Riv. Mat. Univ. Parma (4) **16** (1990), 251-262

GIUSEPPE BUFFONI (*)

Perturbation of a matrix with positive inverse (**)

1 - Introduction

Let

$$A + vB \tag{1.1}$$

be an $n \times n$ real matrix, where A is a nonsingular matrix with positive inverse [6], [3], [2], B a non-zero matrix, with nonnegative part U and nonpositive part $-V$ and v a nonnegative real parameter

$$A^{-1} > 0 \qquad B = U - V \neq 0 \qquad U \geqslant 0 \qquad V \geqslant 0 \qquad v \geqslant 0 . \tag{1.2}$$

The parameter v may be considered as a measure of the size of the perturbation vB of the matrix A. Let

$$Z(u,\ v) = (A + uU - vV)^{-1} \tag{1.3}$$

where u, as v, is a nonnegative real parameter. For $u = v = 0$, we have $Z(0,\ 0) = A^{-1} > 0$; thus, $\det(A + vB) \neq 0$ and $Z(v,\ v) > 0$ in a sufficiently small neighborhood of 0. The purpose of this paper is to find the largest, possibly infinite, number w such that $A + vB$ is nonsingular and $Z(v,\ v) > 0$ in $[0,\ w)$.

Let us first consider the special cases where all the entries of B are either nonpositive or nonnegative. In the case $B = -V \leqslant 0$ $(U = 0)$ we have that [6] (p. 83) $w = 1/r(A^{-1}V) < +\infty$, where $r(\cdot)$ is the spectral radius of the argument matrix. In the case $B = U \geqslant 0$ $(V = 0)$, recently considered

---

(*) Indirizzo: ENEA CREA S. Teresa C.P. 316 - I-19100 La Spezia.
(**) Ricevuto: 27-II-1990.

by the author, we have that the number w may be either finite or infinite; an algorithm to evaluate w when $w < +\infty$ is presented in [1].

We will describe an algorithm, the iterative process defined in Theorem 2, to compute w when $U \geq 0$, $V \geq 0$ and $w < +\infty$; each step of this procedure requires the application of the basic processes (2.4) and/or (2.5), described in Theorem 1, founded on the processes for the two limit cases $V = 0$ and $U = 0$ respectively. In the case $w = +\infty$, the successive approximations form a sequence diverging monotonically to $+\infty$. The results of numerical computations, performed by using a Stieltjes matrix A [6] (p. 85) and a random matrix B, are presented.

2 - The basic processes

Lemma 1. *If $P > 0$ and $Q > 0$, $Q \neq 0$, are $n \times n$ matrices, then both the spectral radius $r(PQ)$ of PQ and the corresponding eigenvector are positive.*

Proof. The matrix Q must have at least one positive entry; thus, as $P > 0$, the matrix PQ has at least one positive column. It follows that $PQ\boldsymbol{x} > 0$, where $\boldsymbol{x}$ is any vector with positive components. From the inequality ([6], p. 47) $0 < \min_i (PQ\boldsymbol{x})_i / x_i \leq r(PQ)$ we have $r(PQ) > 0$. Let $\boldsymbol{y} \geq 0$ be the eigenvector of PQ corresponding to $r(PQ)$. The vector $Q\boldsymbol{y}$ must have at least one component greater than zero; otherwise, from the eigenvalue equation and the fact that $r(PQ) > 0$, we would have $\boldsymbol{y} = 0$. As $P > 0$, from the eigenvalue equation it follows $\boldsymbol{y} = PQ\boldsymbol{y}/r(PQ) > 0$.

Remark. The spectral radius of the matrix QP is again positive: $r(QP) = r(P^T Q^T) > 0$. However, the eigenvector of QP corresponding to $r(QP)$ may have some components equal to zero; in fact, the matrix QP may have only one row with positive entries.

Definitions:

(D0) We denote by w the largest, possibly infinite, number such that $A + vB$ is nonsingular and $Z(v, v) > 0$ in $[0, w)$.

(D1) We denote by u^* the largest, possibly infinite [1], number such that $A + uU$ is nonsingular and $Z(u, 0) > 0$ in $[0, u^*)$.

(D2) We denote by v^* the largest $(v^* = 1/r(A^{-1}V) < +\infty$, Lemma 1) number such that $A - vV$ is nonsingular and $Z(0, v) > 0$ in $[0, v^*)$.

(D3) Let v be a fixed value in $[0, v^*)$; then, $f(v)$ is the largest, possibly infinite [1], positive value of the parameter u such that $A + uU - vV$ is nonsingular and $Z(u, v) > 0$ for u in $[0, f(v))$.

(D4) Let u be a fixed value in $[0, u^*)$; then, $g(u)$ is the largest $(g(u) = 1/r[Z(u, 0)V] < +\infty$, Lemma 1) positive value of the parameter v such that $A + uU - vV$ is nonsingular and $Z(u, v) > 0$ for v in $[0, g(u))$.

Remarks. (i) When $f(v) < +\infty$, $f(v)$ may be computed by the algorithm described in [1]; $1/g(u) = r[Z(u, 0)V]$ may be computed by means of the power method or its variants [4] (p. 25). (ii) $0 < u^* = f(0)$, $0 < v^* = g(0) < +\infty$.

Lemma 2. *Assume* (1.2) *and the definitions* (1.3), (D1)-(D4). *Then*

(2.1) $Z(u, v) > 0 \quad Z_u(u, v) < 0 \quad Z_{uu}(u, v) > 0 \quad$ *for* $0 \leqslant v < v^* \quad 0 \leqslant u < f(v)$

(2.2) $Z(u, v) > 0 \quad Z_v(u, v) > 0 \quad Z_{vv}(u, v) > 0 \quad$ *for* $0 \leqslant u < u^* \quad 0 \leqslant v < g(u)$.

Proof. From the definitions (D1)-(D4) it follows that $Z(u, v) > 0$ for $[0 \leqslant v < v^*, 0 \leqslant u < f(v)]$ or $[0 \leqslant u < u^*, 0 \leqslant v < g(u)]$. By a direct computation of the first and second derivatives of the identity $(A + uU - vV)Z(u, v) = I$ with respect to u and v we obtain an expression for Z_u, Z_{uu}, Z_v and Z_{vv} depending on $Z(u, v)$, U and V. Taking into account (1.2) and the fact that $Z(u, v) > 0$, we obtain the inequalities (2.1) and (2.2).

Remarks. (i) For the matrices $Z(u, 0)$, $Z(0, v)$ and $Z(v, v)$ we have from (2.1) and (2.2):

(2.1)′ $A^{-1} \geqslant Z(u, 0) > 0 \quad Z_u(u, 0) < 0 \quad Z_{uu}(u, 0) > 0 \quad 0 \leqslant u < u^* = f(0)$

(2.2)′ $Z(0, v) > 0 \quad Z_v(0, v) > 0 \quad Z_{vv}(0, v) > 0 \quad 0 \leqslant v < v^* = g(0)$

and at least one entry of $Z(0, v)$ must become infinite as $v \to v^*$;

(2.3) $Z(0, v) \geqslant Z(v, v) \geqslant Z(v, 0) > 0, \qquad 0 \leqslant v < \min(u^*, v^*)$.

(ii) From (2.3) we have $w \geqslant \min(u^*, v^*)$. Let us assume that $\min(u^*, v^*) \leqslant w < \max(u^*, v^*) < +\infty$; then, from (2.1) and (2.2) it follows that w must be the smallest real positive fixed point of one of the two maps $f(v)$ and $g(u)$.

Lemma 3. *Under the assumptions of Lemma 2, it follows that:*
(a) *the function $f(v)$ is non decreasing with v in $[0, v^*)$;*
(b) *the function $g(u)$ is increasing with u in $[0, u^*)$.*

Proof. Part (a). Let $0 \leqslant v_1 < v_2 < v^*$; we have $Z(u, v_i) > 0$, $0 \leqslant u < f(v_i)$, $i = 1, 2$. Now we will show that $Z(u, v_2) \geqslant Z(u, v_1)$, $0 \leqslant u < f(v_1)$, from which it follows the thesis $f(v_2) \geqslant f(v_1)$. For $0 \leqslant u < f(v_1)$ we may write $A + uU - v_2 V = (A + uU - v_1 V)\,[I - (v_2 - v_1) Z(u, v_1) V]$; thus, it is sufficient to show that $(v_2 - v_1) r[Z(u, v_1) V] < 1$ ([6], p. 83). As $0 \leqslant v_1 < v^*$, from the properties (2.1) we have $Z(u, v_1) < Z(0, v_1)$; it follows that $r[Z(u, v_1) V] < r[Z(0, v_1) V] = r[(I - v_1 A^{-1} V)^{-1} A^{-1} V]$; and then

$$(v_2 - v_1) r[Z(u, v_1) V] < (v_2 - v_1) r(A^{-1} V)/[1 - v_1 r(A^{-1} V)] = (v_2 - v_1)/(v^* - v_1) < 1 .$$

Part (b) follows immediatly from (2.1)$'$ and Lemma 1.

Theorem 1. *Assume* (1.2) *and the definitions* (1.3), (D0)-(D4). *Then:*
(a) $w \geqslant \min(u^*, v^*)$.
(b) *The sequence $\{v_k\}$ defined by*

$$v_{k+1} = f(v_k) \qquad k = 0, 1, 2, \dots \qquad v_0 = 0 \tag{2.4}$$

is convergent monotonically to w if $u^* \leqslant w < v^*$. Otherwise, for some $k > 1$, $v_k \geqslant v^*$ and thus $u^* < v^* \leqslant w$.
(c) *The sequence $\{u_k\}$ defined by*

$$u_{k+1} = g(u_k) \qquad k = 0, 1, 2, \dots \qquad u_0 = 0 \tag{2.5}$$

is convergent monotonically to w if $v^* \leqslant w < u^* < +\infty$. Otherwise, for some $k > 1$, $u_k \geqslant u^*$ and thus $v^* < u^* \leqslant w$. When $u^* = +\infty$, it is convergent monotonically to w, if $w < +\infty$; otherwise it is divergent monotonically to $+\infty$.
(d) *When $u^* = v^*$, both the sequences* (2.4) *and* (2.5), *give $v_1 = u_1 = v^* = u^* \leqslant w$.*

Proof. As noted in remark (ii) to Lemma 2, Part (a) follows immediatly from the inequalities (2.3).

Part (b) implies $u^* < v^*$. As $f(v)$ is defined in $[0, \; v^*)$, the term v_{k+1} in (2.4) is defined if $0 \leqslant v_k < v^*$. For the first iteration we have $v_1 = f(v_0) = f(0) = u^* < v^*$. From remark (ii) to Lemma 2, w is the smallest real solution of the equation

$$v = f(v) \tag{2.6}$$

if $u^* \leqslant w < v^*$; in this case, w may be computed by the iterative process (2.4). As $f(v)$ is defined in $[0, \; v^*)$ and it does not decrease with v (Lemma 3), the process (2.4) must produce a monotone non decreasing sequence bounded from above by w and converging monotonically to w. If there are no real solutions of (2.6) in $[0, \; v^*)$, i.e. if $v < f(v)$, the process (2.4) must produce, for some $k > 1$, an iterate $v_k \geqslant v^*$ and then it is stopped; we have $w \geqslant v^*$.

Part (c) implies $v^* < u^*$. Let us first assume $u^* < +\infty$; as $g(u)$ is defined in $[0, \; u^*)$, the term u_{k+1} in (2.5) is defined if $0 \leqslant u_k < u^*$. For the first iteration we have $u_1 = g(u_0) = g(0) = v^* < u^*$. From remark (ii) to Lemma 2, w is the smallest real solution of the equation

$$u = g(u) \tag{2.7}$$

if $v^* \leqslant w < u^*$; in this case, w may be computed by the iterative process (2.5). As $g(u)$ is defined in $[0, \; u^*)$ and increases with u (Lemma 3), the process (2.5) produces a monotone increasing sequence bounded from above by w and converging monotonically to w. If there are no real solutions of (2.7) in $[0, \; u^*)$, i.e. if $u < g(u)$, the process (2.5) must produce, for some $k > 1$, an iterate $u_k \geqslant u^*$ and then it is stopped; we have $w \geqslant u^*$. When $u^* = +\infty$, $g(u)$ is defined in $[0, \; +\infty)$; thus, the sequence $\{u_k\}$ is convergent monotonically to w, if $w < +\infty$; otherwise, it is divergent monotonically to $+\infty$.

Part (d) follows immediately from parts (b) and (c).

Remarks. (i) Theorem 1 assures the computation of w when $\min(u^*, \; v^*) \leqslant w < \max(u^*, \; v^*)$, i.e. when w is sufficiently small. Otherwise, it provides a lower bound for w given by $\max(u^*, \; v^*)$.

(ii) The processes (2.4) and (2.5) are of the fixed point type and have a linear convergence ([5], p. 263). Given the approximations v_k and u_k, both the processes (2.4) and (2.5) require «internal» iterations to obtain the new approximations v_{k+1} and u_{k+1}. The evaluation of $v_{k+1} = f(v_k)$ involves [1] the computation of a sequence of inverses $Z(v_{k_i}, \; v_k)$, $i = 1, \; 2, \ldots, i_0(k)$, to perform the Newton steps for the equation $Z(v, \; v_k) = 0$, and each «internal» iteration i requires $0(n^3)$ operations; however, when $v_{k+1} < +\infty$, the convergence is quadratic [1] and the method is performed successfully for n equal to some tens (few iterations, $i_0(k) \sim$ 5-10, are necessary to obtain an accuracy of at least 12-14 digits, by using a precision of 16 decimal digits). The evaluation of $u_{k+1} = g(u_k)$ involves the computation of only one inverse, $Z(u_k, \ 0)$, and of the spectral radius of the matrix $Z(u_k, \ 0)V$, which is carried out by means of the power method; this iterative process is generally much faster than (2.4).

3 - The successive approximations

Let $m = \max(u^*, \ v^*)$. If $w < m$, Theorem 1 allows us to compute w. If $w > m$, then the matrix

$$A_m = A + mB \tag{3.1}$$

is nonsingular and $Z(m, \ m) > 0$. Therefore we can apply the results of Theorem 1 to the matrix $A_m + vB$ and compute a new approximation of w. We have $(A_m + uU - vV)^{-1} = Z(m+u, \ m+v)$.

Definitions:

(D5) We denote by f_m the largest, possibly infinite, number such that $A_m + uU$ is nonsingular and $Z(m+u, \ m) > 0$ in $[0, \ f_m)$.

(D6) We denote by g_m the largest ($g_m = 1/r[Z(m, \ m)V] < +\infty$, Lemma 1) number such that $A_m - vV$ is nonsingular and $Z(m, \ m+v) > 0$ in $[0, \ g_m)$.

Remark. $f_0 = f(0) = u^*$ and $g_0 = g(0) = v^*$.

Theorem 2. *Assume* (1.2) *and the definitions* (1.3), (3.1), (D0)-(D6). *Let the sequence* $\{m_j\}$ *be given by*

$$m_j = \max(u_j^*, \ v_j^*) \qquad j = -1, \ 0, \ 1, \ldots \qquad u_{-1}^* = v_{-1}^* = 0 \tag{3.2}$$

where

$$u_{j+1}^* = m_j + f_{m_j} \tag{3.3}$$

$$v_{j+1}^* = m_j + g_{m_j} . \tag{3.4}$$

Then, one and only one of the two following mutually exclusive statements is valid:

(a) *for some* $j \geq -1$

$$m_j < \min(u^*_{j+1},\ v^*_{j+1}) \leq w < m_{j+1}\,; \tag{3.5}$$

thus, w may be computed by one of the algorithms (2.4) *or* (2.5) *in Theorem* 1 *and the process* (3.2)-(3.4) *is stopped;*

(b) *the sequence* $\{m_j\}$ *is convergent monotonically to* w, *if* $w < +\infty$, *otherwise it is divergent to* $+\infty$.

Proof. For $j = -1$ we have $m_{-1} = 0$, and, from (3.3)-(3.4), $u^*_0 = u^*$ and $v^*_0 = v^*$; Theorem 1 states that w may be computed by one of the algorithms (2.4) or (2.5) if $m_{-1} = 0 < w < m_0$ otherwise, $w \geq m_0$. The matrix $A + vB$ may be written as $A + vB = A_{m_j} + (v - m_j)B$. For $j = 0,\ 1,\ 2, \ldots$ the situation is analogous to that for $j = -1$, where A_{m_j} and $v - m_j$; replace $A_{m-1} = A_0 = A$ and $v - m_{-1} = v$ respectively. It follows that w may be computed by one of the algorithms (2.4) or (2.5) in Theorem 1 if the inequality (3.5) is satisfied; in this case the sequence $\{m_j\}$ is stopped. Otherwise, $w \geq m_{j+1}$. Let us assume $w < +\infty$ and that the inequality (3.5) never be verified for $j \geq -1$; thus the sequence $\{m_j\}$, with $m_j < w$, converges monotonically and $\lim m_j = m_\infty = w$ as $j \to +\infty$. In fact, (3.2)-(3.4) imply that either $u^*_j \to m_\infty$ or $v^*_j \to m_\infty$; therefore, either $f_{m_j} \to 0$ or $g_{m_j} \to 0$; from (D5)-(D6) it follows that $m_\infty = w$. Finally, it is easy to verify that $m_j \to +\infty$ if $w = +\infty$.

Remark. The application of the process (3.2)-(3.4) needs three levels of successive approximations: (i) the steps of (3.2)-(3.4), (ii) those, at the same level, of (2.4) and/or (2.5) and (iii) those of the «internal» iterations performed at each step of (2.4) and/or (2.5) (see remark (ii) to Theorem 1). Each step of (3.2)-(3.4) requires some steps of the processes (2.4) and/or (2.5); generally, few steps (about 5-10) of (2.4) and/or (2.5) are necessary to gain a new step of (3.2)-(3.4) when convergence is not reached; in the last step of (3.2)-(3.4) very different numbers of iterations (from 10 to 100) of (2.4) and/or (2.5), depending on the special situations, illustrated in the numerical experiments, are necessary to obtain an accuracy of at least 6-8 decimal digits by using a precision of 16 decimal digits. However, we can state that few computational effort is necessary to obtain a good estimate, precisely a lower bound, of w when n is equal to some tens.

4 - Numerical results

Let us consider the matrix

$$A + uB = A + u(U - cV) \tag{4.1}$$

where $c \geq 0$ is a measure of the relative contributions of U and V to the matrix $B = U - cV$. The problem of the computation of $w = w(c)$, depending on c, is illustrated here. Equations (2.6) and (2.7) are now written as $u = f(cu)$, $u = g(u)/c$. We note that in this case $u^* = f(0)$ is independent of c, while $v^*(c) = g(0)/c$ and $w(c)$ depend on c. For c sufficiently small, the matrix V supports the positivity of $Z(u, \ cu)$, in the sense that $w(c)$, $u^* \leq w(c) \leq v^*(c)$, does not decrease with c; $w(c)$ may be computed by the algorithm (2.4). As c increases, the contribution due to the matrix cV, to the perturbation B of A, becomes more weighty than U. First we have $u^* \leq v^*(c) < w(c)$ and then $v^*(c) < u^* < w(c)$; in these situations $w(c)$ cannot be computed by one of the algorithms (2.4) or (2.5). Eventually, we have $v^*(c) < w(c) < u^*$ and $w(c)$ is decreasing with c; $w(c)$ can now be computed by the algorithms (2.5).

As we have seen by means of qualitatively arguments, $w(c)$ is non decreasing when c is sufficiently small, $0 \leq c < c_0$, and it is decreasing when c is sufficiently large. We now show that the matrix (4.1) becomes singular for $c = c_0$ and $u = w(c_0)$. The matrix $Z(u, \ cu)$ increases with c when $0 \leq u < w(c)$; in fact, from the identity $Z(u, \ cu)[A + u(U - cV)] = I$, we obtain $Z_c(u, \ cu) = uZ(u, \ cu)VZ(u, \ cu)$; thus, $Z(u, \ cu) > 0$ and $Z_c(u, \ cu) > 0$. It follows that $w(c)$ does not decrease with c as, for some $c_0 > 0$, $\det[A + w(c)(U - cV)] \to 0$ for $c \to c_0$.

In the numerical experiments we used the matrix $A + vB$ given by

$$A + vb = \frac{1}{h^2} \left| \begin{array}{ccccc} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ \cdot & \cdot & \cdot & & \\ & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & \cdot \\ & & -1 & 2 & -1 \\ 1 & \ldots & \ldots & \ldots & n \end{array} \right| + vh \left| \begin{array}{c} \ldots\ldots \\ \ldots\ldots \\ \ldots\ldots \\ \ldots\ldots \\ \\ k_1 \ldots k_2 \end{array} \right| \begin{array}{c} i_1 \\ \\ \\ i_2 \\ \\ \\ \end{array} \tag{4.2}$$

where $h = 1/(n+1)$, A is a Stieltjes matrix ($-A$ is a discrete analog of the second derivative together with zero boundary conditions), $B = U - cV$ is expressed as in (4.1) and its entries are different from zero only for $i_1 \leqslant i \leqslant i_2$ and $k_1 \leqslant k \leqslant k_2$ (B can be considered as the contribution due to an integral operator); the non zero entries of U and V are random values, assumed uniformly distributed in [0, 1]. The inverse of a matrix and the random numbers have been computed by the IMSL Library subroutines LINV2F and GGUBS respectively; the spectral radius of a matrix by means of the power method. The computations have been performed in double precision (16 decimal digits). Now we describe two sets of numerical experiments.

(NE1) In this set of experiments the matrices U and V are mantained fixed with $n+1=40$, $i_1=k_1=10$, $i_2=k_2=30$; the evaluation of $v^*(c)$ and $w(c)$ has been performed for various values of c. The results, for c in the interval [0, 100], together with the type of the process used in the computations and the number of iterations, are shown in Table I. The trends of u^*, $v^*(c)$ and $w(c)$ versus c are shown in Figure 1. Let c_0 ($c_0 \sim 1.299$) be the value of c for which $w(c_0)=$ maximum. For $0 \leqslant c < c_0$, $w(c)$ is increasing with c and $\det[A+w(c)(U-cV)] \neq 0$; for $c \geqslant c_0$, $w(c)$ is decreasing with c and $\det[A+w(c)(U-cV)]=0$. In Table I only a lower bound for $w(c)$ is reported for $c=1.299$ and $c=1.3$ because the algorithm (3.2)-(3.4) requires many steps of the processes (2.4) and/or (2.5) when c is near c_0. As $c \to c_0$, the number of iterations of the last step of (3.2)-(3.4), to obtain an accuracy of 6-8 decimal digits, increases considerably; however, sufficiently good lower bounds of $w(c)$ may be estimated with less computational effort; for example, the last step of (3.2)-(3.4) gives the following iterates:

$$\begin{array}{llll} \text{for } c=1.25 & u_1=40.1249 & u_{35}=40.5014 & u_{70}=40.5221; \\ \text{for } c=1.50 & u_1=8.1023 & u_{35}=9.5529 & u_{70}=9.5560. \end{array}$$

(NE2) In this set of experiments the random entries of U and V, with fixed $n+1=50$, are generated differently for each case. The non-zero share of the perturbation B goes from a small part to the whole matrix. Two values of c ($c=0.5$ and $c=2$) have been chosen so that $0.5 < c_0 < 2$ (see NE1); in this case few steps ($\leqslant 3$, $j=-1,\ 0,\ 1$) of the process (3.2)-(3.4) are required. The final values of u_j^* and v_j^* of the process (3.2)-(3.4) and those of w ($\min(u_j^*,\ v_j^*) < w < \max(u_j^*,\ v_j^*)$) are given in Table II; these results show the monotonic trend of u_j^*, v_j^* and w versus the size of the non-zero share of the perturbation B.

c	$v^*(c)$	$w(c)$	Number of iterations and iterative process
0.	$+\infty$	$u^* = w = 3.5623$	1(f)
0.1	24.2680	$u^* < 3.8427 < v^*$	6(f)
0.2	12.1340	$u^* < 4.1710 < v^*$	9(f)
0.5	4.8536	$u^* < v^* < 5.6090$	2(m) [1(f) − 9(f)]
1.	2.4268	$v^* < u^* < 13.2676$	4(m) [1(g) − 1(f) − 3(f) − 34(f)]
1.25	1.9414	$v^* < u^* < 40.5230$	16(m) [1(g)...1(f)...7(f) − 85(f)]
1.299	1.8682	$v^* < u^* < 82.05 < w$	−(m)
1.3	1.8668	$v^* < u^* < 78.03 < w$	−(m)
1.5	1.6179	$v^* < u^* < 9.5560$	3(m) [2(g) − 5(g) − 77(g)]
1.9235	1.2617	$v^* < u^* = w = 3.5623$	35(g)
2.	1.2134	$v^* < 3.2017 < u^*$	34(g)
5.	0.4854	$v^* < 0.6455 < u^*$	11(g)
10.	0.2427	$v^* < 0.2770 < u^*$	7(g)
100.	0.0243	$v^* < 0.0246 < u^*$	3(g)

TABLE I

$v^*(c)$ and $w(c)$ for various values of c and the same matrices U and V with $n+1=40$, $i_1 = k_1 = 10$, $i_2 = k_2 = 30$. In the last column, the expression $i(f)$, $i(g)$ and $i(m)$ mean i iterations of the processes (2.4), (2.5) and (3.2)-(3.4) respectively.

c	$(i_1,\ i_2)-(k_1,\ k_2)$	j	u_j^*	v_j^*	w
0.5	(1,49)-(1,49)	0	0.5151	1.8497	1.0924
0.5	(10,40)-(10,40)	1	3.2111	8.4908	3.5128
0.5	(20,30)-(20,30)	1	55.6327	111.0136	66.7242
0.5	(1,25)-(1,25)	1	6.7914	16.3626	7.7851
0.5	(1,10)-(1,10)	1	81.3694	211.7408	93.8788
2.0	(1,49)-(1,49)	1	1.5340	0.7665	1.0558
2.0	(10,40)-(10,40)	0	1.8845	0.7279	1.5291
2.0	(20,30)-(20,30)	0	36.4945	4.1358	8.1412
2.0	(1,25)-(1,25)	1	7.5849	3.2565	3.4672
2.0	(1,10)-(1,10)	0	53.1624	14.8762	30.7650

TABLE II

u^*, v^* and w for various matrices U and V, with $n+1=50$, and various values of $(i_1,\ i_2)-(k_1,\ k_2)$; $c = 0.5,\ 2$.

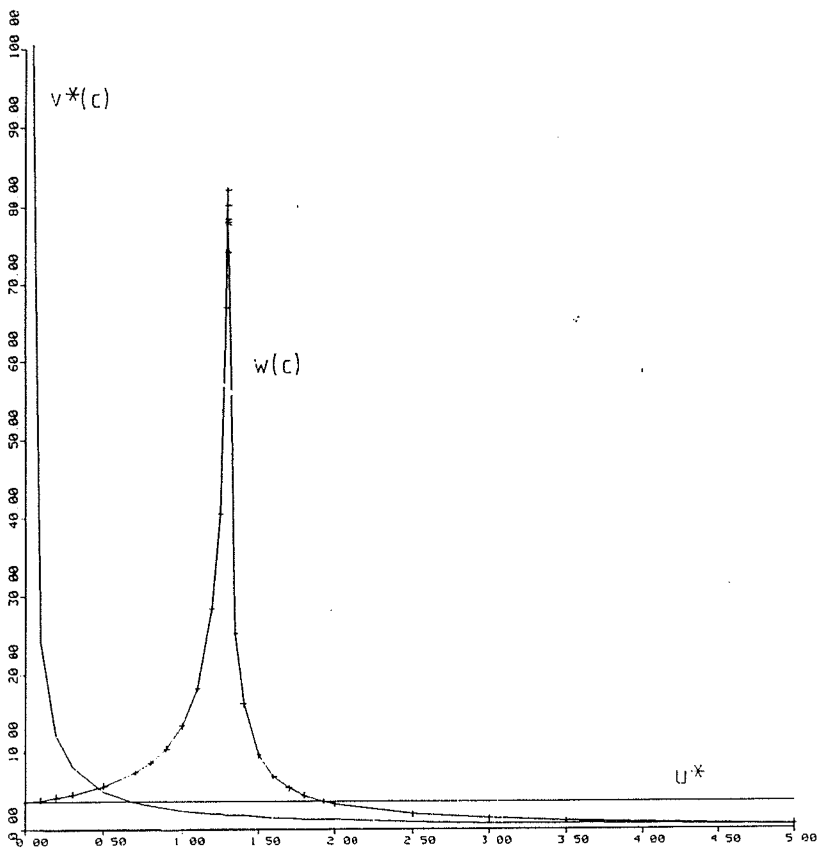

Fig. I

u^*, $v^*(c)$ and $w(c)$ versus c for the matrix $A + u(U - cV)$ with $n+1 = 40$, $i_1 = k_1 = 10$, $i_2 = k_2 = 30$.

References

[1] G. Buffoni, *Nonnegative and skew-symmetric perturbations of a matrix with positive inverse*, Math. Comp., 54 (1990), 189-194.

[2] G. Buffoni and A. Galati, *Matrici essenzialmente positive con inversa positiva*, Boll. Un. Mat. Ital., (4) **10** (1974), 98-103.

[3] K. Fan, *Topological proofs for certain theorems on matrices with non-negative elements*, Monatsh. Math., 62 (1958), 219-237.

[4] G. I. Marcuk, *Metodi del calcolo numerico*, Editori Riuniti, Roma, 1984.

[5] J. R. Rice, *Numerical methods, software and analysis*, Mc Graw-Hill, 1983.

[6] R. S. Varga, *Matrix iterative analysis*, Prentice-Hall, Englewood Cliffs, N.J., 1965.

Sommario

Sia A una matrice reale non singolare con inversa positiva e B una generica matrice reale non nulla. L'inversa di $A + vB$ *sia positiva per* $0 \leqslant v < w < +\infty$ *e almeno un suo elemento sia uguale a zero per* $v = w$ *oppure* $\det(A + wB) = 0$*; in questo lavoro viene descritto un algoritmo per calcolare w e sono presentati alcuni risultati numerici.*

***