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Wiener points and energy decay for a relaxed Dirichlet problem relative to a degenerate elliptic operator (**)

1 - Introduction and notations

1.1 — Aim of the work is to characterize the regular points of the weak solution of the problem

$$
(1.1) \qquad \begin{cases} \mathrm{L}u + \mu u = \nu & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}
$$

where L is a degenerate elliptic operator

$$\mathrm{L}u = -\mathrm{D}_i(a_{ij}(x)\,\mathrm{D}_j u)$$

such that

$$\lambda w(x)|\xi|^2 \leqslant a_{ij}(x)\,\xi_i\,\xi_j \leqslant \Lambda w(x)|\xi|^2$$

$w(x)$ being a nonnegative weight in A_2 Muckenhoupt's class, whereas μ is a Borel measure and ν is a Radon measure. (To get a more precise definition, see, for example, [1] or [7]). In the following $H^p(\Omega, w)$ and $H_0^p(\Omega, w)$ are the usual weighted spaces H^p and H_0^p, while $L^2(\Omega, \mu)$ is

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the space of (equivalence classes) of square integrable functions with regards to μ measure.

We will consider weak solutions of problem (1.1), that is functions $u \in H^1(\Omega,\ w) \cap L^2(\Omega,\ \mu)$ such that

$$\int_\Omega a_{ij} D_j u D_i v \, dx + \int_\Omega uv \, d\mu = \langle v,\ \nu\rangle \qquad \forall v \in H^1(\Omega,\ w) \cap L^2(\Omega,\ \mu).$$

Usually, to shorten the notation, we also denote

$$\int_\Omega a_{ij} D_j u \, D_i v \, dx = a_\Omega(u,\ v).$$

By $B(r,\ x_0)$ we mean the set of points x such that $|x - x_0| < r$ while $\sum = |x| < R_0$ and such as to satisfy $\Omega \subset \sum$. (As a matter of fact $\sum$ is a ball, big enough as to contain the whole Ω.)

1.2 — By $G^{x_0}_{B(R,\,x_0)}$ we mean the Green function $G(x,\ x_0)$ for the Dirichlet problem relative to L in $B(r,\ x_0)$ with singularity in x_0, which is defined as the solution of

$$a_{B(R,\,x_0)}(\varphi,\ G^{x_0}_{B(R,\,x_0)}) = \varphi(x_0) \qquad \forall \varphi \in C_0^\infty(B(R,\ x_0)).$$

Many regularity properties for this function are known.

In particular, let us consider the capacity notion, relative to the operator L, of a set E in Ω, $E \subset \Omega$.

It is defined by

$$\operatorname{cap}(E,\ \Omega) = \inf\{a_\Omega(v,\ v),\ v \in H_0^1(\Omega,\ w),\ v \geqslant 1 \text{ on } E\}.$$

Equivalence relations for the Green-function have then been proved by **[6]**: they have shown, for example, under certain conditions, that

$$\frac{C}{\operatorname{cap}(B(R,\ y),\ B(R_0,\ y))} < G^y_{B(R_0,\,y)} < \frac{C}{\operatorname{cap}(B(r,\ y),\ B(R_0, y))}$$

where C is a constant independent from y, R_0, r, R.

As we work with operators L, which are symmetric, we will also use the so-called *μ-capacity* of E in Ω, defined by

$$\operatorname{cap}_\mu(E,\ \Omega) = \inf\{a_\Omega(v,\ v) + \int_\Omega |v|^2 \, d\mu_E,\ v - 1 \in H_0^1(\Omega,\ w) \cap L^2(\Omega,\ \mu_E)\}$$

for all the sets E which are admissible for the measure μ.

Again, refer to [5] to get a complete definition of admissability and of Borel measure μ_E, restriction of μ.

See, also, [3] for other relations.

1.3 — The regularized Green function $G^{x_0}_{B(R,x_0),\rho}$ is defined as the unique

$$a_{B(R,x_0)}(\varphi,\ G^{x_0}_{B,\rho}) = \frac{1}{w(B(\rho,\ x_0))} \int_{B(\rho,x_0)} \varphi(x)\, w(x)\, \mathrm{d}x \qquad \forall \varphi \in C^\infty_0(B(R,\ x_0)).$$

When $\rho \to 0$, $G^{x_0}_{B,\rho} \to G^{x_0}_B$ weakly in $H^1(B - \overline{B}(r,\ y),\ w)$ and in $H^{1,p}(B_R,\ w)$ $(1 \leqslant p \leqslant 2N/(N-1))$ and uniformly in any compact $K \subset B_R - \{x_0\}$.

1.4 — By $K_n(\Omega)$ we define the space of Radon measures ν such that

$$\lim_{r\to 0} \sup_{x\in\Omega} \int_{\Omega\cap B(r,y)} G^y_\Sigma(x)\, \mathrm{d}\,|\nu| y = 0 \tag{1.2}$$

while $K_{n,\mathrm{loc}}(\Omega)$ is the space of all Radon measures ν such that $\nu \in K_n(\Omega')$, with $\Omega' \subset \Omega$ being Ω' a domain.

We introduce a norm on $K_n(\Omega)$ defined by

$$\|\nu\|_{K_n(\Omega)} = \sup_{x\in\Omega} \int_\Omega G^{x_0}_\Sigma(x)\, \mathrm{d}\,|\nu|\,. \tag{1.3}$$

We can show that $K_n(\Omega)$ is a Banach space with the norm defined by (1.3).

1.5 — Coming finally to our problem, we call *regular* all the points x of Ω, where $u(x)$ is continuous and vanishes.

A criterion will be given, which characterizes such points by a Wiener test, according to a technique developped by Wiener and recently adapted by [6], [7] and [3],among others, to degenerate elliptic operators.

As a matter of fact, if we consider the ball $B(r,\ x_0)$ and we define

$$V(r) = \sup_{B(r,\ x_0)} |u|^2 + \int_{B(r,x_0)} |\mathrm{D}u|^2 G^{x_0}_{B(2q^{-1}r,x_0)} w\, \mathrm{d}x + \int_{B(r,x_0)} |u|^2 G^{x_0}_{B(2q^{-1}r,x_0)} \mathrm{d}\mu$$

the estimate of Wiener modulus ω_μ will be given thanks to a structural estimate of the ratio $V(r)/V(R_0)$ on two homothetic balls with rays r and R_0.

We come up to

$$V(r) = K_1\, \omega^\beta_\mu\, V(R) + K_2 \|\nu\|^2_{K_n(\Omega)}$$

where ω_μ is the Wiener modulus and K_1, K_2 and β are constants which do not depend upon u.

1.6 — Let us point out that dealing with degenerate elliptic operators it is not possible to use the classical Kellog argument on the equivalence of balls and anuli in the Wiener test, due to the presence of points with positive capacity. Therefore, in a certain sense, it has been necessary to invert the procedure normally followed and operate first with the so-called *hole-filling* and then with the Poincaré inequality.

2 - Preliminaries

We prove the following lemma (*Poincaré inequality*).

Lemma 1. *Let us consider $v \in H_0^1(B(r,\ z),\ w)$. We have then*

$$\int_{B(\frac{R}{2},z)} |v|^2 w \,\mathrm{d}x \tag{2.1}$$

$$\leqslant C_q \frac{w(B(\frac{R}{2},\ z))}{\mathrm{cap}_\mu(B(\frac{R}{2},\ z),\ B(R,\ z))} \Big[\int_{B(R,z)} |\mathrm{D}v|^2 w \,\mathrm{d}x + \int_{B(R,z)} v^2 \,\mathrm{d}\mu \Big].$$

To obtain (2.1), we work as in [2], considering the definition previously given for $\mathrm{cap}_\mu(E,\ \Omega)$ and using as test-function

$$\psi = 1 + \frac{v - \bar{v}_q}{\bar{v}_q} \eta .$$

We will need (2.1) in proving Theorem 1. We also have

Lemma 2. *We choose φ as the capacitary potential of $B(sR,\ z)$ in $B(tR,\ z)$ with respect to* L *and we prove the following inequality*

$$\Big| \int_{B(tR,z)} u \varphi G_\rho^z \,\mathrm{d}\nu \Big| \leqslant C \|\nu\|_{K_n(B(tR,z))} \sup_{B(tR,z)} |u| \tag{2.2}$$

G_ρ^z being the regularized Green function and ν a Kato measure, that is a Radon measure such as those considered in **1.4**.

The proof is like the one given in [4].

Lemma 3. *Let us consider u, weak solution of* (1.1). *We obtain*

$$\int_{B(sR,z)} |Du|^2 G^z_{B(\sigma tR,z)} w \, dz + |u|^2 + \int_{B(sR,z)} u^2 G^z_{B(\sigma tR,z)} \, d\mu \tag{2.3}$$

$$\leqslant \frac{C_1 (t-s)^{-2}}{w(B(tR,\ z))} \int_{B(tR,z)-B(sR,z)} |u|^2 w \, dx + C_2 \|\nu\|_{K_n(B(tR,z))} \sup_{B(tR,z)} |u|$$

q.e., where $B(tR,\ z) \subset B(R_0,\ z)$ *and* $s < t$, $\sigma \geqslant 3/2$.

Once again, we reason as in [2], taking care of the terms with μ and ν.

3 - Results

Let us consider

$$\delta(r) = \frac{\mathrm{cap}\,\mu(B(r,\ x_0),\ B(2r,\ x_0))}{\mathrm{cap}(B(r,\ x_0),\ B(2r,\ x_0))} \tag{3.1}$$

$$\omega_\mu(x_0,\ r,\ R) = \exp\left[-\int_r^R \delta(\rho)\frac{d\rho}{\rho}\right]. \tag{3.2}$$

The function $\omega_\mu = \omega_\mu(x_0,\ r,\ R)$ is called *Wiener modulus of the measure in* x_0. It is easily proved that

$$0 \leqslant \delta(\rho) \leqslant 1 \qquad \frac{r}{R} \leqslant \omega_\mu \leqslant 1.$$

We define the Wiener points as those x_0 such that

$$\lim_{r\to 0^+} \omega_\mu(x_0,\ r,\ R) = 0 \tag{3.3}$$

which is equivalent to

$$\int_0^R \delta(\rho)\frac{d\rho}{\rho} = +\infty. \tag{3.4}$$

We then obtain

Theorem 1. *There exist two constants* $K > 0$ *and* $\beta > 0$, *that depend only on the dimensions of the space we are considering, on the ellipticity constants*

λ, Λ and on the weight $w(x)$ such that

$$V(r) \leqslant K\omega_\mu(x_0,\ r,\ R)^\beta V(R) + K\|\nu\|^2_{K_n(B(R,x_0))}\,. \tag{3.5}$$

Finally, we come to the main result of the work.

Theorem 2. *If x_0 is a Wiener point of the μ measure, then*

$$\lim_{r\to 0^+} V(r) = \lim_{x\to x_0} u(x) = u(x_0) = 0 \tag{3.6}$$

that is, x_0 is regular.

4 - Proofs

4.1 — Theorem 1. Let us assume as test-function in the weak form of (1.1) $v = uG^z_\rho\varphi$ being:

G^z_ρ the regularized Green function relative to $G^z = G^z_{B(tR,\,z)}$;
φ the potential of $B(sR,\ z)$ in $B(tR,\ z)$ with respect to L.

Let us consider then, what our problem looks like. We have

$$\int_\Omega a_{ij}\mathrm{D}_j u \mathrm{D}_i(u\varphi G^z_\rho)\,\mathrm{d}x + \int_\Omega u^2\varphi G^z_\rho\,\mathrm{d}\mu = \langle \nu,\ uG^z_\rho\varphi\rangle\,.$$

We easily obtain

$$\int_{B(tR,\,z)} a_{ij}\mathrm{D}_j u\,\mathrm{D}_i u\varphi G^z_\rho + \frac{1}{2}\,\frac{1}{w(B(\rho,\ z))}\int_{B(\rho,\,z)} u^2 w\,\mathrm{d}x + \int_{B(tR,\,z)} u^2\varphi G^z_\rho\,\mathrm{d}\mu$$

$$\leqslant \frac{1}{2}\int_{B(tR,\,z)} a_{ij}\mathrm{D}_j\varphi\,\mathrm{D}_i G^z_\rho u^2\,\mathrm{d}x - \int_{B(tR,\,z)} a_{ij}\mathrm{D}_j u\,\mathrm{D}_i\varphi u G^z_\rho\,\mathrm{d}x + \int_{B(tR,\,z)} uG^z_\rho\varphi\,\mathrm{d}\nu$$

$$\leqslant -2\int_{B(tR,\,z)} a_{ij}\mathrm{D}_j u\,\mathrm{D}_i\varphi u G^z_\rho\,\mathrm{d}x + \int_{B(tR,\,z)} uG^z_\rho\varphi\,\mathrm{d}\nu + \frac{1}{2}\int_{B(tR,\,z)} a_{ij}\mathrm{D}_j\varphi\,\mathrm{D}_i(u^2G^z_\rho)\,\mathrm{d}x$$

$$\leqslant -2\int_{B(tR,\,z)} a_{ij}\mathrm{D}_j u\,\mathrm{D}_i\varphi u G^z_\rho\,\mathrm{d}x + \frac{1}{2}\langle \mathrm{L}\varphi,\ u^2G^z_\rho\rangle + \int_{B(tR,\,z)} uG^z_\rho\varphi\,\mathrm{d}\nu\,.$$

From Lemma 2, recalling the definition of φ, we get

$$\int_{B(tR,z)} a_{ij} D_j u\, D_i u \varphi G^z_\rho + \frac{1}{2}\frac{1}{w(B(\rho,\ z))}\int_{B(\rho,z)} u^2 w\, dx + \int_{B(tR,z)} u^2 \varphi G^z_\rho\, d\mu$$

$$\leqslant -2\int_{B(tR,z)} a_{ij} D_j u\, D_i \varphi u G^z_\rho\, dx + \frac{1}{2}\sup_{B(tR,z)} |u|^2 + C_0 \|\nu\|_{K_n} \sup_{B(tR,z)} |u|\,.$$

Therefore, when $\rho \to 0$ and knowing that $\varphi = 1$ on $B(sR,\ z)$, we have

$$2\lambda \int_{B(sR,z)} |Du|^2 G^z w\, dx + |u|^2 + 2\int_{B(sR,z)} u^2 G^z\, d\mu$$

$$\leqslant \sup_{B(tR,z)} |u|^2 - 4\int_{B(tR,z)} a_{ij} D_j u\, D_i \varphi u G^z\, dx + C_0 \|\nu\|_{K_n(B(tR,z))} \sup_{B(tR,z)} |u|\,.$$

If we simplify and use in the term on the right hand side the Young inequality, we obtain

$$A = \lambda \int_{B(sR,z)} |Du|^2 G^z w\, dx + |u|^2 + \int_{B(sR,z)} u^2 G^z\, d\mu$$

$$\leqslant \sup_{B(tR,z)} |u|^2 + 4\eta\Lambda \sup_{B(tR,z)} |u|^2 \sup_{\partial B(sR,z)} G^z \operatorname{cap}(B(sR,\ z),\ B(tR,\ z))$$

$$+ \frac{\Lambda}{\eta}\int_{B(tR,z)-B(sR,z)} |Du|^2 G^z w\, dx + C_0 \|\nu\|_{K_n(B(tR,z))} \sup_{B(tR,z)} |u|\,.$$

Recalling the equivalence relation for the Green function, previously discussed in **1** (see also [**6**]) we further get

$$A \leqslant (1 + C_1\eta) \sup_{B(tR,z)} |u|^2 + \frac{\Lambda}{\eta}\int_{B(tR,z)-B(sR,z)} |Du|^2 G^z w\, dx + C_0 \|\nu\|_{K_n(B(tR,z))} \sup_{B(tR,z)} |u|$$

where C_1 is a constant which depends only on n, λ/Λ, $w(x)$.

Let us choose now $t = 1 - q$, $s = 2q$, $q \in (0,\ 1/5)$ and take in the previous relation the supremum with $z \in B(qr,\ x_0)$ and finally let us take $\eta = \gamma/C_1$.

If we neglect the first term on the right hand side, we obtain

$$\sup_{B(qR,x_0)} |u|^2 + \sup_{B(qR,x_0)} \int_{B(sR,z)} u^2 G^z\, d\mu \leqslant (1+\gamma) \sup_{B(R,x_0)} |u|^2$$

$$+ \frac{C}{\gamma} \sup_{B(qR,x_0)} \left(\frac{R^2}{w(B(sR,\ z))}\right) \int_{B(R,x_0)-B(qR,x_0)} |Du|^2 w\, dx + C_0 \|\nu\|_{K_n} \sup_{B(R,x_0)} |u|\,.$$

By taking into account the equivalence relations for G and acting as in [3], we finally have

$$\sup_{B(qR,x_0)}|u|^2+K_1\int_{B(qR,x_0)}u^2G^{x_0}_{B(2R,x_0)}\,d\mu\leq(1+\gamma)\sup_{B(R,x_0)}|u|^2$$

$$+\frac{C\Lambda}{\gamma}\int_{B(R,x_0)-B(qR,x_0)}|Du|^2G^{x_0}_{B(2R,x_0)}w\,dx+C_0\|\nu\|_{K_n(B(R,x_0))}\sup_{B(R,x_0)}|u|\,.$$

From Lemma 3 we then get

$$\sup_{B(R,x_0)}|u|^2\geq C_2\lambda\int_{B(qR,x_0)}|Du|^2G^{x_0}_{B(2R,x_0)}w\,dx$$

where $C_2<1$ can be chosen arbitrarily small. Therefore, for arbitrarly $\gamma>0$ (it will be fixed only afterwards) we easily have

$$C_2\lambda\int_{B(qR,x_0)}|Du|^2G^{x_0}_{B(2R,x_0)}w\,dx+\sup_{B(qR,x_0)}|u|^2+K_1\int_{B(qR,x_0)}u^2G^{x_0}_{B(2R,x_0)}\,d\mu$$

$$\leq(2+\gamma)\sup_{B(R,x_0)}|u|^2+\frac{C_1\Lambda}{\gamma}\int_{B(R,x_0)-B(qR,x_0)}|Du|^2G^{x_0}_{B(2R,x_0)}w\,dx+C_0\|\nu\|_{K_n}\sup_{B(R,x_0)}|u|\,.$$

By the usual *hole-filling* argument, consisting in multiplying the whole equation by γ and then in adding to both sides the term

$$C_1\Lambda\int_{B(qR,x_0)}|Du|^2G^{x_0}_{B(2R,x_0)}w\,dx$$

we have

$$(C_1\Lambda+C_2\lambda\gamma)\int_{B(qR,x_0)}|Du|^2G^{x_0}_{B(2R,x_0)}w\,dx+\gamma\sup_{B(qR,x_0)}|u|^2+K_1\gamma\int_{B(qR,x_0)}u^2G^{x_0}_{B(2R,x_0)}\,d\mu$$

$$\leq\gamma(2+\gamma)\sup_{B(R,x_0)}|u|^2+C_1\Lambda\int_{B(R,x_0)}|Du|^2G^{x_0}_{B(2R,x_0)}w\,dx+\gamma C_0\|\nu\|_{K_n}\sup_{B(R,x_0)}|u|\,.$$

We can further estimate the right hand side B in this way

$$\mathrm{B}\leq\gamma(2+\gamma)\sup_{B(R,x_0)}|u|^2+C_1\Lambda\Big[\int_{B(R,x_0)}|Du|^2G^{x_0}_{B(2q^{-1}R,x_0)}w\,dx$$

$$-\frac{C_3}{\Lambda}\,\frac{R^2}{w(B(R,\ x_0))}\int_{B(R,x_0)}|Du|^2w\,dx\Big]+C_0\gamma\|\nu\|_{K_n}\sup_{B(R,x_0)}|u|$$

$$\leq \gamma(2+\gamma) \sup_{B(R,x_0)} |u|^2 + C_1 \Lambda \int_{B(R,x_0)} |\mathrm{D}u|^2 G^{x_0}_{B(2q^{-1}R,x_0)} w \,\mathrm{d}x$$

$$-C_4 \frac{R^2}{w(B(R,\ x_0))} \int_{B(R,x_0)} |\mathrm{D}u|^2 w \,\mathrm{d}x + C_0 \gamma \|\nu\|_{K_n} \sup_{B(R,x_0)} |u| \,.$$

Let us apply now the Young inequality to the last term on the right hand side. It results

$$C_0 \gamma \|\nu\|_{K_n} \sup_{B(R,x_0)} |u| \leq \frac{\alpha}{2} \gamma \sup_{B(R,x_0)} {}^2 |u| + \gamma C_\alpha \|\nu\|^2_{K_n}$$

where α is very small and will be fixed only in the following.

Now, from Lemma 1 and taking into account the decomposition shown before,

$$\mathrm{C} \equiv C_1 \Lambda + C_2 \gamma\lambda \int_{B(qR,x_0)} |\mathrm{D}u|^2 G^{x_0}_{B(2R,x_0)} w \,\mathrm{d}x + \gamma \sup_{B(qR,x_0)} |u|^2 + \gamma K_1 \int_{B(qR,x_0)} u^2 G^{x_0}_{B(2R,x_0)} \,\mathrm{d}\mu$$

$$\leq \gamma(2+\frac{\alpha}{2}+\gamma) \sup_{B(R,x_0)} |u|^2 + C_1 \Lambda \int_{B(R,x_0)} |\mathrm{D}u|^2 G^{x_0}_{B(2q^{-1}R,x_0)} w \,\mathrm{d}x$$

$$-\frac{C_6 \lambda \operatorname{cap}_\mu (B(\frac{R}{2},\ x_0),\ B(R,\ x_0))}{w(B(\frac{R}{2},\ x_0)) \operatorname{cap}(B(\frac{R}{2},\ x_0),\ B(R,\ x_0))} \int_{B(\frac{R}{2},x_0)} u^2 w \,\mathrm{d}x$$

$$+\frac{C_6}{\operatorname{cap}(B(\frac{R}{2},\ x_0),\ B(R,\ x_0))} \int_{B(R,x_0)} u^2 \,\mathrm{d}\mu + K_0 \|\nu\|^2_{K_n(B(R,x_0))}$$

where we have put $\gamma C_\alpha = K_0$ to simplify the notation

$$\mathrm{C} \leq \gamma(2+\frac{\alpha}{2}+\gamma) \sup_{B(R,x_0)} |u|^2 + C_1 \Lambda \int_{B(R,x_0)} |\mathrm{D}u|^2 G^{x_0}_{B(2q^{-1}R,x_0)} w \,\mathrm{d}x$$

$$-\frac{C_6 \lambda}{w(B(\frac{R}{2},\ x_0))} \delta(\frac{R}{2}) \int_{B(\frac{R}{2},\ x_0)} u^2 w \,\mathrm{d}x + \frac{C_4 R^2}{w(B(R,\ x_0))} \int_{B(R,\ x_0)} u^2 \,\mathrm{d}\mu + K_0 \|\nu\|^2_{K_n}$$

and we can also suppose $C_6 = 10 C_2 C$, where C is the constant that appears in the Poincaré inequality.

Recalling once again the equivalence relations for G, we can further estimate

$$\mathrm{C} \leqslant \gamma(2+\frac{\alpha}{2}+\gamma) \sup_{B(R,x_0)} |u|^2 + C_1 \Lambda \int_{B(R,x_0)} |\mathrm{D}u|^2 G^{x_0}_{B(2q^{-1}R,x_0)} w \,\mathrm{d}x$$

$$-\frac{C_6\lambda}{w(B(\frac{R}{2},\ x_0))}\delta(\frac{R}{2}) \int_{B(\frac{R}{2},\ x_0)} u^2 w\,\mathrm{d}x + \Lambda K_2 \int_{B(R,\ x_0)} u^2 G^{x_0}_{(2q^{-1}R,x_0)} \,\mathrm{d}\mu + K_0 \|\nu\|^2_{K_n}.$$

From Lemma 3 we get

$$-10C_2 \frac{C}{w(B(\frac{R}{2},\ x_0))}\delta(\frac{R}{2}) \int_{B(\frac{R}{2},x_0)} u^2 w\,\mathrm{d}x$$

$$\leqslant -5C_2\lambda\delta(\frac{R}{2}) \sup_{B(qR,x_0)} |u|^2 + 5CC_2\lambda\delta(\frac{R}{2})\|\nu\|_{K_n(B(R,x_0))} \sup_{B(R,x_0)} |u|\,.$$

Again we take into account the Young inequality for the second term on the right hand side and we recall that $0\leqslant\delta(r)\leqslant 1$. Therefore

$$5CC_2\lambda\delta(\frac{R}{2})\|\nu\|_{K_n} \sup_{B(R,x_0)} |u| \leqslant 5CC_2\lambda\|\nu\|_{K_n} \sup_{B(R,x_0)} |u| \leqslant \gamma\frac{\alpha}{2} \sup_{B(R,x_0)} |u|^2 + K_\alpha \|\nu\|^2_{K_n}\,.$$

Finally we obtain

$$(C_1\Lambda + C_2\gamma\lambda) \int_{B(qR,x_0)} |\mathrm{D}u|^2 G^{x_0}_{B(2R,x_0)} w\,\mathrm{d}x$$

$$+[\gamma + 5C_2\lambda\delta(\frac{R}{2})] \sup_{B(qR,x_0)} |u|^2 + \gamma K_1 \int_{B(qR,x_0)} u^2 G^{x_0}_{B(2R,x_0)} \,\mathrm{d}\mu$$

$$\leqslant \gamma(2+\alpha+\gamma) \sup_{B(R,x_0)} |u|^2 + C_1\Lambda \int_{B(R,x_0)} |\mathrm{D}u|^2 G^{x_0}_{B(2q^{-1}R,x_0)} w\,\mathrm{d}x$$

$$+K_2\Lambda \int_{B(R,x_0)} u^2 G^{x_0}_{B(2q^{-1}R,x_0)} \,\mathrm{d}\mu + K_3 \|\nu\|^2_{K_n}$$

having chosen $K_3 = K_\alpha + K_0$.

Let us now add to both sides

$$6\frac{C_1}{C_2}\Lambda \sup_{B(qR,x_0)} |u|^2 \qquad \text{and} \qquad \frac{C_1\Lambda}{C_2} K_1 \int_{B(qR,x_0)} u^2 G^{x_0}_{B(2R,x_0)} \,\mathrm{d}\mu\,.$$

In such a case, we get

$$(C_1 \Lambda + C_2 \gamma\lambda) \int_{B(qR,x_0)} |\mathrm{D}u|^2 G^{x_0}_{B(2R,x_0)} w \,\mathrm{d}x$$

$$+[C_1\Lambda + \frac{C_2}{6}(\gamma + 5C_2\lambda\delta(\frac{R}{2}))]\frac{6}{C_2}\sup_{B(qR,x_0)}|u|^2 + [C_1\Lambda + \gamma C_2]\frac{K_1}{C_2}\int_{B(qR,x_0)} u^2 G^{x_0}_{B(2R,x_0)}\,\mathrm{d}\mu$$

$$\leqslant [C_1\Lambda + \frac{C_2}{6}\gamma(2+\alpha+\gamma)]\frac{6}{C_2}\sup_{B(R,x_0)}|u|^2 + C_1\Lambda\int_{B(R,x_0)}|\mathrm{D}u|^2 G^{x_0}_{B(2q^{-1}R,x_0)} w\,\mathrm{d}x$$

$$+(\Lambda K_2 + \frac{C_1K_1}{C_2}\Lambda)\int_{B(R,x_0)} u^2 G^{x_0}_{B(2q^{-1}R,x_0)}\,\mathrm{d}\mu + K_3\|\nu\|^2_{K_n(B(R,x_0))}\,.$$

Let us now fix $\gamma = C_2\lambda\delta(R/2)$ and α such that $2+\alpha+\gamma<3$.
It will be enough to suppose $\alpha < 1 - C_2\lambda\delta(R/2)$ or also $\alpha < 1 - C_2\lambda$.
We can then estimate

$$[C_1\Lambda + C_2^2\lambda\delta(\frac{R}{2})\int_{B(qR,x_0)} |\mathrm{D}u|^2 G^{x_0}_{B(2R,x_0)} w\,\mathrm{d}x$$

$$+[C_1\Lambda + C_2^2\lambda\delta(\frac{R}{2})]\frac{6}{C_2}\sup_{B(qR,x_0)}|u|^2 + [C_1\Lambda + C_2^2\lambda\delta(\frac{R}{2})]\frac{K_1}{C_2}\int_{B(qR,x_0)} u^2 G^{x_0}_{B(2R,x_0)}\,\mathrm{d}\mu$$

$$\leqslant [C_1\Lambda + \frac{C_2^2}{2}\lambda\delta(\frac{R}{2})]\frac{6}{C_2}\sup_{B(R,x_0)}|u|^2 + C_1\Lambda\int_{B(R,x_0)}|\mathrm{D}u|^2 G^{x_0}_{B(2q^{-1}R,x_0)} w\,\mathrm{d}x$$

$$+C_1\Lambda(\frac{K_2}{C_1} + \frac{K_1}{C_2})\int_{B(R,x_0)} u^2 G^{x_0}_{B(2q^{-1}R,x_0)}\,\mathrm{d}\mu + K_3\|\nu\|^2_{K_n(B(R,x_0))}\,.$$

Therefore, if we divide and assume $\Lambda K_2 < \dfrac{C_2^2}{2}\lambda\delta(\dfrac{R}{2})\dfrac{K_1}{C_2}$, we have

$$\int_{B(qR,x_0)} |\mathrm{D}u|^2 G^{x_0}_{B(2R,x_0)} w\,\mathrm{d}x + \frac{6}{C_2}\sup_{B(qR,x_0)}|u|^2 + \frac{K_1}{C_2}\int_{B(qR,x_0)} u^2 G^{x_0}_{B(2R,x_0)}\,\mathrm{d}\mu$$

$$\leq \frac{C_1 \Lambda + \frac{C_2^2}{2}\lambda\delta(\frac{R}{2})}{C_1 \Lambda + C_2^2 \lambda\delta(\frac{R}{2})}\{\int_{B(R,x_0)} |\mathrm{D}u|^2 G^{x_0}_{B(2q^{-1}R,x_0)} w \,\mathrm{d}x$$

$$+\frac{6}{C_2}\sup_{B(R,x_0)}|u|^2 + \frac{K_1}{C_2}\int_{B(R,x_0)} u^2 G^{x_0}_{B(2q^{-1}R,x_0)}\,\mathrm{d}\mu\} + K_4\|\nu\|^2_{K_n}$$

where the meaning of the constant K_4 is clear.

Let us now introduce the auxiliary function

$$\tilde{V}(r) = \int_{B(R,x_0)} |\mathrm{D}u|^2 G^{x_0}_{B(2q^{-1}r,x_0)} w\,\mathrm{d}x + \frac{6}{C_2}\sup_{B(r,x_0)}|u|^2 + \frac{K_1}{C_2}\int_{B(R,x_0)} u^2 G^{x_0}_{B(2q^{-1}r,x_0)}\,\mathrm{d}\mu\,.$$

If we work as in [3] we get

$$\tilde{V}(r) \leq \exp\left(-\beta\int_r^R \delta(\frac{\rho}{2})\,\frac{\mathrm{d}\rho}{\rho}\right) v(R) + K_4\|\nu\|^2_{K_n}$$

where $\beta \in (0,\ 1)$ is a suitable constant which depends only on n, λ/Λ, $w(x)$, q.

Therefore, at the end we obtain

$$V(r) \leq K\omega_\mu(x_0,\ r,\ R)^\beta v(R) + K\|\nu\|^2_{K_n(B(R,x_0))}\,.$$

4.2 — Theorem 2. From Theorem 1 we reason as in [4] and we come to the sufficiency of the Wiener condition for the regularity of x_0.

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Abstract

We prove the sufficiency of a Wiener test for the regular points of the weak solution of a relaxed Dirichlet problem relative to a degenerate elliptic operator.

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