)^2 - (\eta(X) - (\eta(Y))^2 \leqslant 3\,.$$

The inequalities follow easily from (6), (7).

Afterwards, we assume that the manifold M is a *Sasakian space form.* Then the curvature tensor of M is given by

$$R(X,\ Y)Z = \frac{k+3}{4}[(Y \cdot Z)X - (X \cdot Z)Y] \tag{13}$$

$$+ \frac{k-1}{4}[\eta(X)\,\eta(Z)\,Y - \eta(Y)\,\eta(Z)\,X + (X \cdot Z)\,\eta(Y)\,\xi - (Y \cdot Z)\,\eta(X)\,\xi]$$

$$+ \frac{k-1}{4}[Z \cdot FY)FX - (Z \cdot FX)FX + 2(X \cdot FY)FZ]$$

where k is the constant F-sectional curvature of M (see for instance [**1**], p. 97).

From (13) we deduce that the sectional curvature of M is

$$K_{XY} = R(X,\ Y)\,Y\cdot X = \frac{k+3}{4} + \frac{k-1}{4}\,[3(X\cdot FY)^2 - (\eta(X))^2 - (\eta(Y))^2] \tag{14}$$

for any orthonormal vectors $X,\ Y \in T_x$.

Suppose $k < 1$. By use of (14) and of Lemma 1, we have $k \leqslant K_{XY} \leqslant 1$ for any orthonormal vectors X, Y, and taking into account the Theorem 2, we obtain

Proposition 1. *Let M be a Sasakian space form of F-sectional curvature equal to k and $x \in M$.*

(a) *If $k \in [-1,\ -\frac{1}{2})$ then $|K_r| \leqslant 1$ for all planes r of T_x and there exists two planes p, q, for which $|\chi_{pq}| > 1$.*

(b) *If $k \in (-2,\ -1)$ then $|K_r| \leqslant |k|$ for all planes r of T_x and there exists two planes p, q, for which $|\chi_{pq}| > |k|$.*

Remark. As example of Sasakian space form which satisfies the hypotheses of Proposition 1, we can consider the sphere S^{2m+1}, with the deformed structure (studied by S. Tanno)

$$\eta^* = \lambda\eta \qquad \xi^* = \frac{1}{\lambda}\xi \qquad F^* = F \qquad X * Y = X\cdot Y + \lambda(\lambda-1)\,\eta\otimes\eta$$

where λ is a positive constant ([**1**], p. 99).

$(S^{2m+1},\ F^*,\ \xi^*,\ \eta^*,\ *)$ is a Sasakian space form with F^*-sectional curvature $k = \dfrac{4}{\lambda} - 3$ and it satisfies the hipotheses of Proposition 1, for $\lambda \in (\frac{8}{5},\ 4)$.

References

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Summary

See Introduction.

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