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A note on stock price volatility (**)

1 - Introduction

The excess of variance of empirical stock prices, when compared to the variance of the (particular) solution

$$P_t = P_t^F \equiv \sum_1^{\infty}{}_k \frac{E_t D_{t+k}}{(1+r)^k} \tag{1}$$

of the non-homogeneous market equation (with constant return)

$$P_t = E_t(D_{t+1} + P_{t+1})/(1+r), \tag{2}$$

is usually referred to as *volatility* of stock prices, a phenomenon which calls into question market efficiency, that is eq. (1). [E.g., West [8], Braglia [1], and references therein.] Here and below, the superscript «F» is used to indicate that the expression of P_t is given in terms of *fundamentals* of the market. Other notations such as D_t for the real dividend on the stock, r for the real expected return and $E_t(\cdot) \equiv E(\cdot|\Omega_t)$ for the mathematical expectation conditional on the market's period-t information set Ω_t are standard.

Since the first tests by Shiller [6] and LeRoy and Porter [5] in 1981, various attempts have been made to give stock price volatility a satisfactory explanation. However, the problem still remains open. In practice, even if not always explicitly noticed, all these attempts have quite simply (and sometimes arbi-

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trarely) given eq. (1) and extended *perturbed* form, say

$$P_t = P_t^F + H_t \,. \tag{3}$$

As regards the perturbation H_t, market efficiency would require that P_t remains a solution of eq. (2). On the other hand, this request can be satisfied when assuming that [e.g., [4]]

$$H_t = A_t(1+r)^t \qquad \text{with} \qquad E_t A_{t+1} = A_t \tag{4$_a$}$$

or, even if approximately, [e.g., [1]$_1$, [6] and [8]]

$$H_t = E_t \sum_{0}^{\infty}{}_j \frac{D(r_{t+j} - r)}{r(1+r)^{j+1}} \,. \tag{4$_b$}$$

In fact, equation (4a) is the general solution of the homogeneous part of eq. (2) while (4b) follows from a linearization of the expression P_t^F assumes for t-dependent returns. As regards D and r, they play the role of (constant) *mean values* one introduces for dividends and returns [e.g., Braglia [1]$_1$]. However, difficulties of various kinds from both the mathematical and the economic points of view, have led a number of authors to abandon the idea of market efficiency and «invent» new forms of H_t (i.e., P_t) which are not solutions of eq. (2). But, as expected, even this approach is not found free from troubles and criticism. In fact, the same existence of H_t is no longer guaranteed and it becomes difficult both the specification of its form and the assessment of its real practical meaning [e.g., West [8]]. The procedure remains artificial in the absence of a solid theoretical basis from which to deduce H_t. In this paper, a stochastic approach which seems much more familiar in various branches of physics and mathematics (e.g., diffusion theory) than in this particular field of the economic research, will be used to show how a *natural* alternative equation for the perturbation H_t can be obtained.

2 - Alternative approach to the volatility problem

2.1 - Our technique is substantially different from those we have just mentioned. *We have not recourse «a priori» to positions of form* (3). We want to see if one can reveal in a *quite natural way* contributions to the variance of P_t besides that due to P_t^F.

Basically, we found our analysis on the consequences of two empirical facts which in our opinion tend to be underestimated:

(1) the distribution of prices $P(x, t)$ is *not stationary*: it shows a trend (i.e., a drift) [e.g., [6]] and has a variance which is a complicated function of time [e.g., [7]];

(2) the distribution $P(x, t)$ seems to have *non-gaussian* form [e.g., [3] and [2]].

In fact, it can easily be seen that these two observations (when considered *jointly*), indicate that the coefficients of the Fokker-Planck (diffusion) equation for the price distribution function $P(x, t)$ will generally depend on *both x and t*. As mentioned, this conclusion seems to be underestimated, if not quite neglected, by the economic theory relevant to the phenomenon of volatility. However, it leads *automatically* to reveal a contribution to the stock price variance σ_t^2, deriving from the *correlation between trend and prices*, with possible significant consequences on values and temporal law of σ_t^2. In principle, the calculation of such a contribution can be made in a number of different ways. We believe that the following technique has some theoretical advantages and permits a better comprehension of the interpretative model than other more practical approaches.

Consider the price behaviour as a function of time (say, a representation (x, t)). Let $P(s, x, t)\,\mathrm{d}s\,\mathrm{d}x$ be the probability that, at time t, the price will assume a value in $(x, x+\mathrm{d}x)$ with a *slope* (i.e., a variation-velocity) in $(s, s+\mathrm{d}s)$. If, as expected, correlation exists between s and x, instead of accepting eq. (3) we are tempted to expand $P(s, x, t)$ and give it the form

$$P(s, x, t) = \sum_{1}^{\infty}{}_j Q_j(s, t)(-\partial/\partial x)^{j-1} P(x, t) \approx P\{Q_1 - \frac{Q_2}{P}\frac{\partial P}{\partial x}\} \tag{5}$$

which corrects the factorization $P(s, x, t) = P(x, t)\,Q_1(s, t)$ we should assume in the absence of correlation. [Note the substantial difference with the conventional position (3)!] In this equation, Q_1 is the probability density function relevant to the random variable s_t and Q_2 an appropriate function of s and t satisfying the (normalization) condition $\int Q_2(s, t)\,\mathrm{d}s = 0$. In fact, as a *quite natural* consequence, expansion (5) yields the following contribution to the diffusion coefficient $\frac{1}{2}\mathrm{d}\sigma_t^2/\mathrm{d}t$

$$A(t) = \int_{-\infty}^{+\infty} s Q_2(s, t)\,\mathrm{d}s \tag{6}$$

which, in principle, can justify at least in part phenomena such as stock price volatility and temporal behaviour of σ_t^2. Notice that eq. (6) follows from the observation that the «current» in the x-space

$$J(x,\ t) = P(x,\ t)\,W(x,\ t) \equiv P(x,\ t)\Big(\int sQ_1\,\mathrm{d}s - \frac{A(t)}{P}\,\frac{\partial P}{\partial x}\Big) \tag{7}$$

points out a diffusion contribution which is just characterized by the diffusion coefficient $A(t)$. In other words, but quite equivalently, we can say that, because of *external forces* (or, as usually referred to, «fads» effects), we are considering incorrect the description of the temporal behaviour of the price distribution $P(x,\ t)$ as given by a simple Fokker-Planck equation of the form

$$\frac{\partial P(x,\ t)}{\partial t} = -W_0(t)\,\frac{\partial P}{\partial x} + D_0(t)\,\frac{\partial^2 P}{\partial x^2} \tag{8}$$

even if the (averaged unperturbed) coefficients [**1**]$_1$

$$W_0 = W_0(t) \equiv \langle W(x,\ t)\rangle = \frac{\mathrm{d}\langle x\rangle}{\mathrm{d}t} \tag{9$_a$}$$

$$D_0 = D_0(t) \equiv \langle D(x,\ t)\rangle = \frac{1}{2}\,\frac{\mathrm{d}\sigma_t^2}{\mathrm{d}t} \tag{9$_b$}$$

are treated as t-dependent. Thus, *even the usual assumption that the temporal series relative to the r.v. x_t can be represented by the autoregressive process $x_t = \lambda x_{t-1} + v_t$, with $v_t \sim N(0,\ \sigma_v^2)$ and constant parameters, seems a rather severe practical restriction.* [The relation between D_0 and λ can be found in Braglia [**1**]$_1$]. In fact, in the presence of the «external (trading) forces» which may be reasonably assumed to act in practice, the drift term W is expected to become x-dependent and have important consequences on the diffusion coefficient, i.e., σ_t^2. To be a bit more concrete assume, for instance, that W can be represented by the linear form

$$W(x,\ t) = W_0(t) + (A(t)/\sigma_t^2)(x - \langle x\rangle)\,. \tag{10}$$

The general Fokker-Planck equation for $P(x,\ t)$

$$\frac{\partial P(x,\ t)}{\partial t} = \frac{\partial}{\partial x}\Big\{W(x,\ t)\,P(x,\ t) + \frac{1}{2}\,\frac{\partial}{\partial x}\,[2D(x,\ t)\,P(x,\ t)]\Big\} \tag{11}$$

can still be given the form of an equation with coefficients depending on time only. But, as known and already pointed out in [**1**]$_1$, the new equation will have an infinite number of terms and not just two. If we write that

$$\frac{\partial P(x,\ t)}{\partial t} = -\beta_1(t)\,\frac{\partial P}{\partial x} + \frac{1}{2}\beta_2(t)\frac{\partial^2 P}{\partial x^2} + \dots \tag{12}$$

then, in particular, it is found that

$$\beta_1(t) = \langle W(x,\ t)\rangle = W_0(t) \tag{13}$$

which agrees with $(9)_a$, while

$$\begin{aligned} \beta_2(t) &= 2\langle (x - \langle x\rangle)\, W(x,\ t)\rangle + 2D(t) \\ &\approx 2(A(t)/\sigma_t^2)\langle (x - \langle x\rangle)^2\rangle + 2D_0(t) = 2\{D_0(t) + A(t)\} \end{aligned} \tag{14}$$

which disagrees with (9b) and shows how the diffusive process is changed by the dependence of the trend-term on price. Notice that in eq. (14) the x-dependence of D is really still neglected and it is assumed that $D(x,\ t) \approx D_0(t) = \langle D(x,\ t)\rangle$ as in eq. $(9)_b$. The only difference is that now $D_0(t)$ is no longer equal to $\frac{1}{2}\mathrm{d}\sigma_t^2/\mathrm{d}t$. Moreover, notice that eq. (12) will generally have a non-gaussian fundamental solution.

2.2 - As probably expected, under appropriate conditions this result also agrees with eq. (5). In fact, suppose that $P(x,\ t)$ maintains its gaussian form, which agrees with a *two-term truncation* of eq. (12). Then, from (5), it follows that

$$\begin{aligned} P(s,\ x,\ t) &\approx P(x,\ t)\{Q_1(s,\ t) + Q_2(s,\ t)\frac{(x - \langle x\rangle)}{\sigma_t^2}\} \\ &\equiv P(x,\ t)\, Q(s,\ x,\ t)\,. \end{aligned} \tag{15}$$

In accord with (10) and (13), this equation yields

$$W(x,\ t) = \int_{-\infty}^{+\infty} Q(s,\ x,\ t)\, s\, \mathrm{d}s = W_0(t) + (A(t)/\sigma_t^2)(x - \langle x\rangle)\,. \tag{16}$$

Thus, position (10) is consistent with the assumed preservation of gaussian form as usually done in the economic literature, even if the gaussian distribution will not be given by eq. (8), as eq. (14) also reveals.

As easily seen, these results are not simply and directly derived by autoregressions of the form $x_t = \lambda x_{t-1} + v_t$, so widley used in practice. For $v_t \sim N(0,\ \sigma_v^2)$ and constant parameters this equation has the same fundamental and stationary solutions of a special equation of form (8) which preserves the gaussian form of $P(x, t)$. Thus, it appears to be unable to represent the real behaviour of prices [cf. Braglia $[1]_2$]. On the other hand, dependence of trend on x and non-gaussian form of $P(x, t)$ complicate matter as one is forced to assume (proper) dependences on x and/or t of λ and σ_v^2 and possibly non-gaussian forms of v_t with mean different from zero [e.g., Braglia $[1]_1$]. The procedure remains artificial and much less accurate and straightforward than that adopted in this paper. Really, we believe that from a rigorous point of view *the autoregression should be replaced by the iterative form of the integral formal solution of the diffusion (master) equation, a problem which does not seems to have been faced yet* in the economic literature.

3 - Conclusive remarks

We have illustrated here *one* of various possible approaches to the calculation of σ_t^2. Other equivalent techniques (possibly of even more practical interest, when considering the sort of data one is usually concerned with), which are based on the *self-correlation function* relevant to the r.v. s_t are likewise able to put in evidence very important factors of perturbation such as, for instance, the *persistence of «velocity»* that most probably must be assumed at any variation of prices. In fact, as expected, this *persistence will increase the correlation time* and, hence, the variance of prices. In principle, *persistence processes could even explain (explosive) behaviours which can appear in certain periods of time,* or be so important to make the stochastic process divergent. But this is a problem we are planning to consider in details in a successive paper where we shall turn our attention to possible techniques which permit to calculate the contribution $A(t)$ in real situations of special interest.

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Summary

A new model of stock price volatility is discussed which avoids difficulties of other approaches. It is shown how the phenomenon can find a quite natural (possibly partial) interpretation in the correlation between trend and price. The contribution to volatility of the mentioned correlation can be calculated with various techniques one of which is treated here in detail.

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