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Classification of left invariant CR structures on $GL^+(3, \mathbb{R})$ (**)

1 - Introduction

The aim of this paper is to classify the CR structures on $GL^+(3, \mathbb{R})$ which are invariant by left traslation. By an invariant CR structure on a real Lie group we mean a CR structure such that all left multiplications are CR transformations of the group. In the frame of Lie algebras, the problem becomes the following: let $\mathscr{G}$ be the complexification of the real Lie algebra $\mathscr{G}_0$, then we ask for the complex subalgebras q of $\mathscr{G}$ such that

$$q \cap \tau q = \{0\} \qquad q \oplus \tau q \qquad \text{has codimension 1 in } \mathscr{G}$$

where τ is the complex conjugation of $\mathscr{G}$ with respect to $\mathscr{G}_0$.

Such a subalgebra defines CR structure on $\mathscr{G}_0$ that is a subspace $\mathscr{T}$ of codimension 1 in $\mathscr{G}_0$ and an endomorphism J: $\mathscr{T} \to \mathscr{T}$ such that

$$J^2 = -\mathrm{id}$$

$$[X, Y] - [JX, JY] \in \mathscr{T} \quad \text{it} \quad X, Y \in \mathscr{T}$$

$$J([X, Y] - [JX, JY]) = [JX, Y] + [X, JY].$$

The relationship between J and q is given by

$$q = \{X + iJX \colon X \in \mathscr{T}\}$$

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Two invariant CR structures q_1 and q_2 (respectively J_1 and J_2) are said to be equivalent if there exists a Lie endomorphism σ: $\mathscr{G} \to \mathscr{G}$ such that $\sigma q_1 = q_2$ and $\tau\sigma = \sigma\tau$ (respectively a Lie endomorphism σ_0: $\mathscr{G}_0 \to \mathscr{G}_0$, such that $\sigma J_1 = J_2$).

In papers [3], [4], the reader can find many results concerning the analogous problem for complex invariant structures. The case of CR structures has been studied in detail for compact Lie groups and homogeneous compact manifolds by H. Azad, A. Huckleberry, W. Richthoser in [1].

In [2] it is given the classification of a class of CR invariant structures of a Lie group of first cathegory. Here, we want to give the classification for the simpliest group which is not of first cathegory: $GL^+(3, \mathbb{R})$, in order to understand better what happens in these groups and to look in detail the role plaied by the center of the Lie algebra in the construction of CR structures.

2 - Preliminaries

We refer to the paper of T. Sasaki quoted in $[3]_2$, where the author gives the classification of complex invariant structures for $\mathfrak{sl}(3, \mathbb{R})$. There, the reader can find a clear discussion of the decomposition of $\mathfrak{sl}(3, \mathbb{C})$ in roots which respect to Cartan subalgebras. We will use here the same notation as Sasaki and report only the necessary materials for our computation.

$\mathfrak{sl}(3, \mathbb{C})$ has two conjugate classes of Cartan subalgebras represented by $\eta_1 = \mathbb{R}\{h_1, h_2\}$ and $\eta_2 = \mathbb{R}\{\mathrm{i}H_1, \mathrm{i}H_2\}$ where

$$h_1 = E_{11} - E_{22} \qquad h_2 = E_{22} - E_{33}$$

$$H_1 = \frac{\mathrm{i}}{2}(E_{21} - E_{12}) \qquad H_2 = -\frac{\mathrm{i}}{2}(E_{11} + E_{22} - 2E_{33})$$

E_{ij} being the matrix with 1 in the (i, j)-th component and 0 in the others.

The roots of $\mathfrak{sl}(3, \mathbb{C})$ with respect to the complexification $\eta_1^{\mathbb{C}}$ of η_1 are real.

We report here a basis of roots vectors with respect to $\eta_2^{\mathbb{C}}$:

$$e_\alpha = \frac{1}{\sqrt{2}}(E_{13} + \mathrm{i}E_{23}) \qquad e_\beta = \frac{1}{\sqrt{2}}(E_{31} + \mathrm{i}E_{32})$$

$$e_\gamma = \frac{1}{2}(\mathrm{i}(E_{12} + E_{21}) + (E_{11} - E_{22}))$$

$$e_{-\alpha} = \tau e_\beta \qquad e_{-\beta} = \tau e_\alpha \qquad e_{-\gamma} = \tau e_\gamma .$$

Furthermore, denoting $H_\delta = [e_\delta, e_{-\delta}]$ for positive δ we have

$$H_\alpha = H_1 + \mathrm{i}H_2 \qquad H_\beta = H_1 - \mathrm{i}H_2$$

from which se see that $\tau H_\alpha = -H_\beta$.

The values of roots are given by

$$\alpha(H_\alpha) = 2 \qquad \alpha(H_\beta) = -1 \qquad \beta(H_\alpha) = -1 \qquad \beta(H_\beta) = 2 .$$

It will be useful in the following the bracket relations

$$[e_\alpha, e_\beta] = e_\gamma \qquad [e_\beta, e_{-\gamma}] = e_{-\alpha} \qquad [e_{-\gamma}, e_\alpha] = e_{-\beta}$$

$$[e_{-\alpha}, e_\gamma] = e_\beta \qquad [e_\gamma, e_{-\beta}] = e_\alpha \qquad [e_{-\alpha}, e_{-\beta}] = -e_{-\gamma} .$$

Any maximal parabolic subalgebra of $\mathfrak{sl}(3, \mathrm{C})$ is conjugate under an inner authomorphism of $\mathscr{G}_0$ to one of the following

$$p_\alpha = \eta^{\mathrm{C}} + \mathrm{C}\{e_\alpha, e_\beta, e_\gamma, e_{-\alpha}\} \qquad p_\beta = \eta^{\mathrm{C}} + \mathrm{C}\{e_\alpha, e_\beta, e_\gamma, e_{-\beta}\} .$$

When $\eta^{\mathrm{C}} = \eta_1^{\mathrm{C}}$ the corresponding parabolic algebras p coincides with τp; when $\eta^{\mathrm{C}} = \eta_2^{\mathrm{C}}$ then $\dim (p \cap \tau p) = 4$.

Moreover, we recall the following useful propositions.

Proposition 1 [4]. *Any maximal proper subalgebra of a complex semisimple Lie algebra is either parabolic or semisimple.*

Proposition 2 [3]$_2$. $\mathfrak{sl}(3, \mathrm{C})$ *cannot contain a semisimple subalgebra of dimension greater than three.*

Thus, any complex subalgebra q of $\mathfrak{sl}(3, \mathrm{C})$ of dimension greater than 3 must be contained in a parabolic subalgebra p. Moreover, up to τ-stable equivalences σ of $\mathfrak{sl}(3, \mathrm{C})$ we can assume that p is one of p_α, p_β. Let's note that if $\dim (q \cap \tau q) \leq 1$, then we can suppose $\eta^{\mathrm{C}} = \eta_2^{\mathrm{C}}$.

Since the involutive automosrphism $X \to -{}^tX$, transforms p_α in p_β, any complex subalgebra q of $\mathfrak{sl}(3, \mathrm{C})$ of dimension greater than 3 and such that $\dim (q \cap \tau q) \leq 1$ which is contained in p_α is equivalent to one contained in p_β.

3 - Classification of left invariant CR structures

Now, let's remind that $\mathfrak{gl}(3, \mathbb{R}) = \mathbb{R}\mathrm{I} \oplus \mathfrak{sl}(3, \mathbb{R})$ where I denotes the unit matrix. Let q gives an invariant CR structure on $\mathfrak{gl}(3, \mathbb{R})$; i.e. q is a complex subalgebra of $\mathfrak{gl}(3, \mathbb{C})$, $q \cap \tau q = \{0\}$ and $\dim q = 4$. We shall consider separately the following two cases:

(i) $\mathrm{I} \notin q \oplus \tau q$ (in particular we have $\mathfrak{gl}(3, \mathbb{C}) = q \oplus \tau q \oplus \mathbb{C}\mathrm{I}$)

(ii) $$\mathrm{I} \in q \oplus \tau q\,.$$

Case (i).

Let's intersect τq with $\mathfrak{sl}(3, \mathbb{C})$; then, either $\tau q \cap \mathfrak{sl}(3, \mathbb{C}) = \tau q$ and q gives an invariant complex structure of $\mathfrak{sl}(3, \mathbb{R})$ or $\tau q \cap \mathfrak{sl}(3, \mathbb{C}) = \tau\eta$ where η is a complex subalgebra of $\mathfrak{sl}(3, \mathbb{C})$ of dimension three. If this last case occurs, then $\tau\eta$ is contained in $\tau\eta^*$ where η^* is the complex subalgebra of $\mathfrak{sl}(3, \mathbb{C})$ of dimension 4, given by

$$\tau\eta^* = (\tau\eta \oplus \mathbb{C}\mathrm{I}) \cap \mathfrak{sl}(3, \mathbb{C})\,.$$

Now, it is easy to see that $\eta^* \cap \tau\eta^* = \{0\}$, that is η^* is an invariant complex structure of $\mathfrak{sl}(3, \mathbb{R})$. So, given such a complex structure η^*, we look for invariant CR structures in $\mathfrak{gl}(3, \mathbb{R})$ in the following way: start from any complex subalgebra η of dimension three in η^* and consider $\eta \oplus \mathbb{C}(N + a\mathrm{I})$ where N is an element of η^*, which does not belong to η and a is a complex number.

Case (ii).

Since $\mathrm{I} \in \mathscr{T}$, we can consider the following element of $\mathfrak{sl}(3, \mathbb{C})$

$$V = (\mathrm{trace}\,(J\mathrm{I}))\,\mathrm{I} - 3J\mathrm{I}\,.$$

Thus, $V \notin \eta \oplus \tau\eta$ where $\tau\eta = \tau q \cap \mathfrak{sl}(3, \mathbb{C})$, $(\dim \eta = 3)$, moreover $\tau\eta^* = \tau\eta \oplus \mathbb{C}V$ is a subalgebra of dimensione 4 in $\mathfrak{sl}(3, \mathbb{C})$ since $[\tau\eta, V] \subset \tau\eta$ and $\dim(\eta^* \cap \tau\eta^*) = 1$.

Viceversa, starting from a complex subalgebra η^* of $\mathfrak{sl}(3, \mathbb{C})$ of dimension 4 such that $\dim(\eta^* \cap \tau\eta^*) = 1$, and any subalgebra η of η^* of dimension three such that $\eta \cap \tau\eta = \{0\}$, and any real vector V belonging to $\eta^* \cap \tau\eta^*$, we get the following CR structures: $q = \eta \oplus \mathbb{C}(\mathrm{I} - i(V + a\mathrm{I}))\ \forall a \in \mathbb{R}$.

We want to look now for equivalences between CR structures.

Let σ be a τ-stable Lie authomorphism of $\mathscr{G}$, then it is obvious that $\sigma(\mathrm{I}) = r\mathrm{I}$ for some $r \in \mathbb{R}^*$ and that $\sigma(\mathfrak{sl}(3, \mathbb{C}) \subset \mathfrak{sl}(3, \mathbb{C})$. Viceversa any τ-stable Lie authomorphism of $\mathfrak{sl}(3, \mathbb{C})$ can be extended to $\mathfrak{gl}(3, \mathbb{C})$ putting $\sigma(\mathrm{I}) = r\mathrm{I}$ for some real $r \neq 0$.

It is evident that the CR structures arising by cases (i) (ii) are not equivalent, so let's study them separatly.

Case (i).

We have two types of inequivalent CR structures arising from $q \subset \mathfrak{sl}(3, \mathbb{C})$ or $q \not\subset \mathfrak{sl}(3, \mathbb{C})$. For the first type, the problem of equivalence is reduced to that of corresponding invariant complex structures of $\mathfrak{sl}(3, \mathbb{R})$; therefore it has already been solved in $[3]_2$. For the second type, it is obvious that if q and q' are equivalent in $\mathfrak{gl}(3, \mathbb{C})$ then the corresponding complex structures of $\mathfrak{sl}(3, \mathbb{C})$ η^* and $\eta^{*\prime}$ are equivalent in $\mathfrak{sl}(3, \mathbb{C})$, but starting from a complex structure, we can get many inequivalent CR structures as we will see in a moment. Sasaki has classified the complex structures of $\mathfrak{sl}(3, \mathbb{R})$ in two isolated algebras II and III and a family $I(\lambda)$ with $|\lambda| < 1$. We will see what happens starting from each of them.

CR *structures arising from* $I(\lambda) = \{H_\alpha + \lambda H_\beta,\ e_\alpha,\ e_\beta,\ e_\gamma\}$. Since the subalgebras of $I(\lambda)$ of dimension three are $\{e_\alpha, e_\beta, e_\gamma\}$, $\{e_\alpha, e_\gamma, be_\beta + H_\alpha + \lambda H_\beta\}$ and $\{e_\beta, e_\gamma, be_\alpha + H_\alpha + \lambda H_\beta\}$ thus we get the following CR structures

$$A(\lambda, s) = \{H_\alpha + \lambda H_\beta + sI,\ e_\alpha,\ e_\beta,\ e_\gamma\} \qquad |\lambda| < 1 \quad s \in \mathscr{L}$$

where $\mathscr{L} = \{e^{i\vartheta}: -\frac{\pi}{2} < \vartheta \leq \frac{\pi}{2}\}$,

$$B(b, s) = \{e_\alpha,\ e_\gamma,\ be_\beta + 2H_\alpha + H_\beta,\ e_\beta + sI\} \qquad s \in \mathscr{L}.$$

Note that, if $b \neq 0$ $B(b, s)$ is equivalent to $B(1, s')$ for some $s' \in \mathscr{L}$; all the other CR structures are not equivalent.

CR *structures arising from* $II = \{H_\alpha + 2H_\beta,\ e_\alpha + e_{-\alpha},\ e_\beta,\ e_\gamma\}$. Since the subalgebras of II of dimension three are $\{e_\alpha + e_{-\alpha}, e_\beta, e_\gamma\}$ and $\{e_\beta, e_\gamma, a(e_\alpha + e_{-\alpha}) + H_\alpha + 2H_\beta\}$ thus we get the following CR structures:

$$II(s) = \{e_\alpha + e_{-\alpha},\ e_\beta,\ e_\gamma,\ H_\alpha + 2H_\beta + sI\} \qquad s \in \mathscr{L}$$

$$II(b, s) = \{e_\beta,\ e_\gamma,\ b(e_\alpha + e_{-\alpha}) + H_\alpha + 2H_\beta,\ e_\alpha + e_{-\alpha} + sI\} \qquad s \in \mathscr{L} \quad b \in \mathbb{C}.$$

Note that $II(b, s)$ is equivalent to $II(-b, s)$; all the others CR structures are not equivalent.

CR *structures arising from* $III=\{Z=\frac{1}{2}(H_\alpha+2H_\beta-3(e_\alpha+e_{-\alpha})),$ $U=\frac{1}{2}(H_\alpha+2H_\beta+e_\alpha+e_{-\alpha}),\ V=-H_\alpha-e_\alpha+e_{-\alpha}+e_\beta-e_\gamma,\ W=H_\alpha+e_\alpha-e_{-\alpha}+e_\beta$ $-e_\gamma\}$. The complex subalgebras of dimension three of III are easily seen to be

$$\{U,\ V,\ W\} \qquad \{U,\ \pm V+W,\ Z\} \qquad \{V,\ W,\ bU+Z\}$$

$$\{V+bU,\ bU+W,\ -3U+Z\} \qquad \{V+bU,\ -bU+W,\ -3U+Z\}.$$

From the first one, we get a family of CR structures

$$III(s)=\{U,\ V,\ W,\ Z+sI\} \qquad s\in\mathscr{P}.$$

The second one gives no CR structures, while the last three cases give the same CR structures

$$III(b,\ s)=\{V,\ W,\ bU+Z,\ U+sI\} \qquad s\in\mathscr{P} \quad b\in\mathbb{C}.$$

For the equivalence between the CR structures above, let's make only a few remarks. In cases $B(b, s)$ and $II(b, s)$ we have used the authomorphism $\sigma_{r,k}$ of $\mathfrak{sl}(3, \mathbb{C})$ with $r\in\mathbb{R}^*$ and $k\in\mathbb{C}^*$, which will be often used in the sequel:

$$e_\alpha\to ke_\alpha \qquad e_\beta\to\frac{1}{\bar{k}}e_\beta \qquad e_\gamma\to\frac{k}{\bar{k}}e_\gamma \qquad e_{-\alpha}\to\frac{1}{k}e_{-\alpha}$$

$$e_{-\beta}\to\bar{k}e_{-\beta} \qquad e_{-\gamma}\to\frac{\bar{k}}{k}e_{-\gamma} \qquad \mathrm{I}\to r\mathrm{I} \qquad \text{and fixing } \eta^{\mathbb{C}}.$$

Moreover since $B(1, s)$ arise from $I(\frac{1}{2})$, it remains to look that there are not equivalences σ between $B(1, s)$ and $A(\frac{1}{2}, s')$.

Computing $\sigma([H_\gamma, e_\alpha])$ and $\sigma([H_\gamma, e_\gamma])$ it is easy to see that it should hold $\sigma(e_\alpha)=Ae_\alpha$ and $\sigma(e_\gamma)=Ce_\alpha+De_\gamma$ with $|D|=1$. By $\sigma([e_\beta, e_\gamma])=0$ and $\sigma([e_\alpha, e_\beta])=\sigma e_\gamma$ we get $C=0$ and $\sigma(H_\gamma)=H_\gamma$; then it would be $\sigma(e_\beta)$ $=Ee_\alpha+\dfrac{1}{\bar{A}}e_\beta+sr\mathrm{I}$ for some E where $\sigma(\mathrm{I})=r\mathrm{I}$ and $\sigma(H_\beta)=\dfrac{1}{\bar{A}}H_\beta$, which would imply $A=D=1$. So $\eta^{\mathbb{C}}$ should be fixed by σ which would imply $s=0$ while $s\in\mathscr{P}$.

Furthermore, any equivalence between $II(b, s)$ and $II(b', s')$ would imply an equivalence between $\mathfrak{sl}(3, \mathbb{C})\cap II(b, s)$ and $\mathfrak{sl}(3, \mathbb{C})\cap II(b', s')$, while they are not equivalent as abstract Lie algebra (of dimension three) unless

$b = \pm b'$. The same holds for algebras of type $III(b, s)$. $II(b, s)$ is equivalent to $II(-b, s)$ by $\sigma_{-1,-1}$. While $III(b, s)$ is equivalent to $III(-b, s)$, the isomorphism being given by:

$$e_\alpha \to -e_{-\beta} \qquad e_\beta \to H_\alpha + e_\alpha - e_{-\alpha} + e_{-\beta} + e_{-\gamma}$$

$$e_\gamma \to e_{-\beta} + e_{-\gamma} \qquad e_{-\alpha} \to -H_\beta + e_\alpha - e_\beta + e_\gamma + e_{-\beta}$$

$$e_{-\beta} \to -e_\alpha \qquad e_{-\gamma} \to e_\alpha + e_\gamma \qquad H_\alpha \to -H_\alpha + e_\alpha + 2e_{-\beta}$$

$$H_\beta \to -H_\alpha - e_{-\beta} - 2e_\alpha .$$

Let's finally remark that $III(s)$ cannot be equivalent to $III(b, s)$ since $\{U, V, W\}$ is not isomorphic to $\{V, W, bU + Z\}$ (their derived algebra being of different dimension).

Case (ii).

In order to study case (ii) we need first to find the subalgebras η^* of dimension 4 in $\mathfrak{sl}(3, \mathrm{C})$ such that

$$\dim(\eta^* \cap \tau\eta^*) = 1 .$$

As we already noted in the introduction, we can assume that

$$\tau\eta^* \subset \{H_\alpha, H_\beta, e_\alpha, e_\beta, e_{-\beta}, e_\gamma\} = p .$$

Moreover we shall have

$$\eta^* \cap \tau\eta^* = \{H + a e_\alpha + \bar{a} e_{-\beta}\} \qquad H = \bar{H} \qquad a \in \mathrm{C} .$$

It is easy to see that if $\dim(\tau\eta^* \cap \{e_\alpha, e_\beta, e_\gamma\}) \geqslant 2$ then $e_\gamma \in \tau\eta^*$. So, we suppose first that $e_\gamma \in \tau\eta^*$ and we find, after routine computations, the following possibilities for $\tau\eta^*$:

(1) $$\{U = e_\alpha, V = e_\beta, W = e_\gamma, Z = H\}$$

(2) $$\{U = e_\alpha, V = e_\gamma, W = 2H_\alpha + H_\beta, Z = H + a e_\alpha + \bar{a} e_{-\beta}\}$$

(3) $$\left\{U = e_\alpha, V = e_\gamma, W = H_\beta - \frac{\beta(H)}{\bar{a}} e_\beta + \frac{\bar{a}}{\beta(H)} e_{-\beta}, Z = H + a e_\alpha + \bar{a} e_{-\beta}\right\}$$

where $H = \bar{H}$.

When $\dim(\tau\eta^* \cap \{e_\alpha, e_\beta, e_\gamma\}) = 1$, and $e_\gamma \notin \tau\eta^*$, we find the following possibilities for $\tau\eta^*$

(1)′ $\{U = e_\beta,\ V = Ae_\alpha + e_{-\beta},\ W = H_\beta - Ae_\gamma,\ Z = H_\alpha - H_\beta\} \qquad |A| \neq 1.$

(2)′ $\{U = Ae_\alpha + e_\gamma,\ V = A' e_\alpha + \frac{1}{A^2} e_\beta + e_{-\beta},\ Z = H + ae_\alpha + \bar{a}e_{-\beta},\ W = K + Te_\alpha\}$

$$A\beta(H) + \bar{a} = 0 \qquad AA'\alpha(H) = a \qquad K = 2H_\alpha + H_\beta$$

$$T = 3AA' \qquad H = sH_\alpha - \bar{s}H_\beta \qquad |s| = 1.$$

(3)′ $\{U = Be_\beta + e_\gamma,\ V = A' e_\alpha + e_{-\beta},\ W = BH_\beta - A'Be_\gamma + e_\alpha,\ Z = K + Te_\beta\}$

$$K = H_\alpha + 2H_\beta \qquad T = 3BA' \qquad B \neq 0.$$

In this last case, we can take V as real vector when $|A'| = 1$, while, when $|A'| \neq 1$ the real vector can be chosen to be

$$H_\alpha - H_\beta + \bar{m}e_\alpha + me_{-\beta} \qquad mA' - \frac{3}{B} = \bar{m}.$$

Now, we have to find the subalgebras η of dimension three of η^* such that $\eta \cap \tau\eta = \{0\}$ and $[\eta,\ \eta^* \cap \tau\eta^*] \subset \eta$, afterwards take $\{\eta,\ \mathrm{I} + i(Z^* + a\mathrm{I})\}$ where $Z^* \in \eta^* \cap \tau\eta^*$ is real. We will write down the CR structures arising from cases 1, 2, 3, 1′, 2′, 3′ keeping in mind some obvious equivalences given by the authomorphisms of type $\sigma_{r,k}$. Let's denote $H_s = sH_\alpha - \bar{s}H_\beta$ where $s \in \mathscr{S}$; now we get the following CR structures:

Case (1).

$\mathscr{A}(a,\ s) = \{U = e_\alpha,\ V = e_\beta,\ W = e_\gamma,\ \mathrm{I} + i(H_s + a\mathrm{I})\} \qquad$ with $a \in \mathbb{R}$.

Case (2).

$\mathscr{B}(a,\ d,\ s) = \{U = e_\alpha,\ V = e_\gamma,\ W + dZ = 2H_\alpha + H_\beta + d(H_s + e_\alpha + e_{-\beta}),$

$$\mathrm{I} + i(H_s + e_\alpha + e_{-\beta} + a\mathrm{I})\} \qquad \text{with } d \in \mathbb{C},\ a \in \mathbb{R}.$$

Case (3).

$$\mathscr{C}(a, s) = \{U = e_\alpha,\ V = e_\gamma,\ W = H_\beta - \beta(H_s)\, e_\beta + \frac{1}{\beta(H_s)} e_{-\beta},$$

$$\mathrm{I} + \mathrm{i}(H_s + e_\alpha + e_{-\beta} + a\mathrm{I})\} \qquad \text{with } a \in \mathbb{R}.$$

Case (1)′.

$$\mathscr{A}'(A, a) = \{U' = e_\beta,\ V' = Ae_\alpha + e_{-\beta},\ W' = H_\beta - Ae_\gamma,$$

$$\mathrm{I} + \mathrm{i}(H_\alpha - H_\beta + a\mathrm{I})\} \qquad \text{with } a \in \mathbb{R},\ A \geqslant 0,\ A \neq 1.$$

Case (2)′.

$$\mathscr{B}'(c, a, s) = \{U' = e_\alpha - \beta(H_s)\, e_\gamma,\ V' = H_\beta + \frac{1}{\beta(H_s)} e_{-\beta} - \beta(H_s)\, e_\beta,$$

$$W' = c(2H_\alpha + H_\beta + \frac{3}{\alpha(H_s)} e_\alpha) + (-\frac{1}{\alpha(H_s)} e_\alpha + \beta(H_s)\, e_\beta + \frac{1}{\beta(H_s)} e_{-\beta},$$

$$\mathrm{I} + \mathrm{i}Z' = \mathrm{I} + \mathrm{i}(H_s + e_\alpha + e_{-\beta} + a\mathrm{I})\} \qquad \text{where } a \in \mathbb{R},\ c \in \mathbb{C} \cup \infty;$$

in the case $c = \infty$ being $W' = 2H_\alpha + H_\beta + \dfrac{3}{\alpha(H_s)} e_\alpha$.

Case (3)′.

$$\mathscr{C}'(A, a) = \{U' = e_\beta + e_\gamma,\ V' = Ae_\alpha + e_{-\beta},\ W' = H_\beta - Ae_\gamma + e_\alpha,$$

$$\mathrm{I} + \mathrm{i}((H_\alpha - H_\beta + me_{-\beta} + \overline{m}e_\alpha) + a\mathrm{I})\} \qquad \text{where } a \in \mathbb{R},\ mA = \overline{m} + 3,\ |A| = 1,\ A \in \mathbb{C}.$$

The dimensions of the first and second elements of central series of algebras (1) (2) (3) (1)′ (2)′ exclude any equivalence between different families except for types (2) and (2)′ for which we can do the following considerations.

Since their derived algebras are spanned by U, V and U', V' then if there is an equivalence σ, it must be $\sigma U = aU' + bV'$, $\sigma V = lU' + kV'$ and of course $\sigma Z = \rho Z'$ $\rho \in \mathbb{R}$. By $\sigma[U, Z]$ we get $\alpha(H_s) = \rho\beta(H_{s'})$ which implies either $s' = -\overline{s}$ and $\rho = 1$, that is $s' = s = \mathrm{i}$ and $\rho = 1$ or $s' = \overline{s}$ and $\rho = -1$, that is $s' = s$ and $\rho = -1$. Computing $\sigma[U, Z]$ and $\sigma[V, Z]$, it is easy to see that it must be $\rho = -1$ and $\sigma U = a(U' - (1 + \alpha(H_s))\, V')$, $\sigma V = l(U' + (\alpha(H_s) - 1)\, V')$. Moreover, by

$\sigma[W+dZ,\ Z]$ and $\sigma[W+dZ,\ U]$, we get $3+\alpha(H_s)d=0$ or $3c=1$. But if $3+\alpha(H_s)d=0$, then it would be also

$$\sigma(W+dZ) = -\frac{3a}{\alpha(H_s)}U' + \frac{3a}{\alpha(H_s)}(\alpha(H_s)+1)V'$$

which is impossible. If $3c=1$ then

$$\sigma(W+dZ) = -\frac{3a}{\alpha(H_s)}U' + BV' + \frac{3+a(H_s)d}{2}W'$$

where
$$B = \frac{3+\alpha(H_s)d}{2} + \frac{3a}{\alpha(H_s)}(\alpha(H_s)+1).$$

Computing $\sigma[V,\ W+dZ]$ one gets $d=0$ and by $\sigma[V,\ \overline{U}]$, we can conclude that there cannot be equivalences between class (2) and (2)′ (same calculation can be done for $c=\infty$).

We study now the equivalences between algebras in the same family.

Case (1). It is easy to see that there are not equivalences.

Case (2). Any equivalence σ between $\mathscr{B}(a,\ d,\ s)$ and $\mathscr{B}(a',\ d',\ s')$ must be of the following type (since the derived algebra is given by $\{U,\ V\}$):

$$e_\alpha \to ke_\alpha \qquad e_\gamma \to Ce_\alpha + \frac{k}{\bar{k}}e_\gamma \qquad H_s + e_\alpha + e_{-\beta} \to r(H_{s'},\ e_\alpha + e_{-\beta})$$

with $r \in \mathbb{R}$.

So $\sigma H_s = rH_{s'} + (r-k)e_\alpha + (r-\bar{k})e_{-\beta}$. Now, computing $\sigma[H_s,\ e_\alpha]$ and $\sigma[H_s,\ e_{-\beta}]$ we find $H_s = rH_{s'}$, that is $s=s'$ and $r=1$. Computing σH_γ and $\sigma[H_\gamma,\ H_s]$, we obtain either $H_s = \mathrm{i}H_\gamma$ and $C = \mathrm{i}\frac{k}{\bar{k}}(1-\bar{k})$, $C \neq 0$, or $C=0$ and $k=1$. In the last case, the Cartan subalgebra is fixed by σ, so $\sigma(W+dZ) = W+dZ$. In the first case $s=\mathrm{i}$; computing $\sigma[U,\ W+dZ]$ and $\sigma[V,\ W+dZ]$ one obtains $d=d'$.

Case (3). Proceeding as in Case (2) (since the second derived algebra is given by $\{U,\ V\}$) we get $s=s'$ and $r=1$.

Case (1)$'$. Let's compare $\mathscr{A}'(A, a)$ with $\mathscr{A}'(A', a')$. By $\sigma(H_\alpha - H_\beta) = r(H_\alpha - H_\beta)$ with $r \in \mathbb{R}$, it must be $[\sigma(H_\beta - Ae_\gamma), H_\alpha - H_\beta] = 0$ so $\sigma(H_\beta - Ae_\gamma) = k(H_\beta - A' e_\gamma)$. Computing $\sigma[e_\beta, H_\beta - Ae_\gamma]$ we find two possibilities for σ: or $\sigma e_\beta = le_\beta$ and $k = 1$ or $\sigma e_\beta = l(A' e_\alpha + e_{-\beta})$ and $k = -1$. Now bracket relations involving $V' = Ae_\alpha + e_{-\beta}$ show that l must be real and $A = \pm A'$.

Case (2)$'$. Let's compare $\mathscr{B}'(c, a, s)$ with $\mathscr{B}'(c', a', s')$. If $c = -\frac{1}{3}$ and $s = 1$, then the derived algebra is spanned by $U' + \beta(H_s)V' = U' + V'$. In all the other cases, it is spanned by $\{U', V')$. It is easy to see that, if σ is an equivalence thus

$$\sigma(U') = l(e_\alpha - \beta(H_{s'})\, e_\gamma \qquad \sigma(Z') = r(H_{s'} + e_\alpha + e_{-\beta})$$

with $r\gamma(H_{s'}) = \gamma(H_s)$ unless $s' = \mathrm{i}$ and $r = 2\mathrm{Im}s$. Computing $\sigma[V', Z']$ we get $r\beta(H_{s'}) = \beta(H_s)$ from which we have $s' = s$, $r = 1$ and

$$\sigma(V') = C(e_\alpha - \beta(H_{s'})\, e_\gamma) + D(H_\beta + \frac{1}{\beta(H_{s'})} e_{-\beta} - \beta(H_{s'})\, e_\beta)$$

with $C\alpha(H_s) = l - D$. If $s' = \mathrm{i}$ and $r = 2\mathrm{Im}s$, arguing on σ^{-1} as it has been done on σ, we have $s = s'$ and $\sigma(U') = l(e_\alpha - \beta(H_{s'})\, e_\gamma)$. Computing now $\sigma[U', V']$ and $\sigma[V', W']$ one gets $c = c'$.

For what concerns *Case* (3)$'$, these algebras $\mathscr{C}'(A, a)$ are equivalent to those of *Case* (1)$'$ $\mathscr{A}'(A', a)$ by the following equivalence:

$$e_\alpha \to \frac{1}{\bar{k}} e_\alpha \qquad e_\beta \to k(e_\beta - \frac{m^2}{9} e_{-\beta} + \frac{m}{3} H_\beta - \frac{\overline{m}}{3} e_\gamma - \frac{|m|^2}{9} e_\alpha)$$

$$e_\gamma \to \frac{k}{\bar{k}}(e_\gamma + \frac{m}{3} e_\alpha) \qquad e_{-\alpha} \to \bar{k}(e_{-\alpha} - \frac{\overline{m}^2}{9} e_\alpha - \frac{\overline{m}}{3} H_\alpha - \frac{m}{3} e_{-\gamma} - \frac{|m|^2}{9} e_{-\beta})$$

$$e_{-\beta} \to \frac{1}{k} e_{-\beta} \qquad e_{-\gamma} \to \frac{\bar{k}}{k}(e_{-\gamma} + \frac{\overline{m}}{3} e_{-\beta})$$

$$H_\alpha \to H_\alpha + \frac{2}{3}\overline{m} e_\alpha + \frac{m}{3} e_{-\beta} \qquad H_\beta \to H_\beta - \frac{2}{3} m e_{-\beta} - \frac{\overline{m}}{3} e_\alpha \qquad A' = \frac{k}{\bar{k}} A.$$

Then we can conclude with the following

Theorem. *The inequivalent* CR *structures on* $GL^+(3, \mathbb{R})$ *are given by the following*

$$A(\lambda, s) \qquad |\lambda| < 1 \qquad B(1, s) \qquad B(0, s)$$

$$II(s) \qquad II(b, s) \qquad b \in \mathbb{C} \qquad \mathrm{Re}\, b > 0 \ \text{or}\ \mathrm{Re}\, b = 0 \quad \mathrm{Im}\, b \geqslant 0$$

$$III(s) \qquad III(b, s) \qquad s \in \mathscr{P}.$$

Moreover, for the second group case (ii) *we get for* $a \in \mathbb{R}$ *and* $s \in \mathscr{P}$

$$\mathscr{A}(a, s) \qquad \mathscr{B}(a, d, s) \qquad d \in \mathbb{C} \qquad \mathscr{C}(a, s),$$

$$\mathscr{A}'(A, a) \qquad A \geqslant 0 \quad A \neq 1 \qquad \mathscr{B}'(c, a, s) \qquad c \in \mathbb{C} \cup \infty.$$

The first class $A(\lambda, a)$ of *CR* structures, arising in Case (i) and the families (1) and (1)′ satisfy the following condition: there is a Cartan subalgebra X of $\mathfrak{gl}(3, \mathbb{C})$ and a τ-stable subspace X' of X of codimension 1 such that

$$[q, X'] \subset q \qquad q + \tau q + X = \mathscr{G}.$$

All the other families (2), (3), (2)′, do not satisfy the above condition, which appears in [2] as a natural property in the case of a Lie group of first cathegory.

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Sommario

In questo lavoro viene data una classificazione delle CR *strutture invarianti per traslazioni sinistre su* $(\mathfrak{gl}^+(3, \mathbb{R})$.

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