Riv. Mat. Univ. Parma (4) **16** (1990), 331-342

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Minimal and polar projections onto hyperplanes in the spaces l_p and l_∞ (**)

1 - Introduction

In this paper we discuss minimal projections and polar projections (definition below) onto hyperplanes in the spaces l_p, $1 \leq p \leq \infty$. We recall that in a real normed space X, a hyperplane is a subspace V of X of the form $V = f^{-1}(0)$ where $f \in S^*$ (unit sphere of X^*); every projection P: $X \to V$ is of the form $Px = x - f(x)y$ with $f(y) = 1$. We define $\lambda(f^{-1}(0), X) = \lambda_f = \inf\{\|P\| : P: X \to f^{-1}(0)$ is a projection$\}$ and $H(X) = \sup\{\lambda_f, f \in S^*\}$. λ_f is the relative projection constant of $f^{-1}(0)$ in X and $H(X)$ is called the *hyperplane constant* of X (see for ex. [4]$_2$). A projection P onto $f^{-1}(0)$ such that $\|P\| = \lambda_f$ is called *minimal*. Clearly $1 \leq \lambda_f \leq H(x) \leq 2$. Let now X be a space l_p, if $p = 2$ X is a Hilbert space and for any f $\lambda_f = 1$ (and $H(X) = 1$). If $p = 1$ minimal projections onto hyperplanes of l_1 have already been described in [3], so we will study minimal projections only in the case $p \neq 1$ and $p \neq 2$.

Polar projections are defined only in those hyperplanes $V = f^{-1}(0)$ such that the functional f attains its norm; if f is such a functional we say that the projection P defined by $Px = x - f(x)z$ is polar if $\|z\| = f(z) = \|f\| = 1$. Note that there exist polar projections in any hyperplane if and only if X is reflexive and there is unicity if and only if X^* is smooth. Note also that in any Hilbert space the polar projection onto a hyperplane is the orthogonal projection.

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(**) Ricevuto: 24-VII-1990.

We will study polar projections in the spaces l_p, $p \neq 2$. Although the polar projections may appear to be the more natural and simple projections onto a given hyperplane, they are not minimal in general; however they may be used to obtain good estimates for the number $H(l_p)$.

Except for the Hilbert case, the only known infinite dimensional spaces where all polar projections are minimal are the spaces $L_p[0, 1]$ (see $[4]_1$, [5]). We shall also discuss the situation in the spaces $l_p(n)$: 2 is devoted to the space l_1 (where we consider only polar projections); 3 to the spaces l_p $1 < p < \infty$, $p \neq 2$ and 4 to the space l_∞.

2 - The case $X = l_1$

Minimal projections onto hyperplanes of l_1 have been already described in [3], we therefore shall discuss here only the nature of polar projections.

If $Px = x - f(x)z$ defines a projection onto the hyperplane $f^{-1}(0)$, then its norm is $\|P\| = \sup_{n \in N} \{|1 - f_n z_n| + |f_n|(1 - |z_n|)\}$ (see [3], Lemma 3).

P defines a polar projection if and only if $1 = \|z\| = \|f\| = f(z)$. In this case we have

(i) $K = \{j\colon\ |f_j| = \|f\|\} \neq \emptyset$ (ii) $j \in K \Rightarrow \operatorname{sgn} z_j = \operatorname{sgn} f_j$ (iii) $j \notin K \Rightarrow z_j = 0$.

By the above formula we easily obtain that

$$\|P\| = \max\,(\sup_{j \in K}(2 - 2|z_j|),\ (\sup_{j \notin K}(1 + |f_j|))\,.$$

There exists a polar projection of norm 1 (and therefore minimal) if and only if $j \notin K \Rightarrow f_j = 0$ and $\frac{1}{2} \leq |z_j|$ (there are only two possibilities, f has only one nonzero coordinate f_j with $|f_j| = 1$ (and $|z_j| = 1$), or has only two nonzero coordinates f_{j_1}, f_{j_2} (and $|z_{j_1}| = |z_{j_2}| = \frac{1}{2}$).

If the set K is infinite we have $\|P\| = 2$.

Note that there exists a polar projection P with $\|P\| < 2$ if and only if $\min_{j \in K} |z_j| > 0$ and $\sup_{j \notin K} |f_j| < 1$.

If $\operatorname{card}(K) = n$ then we can see that

$$\|P\| \geqslant \max\left(2 - \frac{2}{n}, \sup_{j \notin K}(1 + |f_j|)\right)$$

$$\min\{\|P\|, P \text{ is polar}\} = \max\left\{2 - \frac{2}{n}, \sup_{j \notin K}(1 + |f_j|)\right\} = \alpha_n$$

if $n > 1$ the norms of the polar projections fill the interval $[\alpha_n, 2]$.

3 - The case $X = l_p$

X will denote l_p or $l_p(n)$, $1 < p < \infty$, $p \neq 2$. Note the following well known facts which we recall without proof.

Let $\varepsilon_i = \pm 1$, $\varepsilon = \{\varepsilon_i\}$, $\{f_i\} = f \in S^*$ and $f_\varepsilon = \{\varepsilon_i f_i\}$ then $\lambda_f = \lambda_{f_\varepsilon}$. For any π: $N \to N$ 1-1 onto let $f_\pi = \{f_{\pi_i}\}$ then $\lambda_f = \lambda_{f_\pi}$. If $f \in S^*$ with $f_i > 0$ there is a rearrangement π such that $f_\pi = \{f_{\pi_i}\}$ is non-increasing.

Assume that for $f \in S^*$ $\{i \in N: f_i \neq 0\} = \{\nu_j\}_{j \in N}$ $(\nu_1 < \nu_2 < \ldots \leqslant \nu_j \ldots)$, then $\lambda_f = \lambda_{\bar{f}}$ where $\bar{f} = \{f_{\nu_i}\}$. If only $f_{\nu_i} \neq 0$, $j = 1, \ldots, n$, define $\bar{f} \in (l_p(n))^* = l_q(n)$ by $\bar{f} = (f_{\nu_1}, \ldots, f_{\nu_n})$: then $\lambda_f = \lambda(f^{-1}(0), l_p) = \lambda(\bar{f}^{-1}(0), l_p(n))$.

Using the natural embedding of $l_p(m)$ in $l_p(n)$, $m < n$, and of $l_p(n)$ in l_p we see that $1 = H(l_p(2)) \leqslant H(l_p(3)) \leqslant \ldots \leqslant H(l_p(n)) \leqslant H(l_p)$.

Theorem 1.3. *$H(l_p) = H(L_p[0, 1]) = \Lambda_p$ where $\Lambda_p = \max_{t \in [0,1]} \varphi_p(t)$ and*

$$\varphi_p(t) = \left[t^{\frac{1}{p-1}} + (1-t)^{\frac{1}{p-1}}\right]^{\frac{p-1}{p}} \left[t^{p-1} + (1-t)^{p-1}\right]^{\frac{1}{p}}.$$

Proof. Rolewicz has shown in [5] that $H(l_p) \leqslant H(L_p[0, 1])$ and Franchetti in $[4]_1$ that $H(L_p[0, 1]) = \Lambda_p$, thus we need only to prove that $H(l_p) \geqslant \Lambda_p$.

Let $X = l_p(n)$, $f \in S^* = \dfrac{(1, 1, \ldots, 1)}{n^{1/q}}$; the minimal projection P onto $f^{-1}(0)$ is given by the formula $Px = x - f(x)z$ where $z = \dfrac{(1, 1, \ldots, 1)}{n^{1/p}}$ (the fact that P is the minimal projection is due to the unicity and the simmetry of f; it could be proved by standard argument, see for ex. [5]). Using Theorem 2 from $[4]_1$ one can deduce that if $x \in S$ is such that $\|Px\| = \|P\|$, then x takes only two different values say $x_1 = x_2 = \ldots = x_k = \alpha > 0$ and $x_{k+1} = \ldots = x_n = -\beta < 0$. It then follows that $\|P\| = \|Px\| = \max_{0 \leqslant k \leqslant n} \varphi_p\left(\frac{k}{n}\right) \stackrel{\text{def}}{=} \Lambda_p(n)$ (the computation goes as follows; the optimal x must satisfy the conditions: $k\alpha^p + (n-k)\beta^p = 1$ ($\|x\| = 1$), $k\alpha^{p-1} - (n-k)\beta^{p-1} = 0$ (an orthogonality condition which is necessary for optimality, see [4]$_1$). Thus $\|P\|$ is the maximum value of $\|x - f(x)z\|$ where $f(x) = n^{-1/q}(k\alpha - (n-k)\beta)$).

Of course we have $\Lambda_p(n) \leq \Lambda_p$. In fact there is a unique $\tau_p \in (0, \frac{1}{2})$ and a unique $\tau'_p \in (\frac{1}{2}, 1)$ such that $\varphi_p(\tau_p) = \varphi_p(\tau'_p) = \Lambda_p$.

Call k_n an integer $k \leq n$ such that $\frac{k}{n} \leq \frac{1}{2}$ and $\varphi_p(\frac{k_n}{n}) = \Lambda_p(n)$; cleary we have $\frac{k_n}{n} \to \tau_p$, hence $\Lambda_p(n) \to \Lambda_p$. We now have $H(l_p) \geq H(l_p(n)) \geq \Lambda_p(n)$ which implies that $H(l_p) \geq \Lambda_p$.

Remarks. If τ_p is irrational then $\Lambda_p(n) < \Lambda_p$ $\forall n$. For example this is the case for $p = 3$, here $\tau_3 = \frac{1}{2} - \frac{\sqrt{1 + 2\sqrt{7}}}{6}$.

It is interesting to note that $\Lambda_p(n)$ is not in general monotone, for ex. $\Lambda_3(n)$ increases $2 \to 12$, decreases $12 \to 18$, increases $18 \to 25$ etc. Define $\Lambda_p^*(n) = \sup_{k \leq n} \Lambda_p(k)$; obviously $\Lambda_p^*(n)$ is monotone and $H(l_p(n)) \geq \Lambda_p^*(n)$.

Problem: is it true that $H(l_p(n)) = \Lambda_p^*(n)$?

One can see that when p runs over (1, 2), then τ_p runs with continuity in $(0, \frac{1}{2})$. We thus see that there exists $r \in (1, 2)$ such that the corresponding τ_r are rational, i.e. $\exists r \in (1, 2)$ and $n(r) \in N$ such that $H(l_r(n(r))) = \Lambda_p$.

Recall that if $\dim X = n$ we have $H(X) \leq 2 - \frac{2}{n}$. Assume that $\Lambda_p > 2 - \varepsilon$ (recall that $\Lambda_p \to 2$ for $p \to 1$) and τ_p is rational so that in fact $\Lambda_p = \Lambda_p(n)$; then it must be $2 - \varepsilon \leq 2 - \frac{2}{n}$ i.e. $n > \frac{2}{\varepsilon}$ (this is to show that $n = n(p)$ must be «large» in order to have $H(l_p(n)) = \Lambda_p$).

We remark also that if τ_p is rational then for n large we have $H(l_p(n)) = \Lambda_p$, consequently also $\Lambda_p = H(l_p)$ and $\sup_f \lambda_f = H(l_p)$ is attained by a functional of the type $f = c(\underbrace{1, 1, \ldots, 1}_{n\ times}, 0, 0, \ldots)$.

Problem: if τ_p is irrational is $\sup_f \lambda_f$ attained?

Polar projections. Due to reflexivity, rotundity and smoothness of X, the family V of all hyperplanes in X can be indexed by S: $V = \{J_z^{-1}(0),\ z \in S\}$ ($J_z \in S(X^*)$, $J_z(z) = 1$). There is a unique polar projection onto $J_z^{-1}(0)$, namely the projection P_z defined by $P_z x = x - J_z(x) z$. P_z is not in general the minimal projection onto $J_z^{-1}(0)$ but it does have special properties. Let $K(X)$ be the radial constant of the space X

$$K(X) = \sup_{x \neq y} \frac{\|Rx - Ry\|}{\|x - y\|} \qquad Rx = \begin{cases} x & \text{if } \|x\| \leqslant 1 \\ \dfrac{x}{\|x\|} & \text{if } \|x\| > 1 . \end{cases}$$

In $[4]_1$ it is proved that $K(X) = \sup\{\|P_z\|,\ z \in S\}$ (P_z polar). Since it is also known that $H(X) \leqslant K(X)$, $K[l_p(2)] = \Lambda_p$, $K[l_p[0, 1]] = \Lambda_p$ and since obviously $Y \subset X \Rightarrow K(Y) \subset K(X)$ we have for $n \geqslant 2$ $K(l_p(n)) = K(l_p) = \Lambda_p$ (see [4]). Thus in our spaces X for any polar projection P_z we have $\|P_z\| \leqslant \Lambda_p$.

Theorem 2.3. *Let* $z = \{z_n\}_{n \in I}$ *(*I *may be* $N (X = l_p)$ *or* $\{1, 2, \ldots, s\}$ $(X = l_p(s))$*) and* P_z *be the corresponding polar projection* $(P_z x = x - J_z(x) z)$*; if* A *is any subset of* I *let* $\gamma_A = \sum_{i \in A} |z_i|^p$*; then* $\|P_z\| \geqslant \varphi_p(\gamma_A)$ *(*φ_p *is defined in Theorem* 1.3).

Proof. Call $\alpha(\lambda)$, $\beta(\lambda)$ the positive solution of

$$\lambda \alpha^p + (1 - \lambda) \beta^p = 1 \qquad \lambda \alpha^{p-1} - (1 - \lambda) \beta^{p-1} = 0 .$$

Then, if $c(\lambda) = \lambda \alpha - (1 - \lambda) \beta$, we have

$$\lambda(\alpha(\lambda) - c(\lambda))^p + (1 - \lambda)(\beta(\lambda) + c(\lambda))^p = [\varphi_p(\lambda)]^p .$$

Let us define $\delta = \{\delta_n\}$ with $\delta_n = \begin{cases} \alpha(\gamma_A) & n \in A \\ -\beta(\gamma_A) & n \in I \setminus A \end{cases}$ and $z\delta$ by $(z\delta)_n = z_n \delta_n$. We have

$$\|z\delta\|^p = \alpha^p \sum_{i \in A} |z_i|^p + \beta^p \sum_{i \in I \setminus A} |z_i|^p = \gamma_A \alpha(\gamma_A)^p + (1 - \gamma_A) \beta(\gamma_A)^p = 1$$

$$J_z(z\delta) = \sum_i |z_i|^p \delta_i = \alpha \sum_{i \in A} |z_i|^p - \beta \sum_{i \in I \setminus A} |z_i|^p = c(\gamma_A)$$

$$\|P(z\delta)\| = \|z\delta - c(\gamma_A) z\| = \varphi_p(\gamma_A) .$$

Remarks. Note that $\varphi_p(t) \geqslant 1$ and $\varphi_p(t) = 1$ iff $t \in \{0, \frac{1}{2}, 1\}$ thus $\sup_A \varphi_p(\gamma_A)$ is in general a non-trivial lower bound for $\|P_z\|$.

Assume that z has at most 2 coordinates different from zero (we can assume that they are positive) $z_1 \geqslant z_2 \geqslant 0$; since the values taken by γ_A are z_1^p, z_2^p, $z_1^p + z_2^p$

we can have the trivial case only if $z_1^p = z_2^p = \frac{1}{2}$; $(z_1^p + z_2^p) = 1$ i.e. $z_1 = z_2 = \dfrac{1}{z^{1/p}}$; consequently the polar projection may have norm 1 only if (after reordering) $z = (1, 0, \ldots)$ or $z = (\dfrac{1}{2^{1/p}}, \dfrac{1}{2^{1/p}}, 0, 0, \ldots)$. In these cases we actually have $\|P_z\| = 1$.

Corollary 1.3. $\|P_z\| = 1$ *if and only if* z *is of the above form* (see also [2]).

Corollary 2.3. *We have that* $\|P_z\| = \Lambda_p$ *if* $\inf_A |\gamma_A - \tau_p| (= \inf_A |\gamma_A - \tau'_p|) = 0$.

It is not difficult to select $z \in S$ such that $\|P_z\| = \Lambda_p$; in fact by Corollary 2.3 it is enough to find $A \subset N$ such that $\gamma_A = \sum_{i \in A} |z_i|^p = \tau_p$; that many choices for $z = (z_1, z_2, \ldots, z_n, \ldots)$ are possible can be seen from the following elementary result on positive series: assume that $a_i > 0$, $\sum_{i=1}^{\infty} a_i = 1$ and that for any $n a_n \leq \sum_{i=1}^{\infty} a_{n+i}$; then $\forall \lambda \in (0, 1]$ $\exists A \subset N$: $\sum_{i \in A} a_i = \lambda$.

We conjecture that the polar projection P_z is minimal if and only if the nonzero coordinates of z are equal in absolute value (and therefore finite in number).

Problem: is it true that $\|P_z\| = \sup_{A \subset N} \varphi_p(\gamma_A)$?

4 - The case $X = l_\infty$

Let f be a functional defined on l_∞ and set $\tilde{h} = f_{|c_0} = (h_1, h_2, \ldots, h_n, \ldots)$, then $\tilde{h} \in l_1$; we shall denote by h the natural extension of $\tilde{h}$ to all l_∞ ($h(x) = \sum_i h_i x_i$, $x = (x_1, \ldots, x_n, \ldots) \in l_\infty$). If we define $g = f - h$ we obtain a canonical decomposition of an element $f \in (l_\infty)^*$ in the form: $f = g + h$, with $g \in (c_0)^\perp$ (the set of functionals vanishing on c_0). We shall always use the letters h and g for such a decomposition, meaning that h is the «l_1 part» and g the «$(c_0)^\perp$ part».

We shall use the following well known result

Lemma 1.4. *Let* $f = g + h$, *then* $\|f\| = \|h\| + \|g\|$, *consequently* f *attains its norms in* S *if and only if* h *and* g *attain simultaneously their norm.*

If $z \in S$ note that $h(z) = \|h\|$ if and only if $z_i \in [-1, 1]$ for $i \in T$ and $z_i = \operatorname{sgn} h_i$ for $i \notin T$, here $T = \{i \in N: h_i = 0\}$. If $g = 0$, then f attains always its norm (uniquely if and only if $T = \emptyset$).

The following Lemma is taken from $[1]_2$ a proof is given here for completeness.

Lemma 2.4 ($[1]_2$). *Let the projection $P: l_\infty \to f^{-1}(0)$ be defined by $Px = x - f(x)z$ ($\|f\| = f(z) = 1$), then*

$$(*) \qquad \|P\| = \sup_{j \in N} \{|1 - h_j z_j| + |z_j|(1 - |h_j|)\}.$$

Proof. Let $x \in S$ we have

$$(Px)_j = x_j - h(x) z_j - g(x) z_j = \sum_{k=1}^{\infty} (\delta_{kj} - h_k z_j) x_k - g(x) z_j$$

$$|(Px)_j| \leqslant \sum_{k=1}^{\infty} |\delta_{kj} - h_k z_j| + \|g\||z_j| \quad \text{i. e.} \quad \|P\| \leqslant \sup_{j \in N} \{|1 - h_j z_j| + |z_j|(\|g\| + \|h\| - |h_j|)\}.$$

Given $\varepsilon > 0$ let $x^\varepsilon \in S$ be such that $g(-x^\varepsilon) > \|g\| - \varepsilon$. Fix $j \in N$ and for any $n > j$ define x^n by

$$(x^n)_i = \begin{matrix} \operatorname{sgn}(\delta_{ij} - h_i z_j) & \text{for } i \leqslant n \\ x_i^\varepsilon \operatorname{sgn} z_j & \text{for } i > n \end{matrix}$$

and note that $g(x^\varepsilon) = \operatorname{sgn} z_j g(x^n)$. We have

$$\|P\| \geqslant \|Px^n\| \geqslant (Px^n)_j = \sum_{k=1}^{\infty} |\delta_{kj} - h_k z_j| - |z_j| \sum_{k=n+1}^{\infty} h_k x_k^\varepsilon - |z_j| g(x^\varepsilon).$$

Thus we have

$$\|P\| \geqslant \sum_{k=1}^{\infty} |\delta_{kj} - h_k z_j| - |z_j|(\|g\| - \varepsilon) \quad \text{i. e.} \quad \|P\| \geqslant \sup_{j \in N} \{|1 - h_j z_j| + |z_j|(\|g\| + \|h\| - |h_j|)\}.$$

Lemma 3.4. *Assume that $1 > 2\|h\|_\infty$ and set $\nu = \{\|g\| + \sum_{i=1}^{\infty} \frac{|h_i|}{1 - 2|h_i|}\}^{-1}$, then for any projection $P: l_\infty \to f^{-1}(0)$ we have $\|P\| \geqslant 1 + \nu$ ($1 - 2|h_i| > 0$ $\forall i$ since $\|h\|_\infty$ is attained).*

Proof. If $h=0$, then $\|g\|=\nu=1$ and by $(*)$ $\|P\|\geqslant 1+\|z\|\geqslant 2=1+\nu$. Let $Px=x-f(x)z$ where $f(z)=1$, $f=h+g$ with $h\neq 0$; assume if possible that $\|P\|<1+\nu$. By $(*)$ we have

$$1-|h_i z_i|+|z_i|(1-|h_i|)\leqslant\|P\|<1+\nu \quad \text{i. e.} \quad |z_i|<\nu(1-2|h_i|)^{-1}.$$

Setting $z^p=(0,\dots,0,\ z_{p+1},\dots,z_n,\dots)$ and $h^p=(0,\dots,0,\ h_{p+1},\dots,h_n,\dots)$ we have

$$|g(z)|=|g(z^p)|\leqslant\|g\|\,\|z^p\|\leqslant\frac{\nu\|g\|}{1-2\|h^p\|_\infty}.$$

Letting $p\to\infty$ we get $|g(z)|\leqslant\nu\|g\|$. Moreover

$$|h(z)|\leqslant\sum_i|h|_i|z_i|<\nu\sum_i\frac{|h_i|}{1-2|h_i|}$$

(note that the inequality is strict since $h\neq 0$). We have

$$1=f(z)\leqslant|h(z)|+|g(z)|<\nu[\|g\|+\sum_i\frac{|h_i|}{1-2|h_i|}]=\nu\nu^{-1}=1, \quad \text{a contradiction.}$$

Theorem 1.4. *$\lambda_f=1\Leftrightarrow 1\leqslant 2\|h\|_\infty$, moreover $\lambda_f=1\Rightarrow f^{-1}(0)$ is 1-coplemented, if $\lambda_f=1$ there is a unique norm one projection if and only if $|h_i|\geqslant\frac{1}{2}$ for exactly one index i.*

Proof. By Lemma 3.4 if $1>2\|h\|_\infty$ then $\lambda_f\geqslant 1+\nu>1$, hence $\lambda_f=1\Rightarrow 1\leqslant 2\|h\|_\infty$. If $1\leqslant 2\|h\|_\infty$ we have a norm one projection taking $z=(z_i)$ with $z_i=\delta_{ij}h_j^{-1}$ where j is such that $|h_j|=\|h\|_\infty$. We see that $Px=x-f(x)z$ defines a projection ($f(z)=h(z)=1$ since $g(z)=0$ being z in c_0); applying Lemma 2.4 we see that $\|P\|=1$. The assertion on unicity follows also easily.

Remarks. The fact that $1\leqslant 2\|h\|_\infty\Leftrightarrow f^{-1}(0)$ is 1-complemented was already proved in $[1]_1$.

We note also that Theorem 1.4 is the parallel in l_∞ of Theorem 1 in c_0 proved in [3].

Theorem 2.4. *Assume that $1>2\|h\|_\infty$, then $\lambda_f=1+\nu$.*

Proof. If $h=0$, by (1) $\|P\| = 1+\|z\|$ and so $\lambda_f = 2 = 1+\nu$. We therefore assume that $h \neq 0$. Set $\nu_n = (\|g\| + \sum_{i=1}^{n} \frac{|h_i|}{1-2|h_i|})^{-1}$ $\nu_n \to \nu$ $0<\nu<1$. Choose $x^n \in S$ such that $g(x^n) \to \|g\|$ and define $z^n = \begin{matrix} \nu \operatorname{sgn} h_i/(1-2|h_i|) & \text{for } i \leq n \\ \nu x_i^n & \text{for } i > n \end{matrix}$.

We have

$$f(z^n) = h(z^n) + g(z^n) = \nu\Big(\sum_{i=1}^{n} \frac{|h_i|}{1-2|h_i|} + \sum_{i=n+1}^{\infty} h_i x_i^n + g(x^n)\Big) = \nu(\nu_n^{-1} + \sigma_n)$$

hence $f(z^n) \to 1$ since $\sigma_n \to 0$. Let now P_n be the projection defined by $P_n y = y - f(y)\frac{z^n}{f(z^n)}$. By Lemma 2.4 we have $\|P_n\| = \sup_{j \in N} A(j)$ where

$$A(j) = \begin{cases} \left|1 - \dfrac{\nu|h_j|}{f(z^n)(1-2|h_j|)}\right| + \dfrac{\nu(1-|h_j|)}{f(z^n)(1-2|h_j|)} & \text{for } j \leq n \\[2ex] \left|1 - \dfrac{\nu h_j x_i^n}{f(z^n)}\right| + \dfrac{\nu|x_i^n|(1-|h_j|)}{f(z^n)} & \text{for } j > n. \end{cases}$$

Since $\max_{j \in N} \frac{|h_j|}{1-2|h_j|} < \frac{1}{\nu}$ (easy to see) and $f(z^n) \to 1$, we have for n large $A(j) = 1 + \frac{\nu}{f(z^n)} = 1+\nu+\varepsilon_n$ for $j \leq n$ and $A(j) \leq 1+\nu+\varepsilon_n$ for $j > n$, where $\varepsilon_n \to 0$. We thus have that for any $\varepsilon > 0$ $\exists n_\varepsilon$ such that $\|P_{n_\varepsilon}\| < 1+\nu+\varepsilon$. This means that $\lambda_f \leq 1+\nu$, by Lemma 3.4 the proof is complete.

Corollary 1.4. $\lambda_f = 2 \Leftrightarrow h = 0$.

Proof. If $h \neq 0$ $\nu^{-1} > \|g\| + \|h\| = 1$ hence $\lambda_f = 1+\nu < 2$. If $h=0$ then $\nu = \|g\| = 1$ hence $\lambda_f = 2$.

Theorem 3.4. *Assume that* $1 > 2\|h\|_\infty$ *(hence* $\lambda_f > 1$*), then* $f^{-1}(0)$ *admits a minimal projection if and only if* f *attains its norm.*

Proof. If: let $x \in S$ be such that $h(x) = \|h\|$, $g(x) = \|g\|$ then $x_i = \operatorname{sgn} h_i$ if $h_i \neq 0$ and $|x_i| \leq 1$ if $h_i = 0$. Let us define z by $z_i = \frac{\nu x_i}{1-2|h_i|}$ and w by $w_i = \nu x_i$ then $(z-w) \in c_0$; $h(z) = \nu \sum_i \frac{|h_i|}{1-2|h_i|}$, $g(z) = g(w) = \nu\|g\|$ so that $f(z) = 1$ and consequently $Px = x - f(x)z$ defines a projection. By Lemma 2.4 we have

$$\|P\| = \sup_{j \in N} \{|1 - \frac{\nu h_j x_j}{1-2|h_j|}| + \frac{\nu |x_j|}{1-2|h_j|}(1-|h_j|)\}.$$

If $T = \{j \in N\colon\ h_j = 0\}$ we have

$$\|P\| = \max(\sup_{j \in T}(1+\nu|x_j|),\ \sup_{j \in N \setminus T}[|1 - \frac{\nu h_j}{1-2|h_j|}| + \frac{\nu}{1-2|h_j|}(1-|h_j|)])$$

$$= \max(\sup_{j \in T}(1+\nu|x_j|),\ 1+\nu) = 1+\nu.$$

Only if: assume that $Px = x - f(x)z$ with $f(z) = 1$ defines a minimal projection, so that $\|P\| = 1 + \nu$. As in the proof of Lemma 3.4 we have $|z_i| \leq \nu(1-2|h_i|)^{-1}$, $|g(z)| \leq \nu\|g\|$, $|h(z)| \leq \nu \sum_i \frac{|h_i|}{1-2|h_i|}$ and also $1 = h(z) + g(z) \leq |h(z)| + |g(z)| = \nu\nu^{-1} = 1$ therefore $g(z) = \nu\|g\|$, $h(z) = \nu \sum_i \frac{|h_i|}{1-2|h_i|}$; if $h_i \neq 0$ $z_i = \frac{\nu \operatorname{sgn} h_i}{1-2|h_i|}$. We now define a by

$$a_1 = \begin{cases} 0 & \text{if } h_i = 0 \\ \nu \operatorname{sgn} h_i - z_i & \text{if } h_i \neq 0 \end{cases}$$

and note that $a \in c_0$. We have $g(z+a) = g(z) = \nu\|g\|$; $\|z+a\| = \nu$; $h(z+a) = \nu \sum_i |h_i| = \nu\|h\|$; thus g and h attain their norm on S in the point $\frac{z+a}{\nu}$.

Corollary 2.4. *If f attains its norm and $\lambda_f > 1$, then there is a unique minimal projection P_z onto $f^{-1}(0)$ if and only if $h_i \neq 0$ for every i; P_z is determined by $z = (z_i)$ with $z_i = \frac{\nu \operatorname{sgn} h_i}{1-2|h_i|}$.*

Remarks. Let $h \in l_1$, using Theorem 1.2 in [3] and our results we see that $\lambda(h^{-1}(0),\ c_0) = \lambda(h^{-1}(0),\ l_\infty)$ (in the second term h is considered as an element of $(l_\infty)^*$). It can also be seen that when f runs over $S(l_\infty)^*$, λ_f fills the interval $[1,\ 2]$.

Polar projections. Recall that polar projections are defined only on the hyperplanes $f^{-1}(0)$ such that f attains its norm. If f is such a functional $Qx = x - f(x)z$ defines a polar projection if and only if $\|f\| = f(z) = \|z\| = 1$. If Q

is such a projection by (1) we have $(f=h+g)\|Q\| = \sup_{j\in N} A_j$ where

$$A_j = \begin{matrix} 1+|z_j| & \text{for } j\in N\setminus D \\ 2-2|h_j| & \text{for } j\in D \end{matrix} \qquad D=\{i\in N:\ h_i\neq 0\}.$$

Theorem 4.4. *Assume that $f=h+g$ is a norm one functional attaining its norm, then:*

(i) *If $g\neq 0$ every polar projection Q has norm 2, consequently Q is minimal if and only if $h=0$.*

(ii) *If $g=0$ the norms of the polar projections Q fill the closed interval $[a, 2]$ where $a=\max(1, \max_{j\in D}(2-2|h_j|))$. There exists a polar projection Q which is minimal,* i.e. *$a=\lambda_f$, if and only if D is finite and $|h_i| = \frac{1}{n}$ where $n=\operatorname{card}(D)$; in such a case $\|Q\|=1$ if $n\leqslant 2$, $\|Q\| = 2-\frac{2}{n}$ for $n>2$.*

Proof. (i) If D is infinite $\sup_{j\in D}(2-2|h_j|)=2$ since $\sum|h_i|<\infty$; if D is finite let $p>\max_{j\in D} j$ and observe that $0<\|g\|=g(z)=g(z^p)$ $(z^p=(0,\ldots,0,\ z_{p+1},\ldots,z_n,\ldots))$ $g(z^p)\leqslant\|g\|\,\|z^p\|$ hence $\|z^p\|=1$ and this implies that $\sup_{j>p}(1+|z_j|)=2$. The unicity assertion follows from Corollary 1.4.

(ii) If D is infinite $a=2$. We therefore assume that D is finite. Since for $i\in N\setminus D$ $|z_i|$ can be any number in $[0,\ 1]$ we see that the norms of the polar projections Q do indeed fill the interval $[a, 2]$. Since $\|h\|=1$ we see that $\max(1,\ \max(2-2|h_j|))=1$ if and only if $\operatorname{card}(D)\leqslant 2$, in this case $\|Q\|=1$ as it is immediate to see. If $\operatorname{card}(D)>2$ there is a polar Q with $\|Q\|=2-2\inf_{j\in D}|h_j| = 2-2|h_k|$ for a $K\in D$. For any $i\in D$ we have

$$\frac{|h_i|}{1-2|h_k|}\leqslant\frac{|h_i|}{1-2|h_i|}\quad\text{thus}\quad\frac{1}{1-2|h_k|}\leqslant\sum_i\frac{|h_i|}{1-2|h_i|}=\nu^{-1};\ \ 1+\nu\leqslant 2-2|h_k|$$

and we have equality if and only if $|h_i|=|h_k|$ $i\in D$.

Remark. Using [3] and recalling that in c_0 a functional $f\in l_1$ attains its norm if and only if $D=\{i:\ f_i\neq 0\}$ is finite, we see that for polar projections Q in c_0 we have a statement exactly equal to (ii) of Theorem 4.4.

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Summary

Minimal projections and polar projections onto hyperplanes in the spaces l_p, $1<p\leqslant\infty$ *are discussed. Complete results are obtained for* $p=\infty$, *in the other cases estimates are deduced with finite dimensional methods.*

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