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Möbius transformations and surface groups (***)

1 - Fuchsian groups

For the theory of Fuchsian groups we refer to [2], [3], [4], [5]. Now we recall some definitions and notations to make the reading clear. Given the complex plane \( C \), we set \( \overline{C} = C \cup \{ \infty \} \) (the 2-sphere). By \( \Delta, \partial \Delta \) we denote the open unit disc and the unit circle in \( C \) respectively.

The hyperbolic plane is the pair \((\Delta, \delta)\), where is the metric on \( \Delta \) derived from the differential

\[
ds = \frac{2|dz|}{1 - |z|^2}, \quad z \in \Delta.
\]

The lines of the hyperbolic plane are the half circles or (euclidean) half lines orthogonal to \( \partial \Delta \). Furthermore the hyperbolic angle is just the euclidean angle.

Given two distinct points \( z_1, z_2 \in \Delta \), let \( z_3, z_4 \) be the limit points on \( \partial \Delta \) of the unique hyperbolic line through \( z_1 \) and \( z_2 \). If the points follow in the order \( z_1, z_2, z_3, z_4 \), then the distance \( \delta(z_1, z_2) \) is given by the formula

\[
\delta(z_1, z_2) = \log [z_4, z_3, z_1, z_2]
\]

where \([z_4, z_3, z_1, z_2]\) represents the cross-ratio of these points.

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Let $G$ be a group of homeomorphisms of $\Delta$ onto itself. We say that $G$ acts discontinuously on $\Delta$ if, for every compact set $K \subset \Delta$, $\mathcal{V}(K) \cap K$ is empty except for a finite number of $\mathcal{V} \in G$.

Two configurations $C$, $C'$ in $\Delta$ (points, curves, regions, etc.) are said to be congruent with respect to $G$ if there exists an element $\mathcal{V} \in G$ such that $\mathcal{V}(C) = C'$. A fundamental region of $G$ is a closed simply connected subset $F \subset \Delta$ such that $\Delta$ is the union of the images $\mathcal{V}(F)$, $\mathcal{V} \in G$, and any point common to $\mathcal{V}_1(F)$ and $\mathcal{V}_2(F)$ (where $\mathcal{V}_1$, $\mathcal{V}_2 \in G$, $\mathcal{V}_1 \neq \mathcal{V}_2$) lies on the boundary of both. All the congruent sets $\mathcal{V}(F)$, $\mathcal{V} \in G$, give a tessellation $\Pi_G$ of $\Delta$. Any curve common to two distinct sets $\mathcal{V}_1(F)$ and $\mathcal{V}_2(F)$ is said to be a side of $\Pi_G$. The vertices of $\Pi_G$ are the end-points of sides of $\Pi_G$. Finally the quotient space $\Delta/G$ is defined as the set of orbits $Gz = \{\mathcal{V}(z)/\mathcal{V} \in G\}$ of points $z \in \Delta$.

A linear fractional transformation is a map $f: \overline{C} \to \overline{C}$ of the form

$$f(z) = \frac{az + b}{cz + d}$$

where $a$, $b$, $c$, $d \in C$ and $ad - bc \neq 0$. The number $ad - bc$ is called the determinant of $f$, written $\det(f)$. Obviously $\det(f)$ can always be assumed equal to 1 if the numerator and denominator of the fraction are divided by $\pm \sqrt{ad - bc}$. The map $f$ is said to be a complex Möbius transformation if $b = \bar{c}$ and $d = \bar{a}$, where $\bar{x}$ is the conjugate of $x \in C$. In the last case, $f$ carries $\Delta$ and $\partial \Delta$ into itself respectively. The group of all complex Möbius transformations is denoted by $\mathbb{M}$.

A Fuchsian group is a subgroup $G$ of $\mathbb{M}$ which acts discontinuously on $\Delta$.

Let now $\Sigma_g$ be the closed connected orientable surface of genus $g$ ($g \geq 2$). It is well-known that $\Sigma_g$ can be obtained as the quotient space $\Delta/G_g$, $G_g$ being a Fuchsian group, named the surface group (see [1], p. 58; [3], p. 200). The group $G_g$ is isomorphic to the fundamental group of $\Sigma_g$. Furthermore $G_g$ induces on $\Delta$ a tessellation $\Pi_g$ whose congruent tiles are regular $4g$-gonal regions with interior angles equal to $\pi/2g$.

In the present paper, we describe a simple geometric construction of the surface group $G_g$ presented in terms of $2g$ complex Möbius transformations as generators and of one relation among them. The relation is directly computed by using the equations of the given generators. Our construction allows us to write the equation of any transformation of the group $G_g$. Furthermore we give a recurrence formula to obtain all the vertices of the tessellation $\Pi_g$. It seems that these facts have, as yet, not been listed in the literature.
2 - Geometric constructions

Given two points \( X, \ Y \in C \) the symbol \( \overline{X Y} \) represents the euclidean distance between \( X \) and \( Y \). In an euclidean triangle \( XYZ \) of \( C \), we denote by \( \overline{X \hat{Y} Z} \) the angle at \( Y \). If \( X \) is a point of the unit circle \( \partial \Delta \), we denote by \( X^* \) its antipodal point. Obviously \( X^{**} = X \).

Let now \( a, \ b \) be two real numbers such that \( a < 0, \ b > 0 \) and \( a^2 + b^2 < 1 \). We consider the following distinct points of \( \Delta \)

\[
A = a + ib \quad A' = -a + ib \quad B = a - ib \quad B' = -a - ib.
\]

The unique complex Möbius transformation \( T \), carrying \( A, \ B, \ 1 \) into \( A', \ B', \ 1 \) respectively, has the equation

\[
T(z) = \frac{\alpha z + \beta}{\beta z + \alpha} \quad z \in \Delta
\]

where \( \alpha, \ \beta \in \mathbb{R} \) (real numbers)\n
\[
\alpha = \frac{a^2 + b^2 + 1}{\sqrt{(a^2 + b^2 + 1)^2 - 4a^2}} \quad \beta = \frac{2a}{\sqrt{(a^2 + b^2 + 1)^2 - 4a^2}}.
\]

Let \( \mathcal{G}_1 \) be the euclidean circle in \( C \) with center \( C_1 = -\varphi \in \mathbb{R} \ (\varphi > 1) \) and radius \( \sqrt{\varphi^2 - 1} \), i.e. the equation of \( \mathcal{G}_1 \) is

\[
z\bar{z} + \varphi z + \varphi \bar{z} + 1 = 0.
\]

For a fixed integer \( g \ (g \geq 2) \), let \( \mathcal{G}_2 \) be the euclidean circle in \( C \) with center

\[C_2 = -\varphi \cos(\pi/2g) - \varphi i \sin(\pi/2g)\]

and radius \( \sqrt{\varphi^2 - 1} \), i.e. the equation of \( \mathcal{G}_2 \) is

\[
z\bar{z} + \varphi e^{-(\pi/2g)i} z + \varphi e^{(\pi/2g)i} \bar{z} + 1 = 0.
\]

Obviously \( \mathcal{G}_1, \ \mathcal{G}_2 \) are orthogonal to the unit circle \( \partial \Delta \) (\( z\bar{z} = 1 \)). Now we assume that the point \( B \) belongs to the intersection \( \mathcal{G}_1 \cap \mathcal{G}_2 \). As a direct consequence, we can express \( \varphi, a, \ b \) as functions of \( g \).

If \( \partial \) is the origin of \( C \), let \( \partial B \) be the straight half line in \( C \) beginning at \( \partial \) and passing through \( B \). We denote by \( H \) the intersection point between \( \partial B \) and the straight line with ends \( C_1 \) and \( C_2 \). The triangles \( C_1 \partial H, \ C_1 BH, \ C_2 \partial H \) and \( C_2 BH \) have a right angle at \( H \). We compare the triangles \( C_1 \partial H \) and \( C_1 BH \).
which have the side $C_1H$ in common. Then we have

(1) \[ \overline{C_1H} = \sqrt{\rho^2 - 1} \cos(\pi/4g) \]

in the triangle $C_1BH$ and

(2) \[ \overline{C_1H} = \rho \sin(\pi/4g) \]

in the triangle $C_1\mathcal{O}H$. Equating (1) and (2) and using the formula \( \cos(\pi/2g) = \cos^2(\pi/4g) - \sin^2(\pi/4g) \), we can determine $\rho$ uniquely as a function of the fixed integer $g$, i.e.

(3) \[ \rho = \frac{\cos(\pi/4g)}{\sqrt{\cos(\pi/2g)}}. \]

By simple geometric arguments, the following relations are also true:

\[
\begin{align*}
C_1 \mathcal{O}H &= C_2 \mathcal{O}H = H\mathcal{C}_1B = H\mathcal{C}_2B = \pi/4g \\
(\sideset{\h}{/} C_1 \mathcal{O} \mathcal{C}_2 &= \pi/2g \quad C_1 \mathcal{B} \mathcal{C}_2 = \pi - \pi/2g \\
H\mathcal{C}_1 \mathcal{O} &= H\mathcal{C}_2 \mathcal{O} = \pi/2 - \pi/4g
\end{align*}
\]

(\sideset{\h}{/} \overline{\mathcal{O}H} = \rho \cos(\pi/4g) \quad \overline{HB} = \sqrt{\rho^2 - 1} \sin(\pi/4g) \]

(\sideset{\h}{/} \overline{\mathcal{O}B} = \overline{\mathcal{O}H} - \overline{HB} = \sqrt{\cos(\pi/2g)} \]

(\sideset{\h}{/} \overline{C_1B} = \overline{C_2B} = \text{radius of } \mathcal{G}_1(G_2) = \frac{\sin(\pi/4g)}{\sqrt{\cos(\pi/2g)}}. \]

This implies that

(4) \[ a = -\overline{\mathcal{O}B} \cos(\pi/4g) = -\sqrt{\cos(\pi/2g)} \cos(\pi/4g) \]

(5) \[ b = \overline{\mathcal{O}B} \sin(\pi/4g) = +\sqrt{\cos(\pi/2g)} \sin(\pi/4g). \]

If $K$ is the intersection point between $\mathcal{G}_1$ and the real axes, i.e.

\[ K = -\frac{\cos(\pi/4g) - \sin(\pi/4g)}{\sqrt{\cos(\pi/4g) + \sin(\pi/4g)}} = -\sqrt{\cotg(\pi/4g + \pi/4)} \]

it follows that $T(K) = -K$. Moreover $T$ carries each point $z = \gamma + i\epsilon \in \mathcal{G}_1$ into its
symmetric $z' = -\gamma + i\epsilon \in G_1'$ with respect to the imaginary axes, $G_1'$ being the circle of equation $\bar{z}z - \rho z - \bar{\rho}z + 1 = 0$ (see fig. 1).

![Diagram](image)

**Fig. 1.**

Using (4) and (5), we also have

$$\alpha/\beta = \frac{a^2 + b^2 + 1}{-2a} = \frac{1 + \cos(\pi/2g)}{2\sqrt{\cos(\pi/2g) \cos(\pi/4g)}} = \rho$$

since $1 + \cos(\pi/2g) = 2\cos^2(\pi/4g)$. Thus we can write the equation of $T$ as follows

$$T(z) = \frac{\rho z + 1}{z + \rho} \quad z \in \Delta.$$

Let now $P_g$ be the regular $4g$-gonal region in $\Delta$ whose internal angles are
equal to $\pi/2g$ and whose vertices are the following

\begin{equation}
B_k = -\sqrt{\cos(\pi/2g)} \cos\left(\frac{(2k + 1)}{4g} \pi\right) - i \sqrt{\cos(\pi/2g)} \sin\left(\frac{(2k + 1)}{4g} \pi\right) \\
(k \in Z, -2g < k \leq 2g).
\end{equation}

Obviously $B_{-1} = A$, $B_0 = B$, $B_{2g-1} = E'$ and $B_{2g} = A'$. The $2g$ complex Möbius transformations (also named *translations*) of $\Delta$, each of them carrying one face of $P_g$ into its opposite face along lines joining the mid-points of pairs of opposite sides, are

\begin{equation}
T_h = R_{-\kappa/2g} TR_{h\kappa/2g} \quad (h = 1, 2, \ldots, 2g)
\end{equation}

where $R_\theta(z) = e^{i\theta}z$ is the rotation around the origin $0$ by the oriented angle $\theta$. Here we adopt the convention that $UV$ means «apply first $V$, then $U$». Thus we have

\begin{equation}
T_h(z) = \frac{e^z + e^{(-h/2g)i}}{e^{h/2g}i} z + \rho \\
z \in \Delta \quad \det(T_h) = \rho^2 - 1.
\end{equation}

The surface group $G_g$ is the subgroup of $M$ generated by $T_1, T_2, \ldots, T_{2g}$. The regular $4g$-gonal region $P_g$ is a fundamental region of $G_g$. The set of all the images $S(P_g), S \in G_g$, gives a regular tessellation $\Pi_g$ of type $(4g, 4g)$ in $\Delta$ (also compare [1], p.52). There are exactly $4g$ tiles $S(P_g), S \in G_g$, at each vertex of $\Pi_g$. The area of $P_g$ is $4(g-1)\pi$. The quotient space $\Delta/G_g = P_g/G_g$ is the closed connected orientable surface $\Sigma_g$ of genus $g$, which is triangulated by a complex with one vertex, $2g$ edges and one face. Furthermore $G_g$ is the fundamental group of $\Sigma_g$. The relations of $G_g$ are obtained by running around each vertex of $P_g$. By simmetry we have exactly one relation which is studied in the next section.

The transformation $T_h$ ($h = 1, 2, \ldots, 2g$) has two fixed points: $\xi_{1,h} = e^{(-h/2g)i}$, $\xi_{2,h} = -e^{(-h/2g)i}$ such that $\xi_{j,h} \in \partial \Delta$ ($j = 1, 2$) and $\xi_{1,h}^* = \xi_{2,h}$. Thus the equation of $T_h$ may be written as follows

\begin{equation}
\frac{z' - \xi_{1,h}}{z' - \xi_{2,h}} = K(h) \frac{z - \xi_{1,h}}{z - \xi_{2,h}}
\end{equation}
where

\[ z' = T_h(z) \quad K(h) = \frac{\rho - e^{i(\rho n/2g)^i} \xi_{1,h}}{\rho - e^{i(\rho n/2g)^i} \xi_{2,h}} \]

is the multiplier of \( T_h \) (see [2], p. 15). The value of \( K(h) \) determines the character of the transformation, i.e. \( T_h \) is a hyperbolic transformation since \( K(h) = (\rho - 1)/(\rho + 1) \) is real (see [2], p. 18). Note that the transformation \( T^n_h \) (\( T_h \) repeated \( n \) times) has equation

\[ \frac{z' - \xi_{1,h}}{z' - \xi_{2,h}} = \left( \frac{\rho - 1}{\rho + 1} \right)^n \frac{z - \xi_{1,h}}{z - \xi_{2,h}} \]

whence

\[ T^n_h(z) = \frac{((1 + \rho)^n + (\rho - 1)^n)z + (1 + \rho)^n e^{i(\rho n/2g)^i} - (\rho - 1)^n e^{-i(\rho n/2g)^i}}{((1 + \rho)^n e^{i(\rho n/2g)^i} - (\rho - 1)^n e^{-i(\rho n/2g)^i})z + (1 + \rho)^n + (\rho - 1)^n} \]

Obviously \( \lim_n T^n_h = \text{identity since} \lim_n K^n(h) = 0 \).

The isometric circle \( I_h \) (see [2], p. 23) of \( T_h \) (i.e. the complete locus of points in the neighbourhood of which lengths and areas are unaltered in magnitude) is given by the following equation

\[ I_h: \frac{e^{i(\rho n/2g)^i}}{\rho^2 - 1} z + \frac{\rho}{\rho^2 - 1} = 1 \]

and therefore

\[ I_h: \bar{z}z + \rho e^{i(\rho n/2g)^i}z + \rho e^{-i(\rho n/2g)^i} \bar{z} + 1 = 0. \]

Thus \( I_h \) is the circle with center \(-\rho e^{-i(\rho n/2g)^i}\) and radius \(\rho^2 - 1\). The isometric circle \( I'_h \) of the inverse transformation \( T_h^{-1} \) is given by

\[ I'_h: \frac{e^{i(\rho n/2g)^i}}{\rho^2 - 1} z - \frac{\rho}{\rho^2 - 1} = 1 \]

whence

\[ I'_h: \bar{z}z - \rho e^{i(\rho n/2g)^i}z - \rho e^{-i(\rho n/2g)^i} \bar{z} + 1 = 0. \]

Thus \( I'_h \) has center \(\rho e^{i(\rho n/2g)^i}\) and radius \(\rho^2 - 1\). The equation of the straight line
through the centers of the isometric circles is

\[ e^{(h/r/2g)}i_z - e^{(-h/2g)}i_z = 0. \]

Let \( L \) be the straight line orthogonal to the line (9) and containing the mid-point between the centers of \( I_h \) and \( I_h'. \) Then the equation of \( L \) is

\[ e^{(h/r/2g)}i_z + e^{(-h/2g)}i_z = 0. \]

Let \( J_{I_h} \) and \( J_L \) be the inversions (see [2], p. 10) in \( I_h \) and \( L \) respectively, i.e.

\[ J_{I_h}(z) = \frac{e^{(h/r/2g)}i_z - 1}{z + e^{(h/r/2g)}i_z} \quad J_L(z) = -e^{(-h/2g)}i_z. \]

Then the transformation \( T_h \) is just the composition \( J_LJ_{I_h} \), i.e. it is equivalent to an inversion in \( I_h \) followed by a reflection in \( L \).

3 - The relation

The relation of the group \( G_g \) is the following

\[ T_1 T_2^{-1} \ldots T_{2g-1} T_{2g}^{-1} T_1 T_2 \ldots T_{2g-1} T_{2g} = I \]

where \( I \) is the identity of \( \Delta \). Here we directly compute (10) by using the equations of the generators \( T_1, T_2, \ldots, T_{2g} \) of \( G_g \) (see (7), (8)). Substituting in the left side of (10)

\[ T_h = R_{-h/2g} TR_{h/2g} \quad T^{-1} = R_\pi TR_\pi \]

we obtain

\[ R_{-\pi/2g} TR_{\pi/2g} R_{-2\pi/2g} R_\pi TR_\pi R_{2\pi/2g} \ldots R_{-2g-1/2g} R_\pi TR_\pi R_{2g-1/2g} R_{-2g} TR_{2g} R_{-2g} \]

\[ = R_{-\pi/2g} (TR_{\pi/2g})^{2g} R_{\pi/2g}. \]

The fixed points of the linear fractional transformation

\[ S(z) = (TR_{\pi/2g})(z) = \frac{-e^{(-\pi/2g)}i_z + 1}{-e^{(-\pi/2g)}i_z + -1} \]

are

\[ \eta_1 = \frac{1}{\sqrt{\cos(\pi/2g)}} e^{(\pi/4g)i} \quad \eta_2 = \sqrt{\cos(\pi/2g)} e^{(\pi/4g)i}. \]
Thus the multiplier $K_S$ of $S$ is given by

$$K_S = \frac{\rho - \eta_1}{\rho - \eta_2} = \frac{\rho - \frac{\sqrt{\cos(\pi/2g)}}{e^{(\epsilon/4g)i}}}{\rho - \frac{1}{e^{(\epsilon/4g)i}}} = e^{(\epsilon/2g)i},$$

whence $S$ is an elliptic transformation (see [2], p. 19). The transformation $S^{4g}$ ($S$ repeated 4 times) has equation

$$\frac{z' - \eta_1}{z' - \eta_2} = K_S^{4g} \frac{z - \eta_1}{z - \eta_2} = z' = S^{4g}(z)$$

so that $S^{4g} = I$ (identity) since $K_S^{4g} = e^{2\pi} = 1$. This proves the formula (10).

The isometric circle $I_S$ of $S$ is given by

$$I_S: \left| \frac{-e^{-\pi(\epsilon/2g)i}z + \rho}{(-\rho^2 + 1)e^{-\pi(\epsilon/2g)i}} \right| = 1$$

whence

$$I_S: z\bar{z} - \rho e^{(-\pi/2g)i}z - \rho e^{(\pi/2g)i}\bar{z} + 1 = 0.$$  

Thus $I_S$ is the circle with center $\rho e^{(\pi/2g)i}$ and radius $\rho^2 - 1$. Let now $J_{lS}$ and $J_l$ be the inversions in $I_S$ and $l$ (straight line of equation $e^{(\pi/2g)i}z + e^{(\pi/2g)i}\bar{z} = 0$) respectively, i.e.

$$J_{lS}(z) = \frac{\rho e^{(\pi/2g)i}\bar{z} - 1}{\bar{z} - \rho e^{(-\pi/2g)i}} \quad J_l(z) = \bar{z} e^{(\pi/2g)i}. $$

Then the elliptic transformation $S$ is equivalent to an inversion in $I_S$ followed by a reflection in $l$, i.e. $S = J_lJ_{lS}$.

4 - The composition law

In order to obtain the equation of an arbitrary transformation of the group $G_g$, we shall now consider the successive performance of the complex Möbius transformations $T_1, T_2, \ldots, T_{2g}$. First we introduce some notations to simplify the writing of the formulae. We use the symbol $[a_1 \; \; a_2]$ to represent the most
general transformation

\[ z' = \frac{\alpha_1 z + \bar{\alpha}_2}{\alpha_2 z + \bar{\alpha}_1} \quad |\alpha_1|^2 - |\alpha_2|^2 \neq 0 \]

which carries \( \partial \Delta \) and \( \Delta \) onto itself respectively. Thus, for each \( h = 1, 2, \ldots, 2g \), we set

\[ T_h = [\rho; e^{(h\pi/2g)i}] \]

Let now \( C^u_t (t \leq n) \) be the set of all the simple combinations of \( n \) elements taken \( t \) to \( t \). We use the notation \( \sum_{q_1, q_2, \ldots, q_t} \) to represent a sum taken over all the simple combinations \( (q_1, q_2, \ldots, q_t) \in C^u_t \), where \( q_1 < q_2 < \ldots < q_t \). Then we have

\[ T_{h_1, h_2}(z) = T_{h_1} T_{h_2}(z) = \frac{(\rho^2 + \rho e^{(h_2 - h_1)\pi/2g}i) z + \rho e^{(-h_1\pi/2g)i} + \rho e^{(-h_2\pi/2g)i}}{(\rho e^{(h_1\pi/2g)i} + \rho e^{(h_2\pi/2g)i}) z + \rho e^{(h_1 - h_2)\pi/2g}i + \rho^2} \]

whence

\[ T_{h_1, h_2} = [\rho^2 + \rho e^{(h_2 - h_1)\pi/2g}i; \rho e^{(h_1\pi/2g)i} + \rho e^{(h_2\pi/2g)i}] \].

Going on like this, we easily obtain

\[ T_{h_1, \ldots, h_{2p}} = [\rho^{2p} + \rho^{2p-2} \left( \sum_{s, r} e^{(h_r - h_s)\pi/2g}i \right) + \ldots + \rho^2 \left( \sum_{q_1, \ldots, q_{2p}} e^{(h_{q_1} - h_{q_2})\pi/2g}i \right); \]

\[ \rho^{2p-1} \left( \sum_{r} e^{(h_r\pi/2g)i} \right) + \rho^{2p-3} \left( \sum_{s, r, t} e^{(h_r - h_s + h_t)\pi/2g}i \right) \]

\[ + \ldots + \rho \left( \sum_{q_1, \ldots, q_{2p-1}} e^{(h_{q_1} - h_{q_2} + \ldots + h_{2p-1})\pi/2g}i \right) \].

This also yields the formula for the odd case since \( T_{h_1, \ldots, h_{2p+1}} = T_{h_1, \ldots, h_{2p}} T_{h_{2p+1}} \).

5 - The tessellation \( \Pi_g \)

In this section we give a recurrence formula to obtain all the vertices of the regular tessellation \( \Pi_g \) induced on \( \Delta \) by the group \( G_g \). Let us consider the ver-
MÖBIUS TRANSFORMATIONS AND SURFACE GROUPS

\[ B_{2g} = \sqrt{\cos(\pi/2g)} \cos(\pi/4g) + i \sqrt{\cos(\pi/2g)} \sin(\pi/4g). \]

\[ B_{2g-1} = \sqrt{\cos(\pi/2g)} \cos(\pi/4g) - i \sqrt{\cos(\pi/2g)} \sin(\pi/4g) \]

Obviously \( B_{2g} \) and \( B_{2g-1} \) belong to the circle \( O_1 \) whose equation is (see fig. 1)

\[ z \zeta - \rho z - \rho \zeta + 1 = 0. \]

If \( x \) is the real part of \( z \) and \( iy \) is its imaginary part, then \( O_1 \) is also represented by the equation

\[ x^2 + y^2 - 2\rho x + 1 = 0. \]

Let now \( X, Y \) be the points in which the circle \( O_1 \) intersects the unit circle \( \mathbb{A} \) \((\zeta z = 1)\), i.e.

\[ X = \frac{\sqrt{\cos(\pi/2g)}}{\cos(\pi/4g)} + i \tan(\pi/4g) \]
\[ Y = \frac{\sqrt{\cos(\pi/2g)}}{\cos(\pi/4g)} - i \tan(\pi/4g). \]

Now we calculate the value of the cross-ratio

\[ [X, Y, B_{2g}, B_{2g-1}] = \frac{B_{2g} - X}{B_{2g} - Y} \cdot \frac{B_{2g-1} - Y}{B_{2g-1} - X}. \]

We have

\[ [X, Y, B_{2g}, B_{2g-1}] \]

\[ \frac{[\sqrt{\cos(\pi/2g)} \cos(\pi/4g) - \sqrt{\cos(\pi/2g)} \sin(\pi/4g) - \tan(\pi/4g)]^2}{[\sqrt{\cos(\pi/2g)} \cos(\pi/4g) - \sqrt{\cos(\pi/2g)} \sin(\pi/4g) + \tan(\pi/4g)]^2} \]

\[ = \frac{\cos(\pi/4g) - \sqrt{\cos(\pi/2g)}}{\cos(\pi/4g) + \sqrt{\cos(\pi/2g)}} = \frac{\rho - 1}{\rho + 1} \]

since \( \cos(\pi/2g) + 1 = 2 \cos^2(\pi/4g) \) and \( \rho \) satisfies (3).
Let now $X_k$ be the point of the circle $G'_1$ such that
\[
\delta(X_k, B_{2g}) = k \delta(B_{2g-1}, B_{2g})
\]
where $\delta(A, B)$ denotes the hyperbolic distance between the points $A, B \in \Delta$. Since the Möbius transformations of the surface group $G_\rho$ preserve the hyperbolic distance, an arbitrary vertex of $\Pi_\rho \cap G'_1$ must be a point $X_k$ for a suitable integer $k$. Then we have
\[
\delta(X_k, B_{2g}) = \log |X, Y, X_k, B_{2g}| = k \delta(B_{2g-1}, B_{2g})
= k \log |X, Y, B_{2g}, B_{2g-1}| = \log |X, Y, B_{2g}, B_{2g-1}|^k
\]
whence
\[
|X, Y, X_k, B_{2g}| = |X, Y, B_{2g}, B_{2g-1}|^k = (\frac{\rho - 1}{\rho + 1})^k
\]
\[
\frac{X_k - X}{X_k - Y} = (\frac{\rho - 1}{\rho + 1})^k \frac{B_{2g} - X}{B_{2g} - Y}.
\]

By an algebraic calculation and making use of some trigonometric formulae, we can obtain the point $X_k = x_k + i y_k$ expressed in terms of $\rho$ and $k$. The real part $x_k$ of $X_k$ is given by the following formula
\[
(12) \quad x_k = \frac{(\rho - 1)^{2k+1} + (\rho + 1)^{2k+1}}{\rho[(\rho - 1)^{2k+1} + (\rho + 1)^{2k+1}] + 2(\rho + 1)^k(\rho - 1)^k+1}.
\]

Since $X_k$ belongs to $G'_1$, the pair $(x_k, y_k)$ satisfies (11). Thus we find easily that
\[
(13) \quad y_k = \frac{\sqrt{\rho^2 - 1}[(\rho + 1)^{2k+1} - (\rho - 1)^{2k+1}]}{\rho[(\rho - 1)^{2k+1} + (\rho + 1)^{2k+1}] + 2(\rho + 1)^k(\rho - 1)^k+1}.
\]

Putting $\sigma = (\rho - 1)/(\rho + 1)$, it follows that $\rho = (1 + \sigma)/(1 - \sigma)$, $\rho - 1 = (2\sigma)/(1 - \sigma)$, $\rho + 1 = 2/(1 - \sigma)$ and $\sqrt{\rho^2 - 1} = (2\sqrt{\sigma})/(1 - \sigma)$. Substituting these relations in (14) and (15), we obtain
\[
(12)' \quad x_k = \frac{(1 - \sigma)(1 + \sigma^{2k+1})}{(1 + \sigma)(1 + \sigma^{2k+1}) + 4 \sigma^{k+1}}
\]
\[
(13)' \quad y_k = \frac{2 \sqrt{\sigma} (1 - \sigma^{2k+1})}{(1 + \sigma)(1 + \sigma^{2k+1}) + 4 \sigma^{k+1}}
\]
\[
\sigma = \frac{\cos(\pi/4g) - \sqrt{\cos(\pi/2g)}}{\cos(\pi/4g) + \sqrt{\cos(\pi/2g)}}.
\]

Now we give a recurrence formula to obtain all the centers of the circles orthogonal to \( \partial \Delta \) and containing sides of \( II_g \).

Let \( G \) be a circle which is orthogonal to the unit circle \( \partial \Delta \). The equation of \( G \) has the form

\[
z\overline{z} - Cz - \overline{C} + 1 = 0
\]

where \( C \) is a complex number. This circle has center \( C \) and radius \( \sqrt{|C|^2 - 1} \).

Since we shall be interested only in real circles, we shall require that \( |C|^2 > 1 \). Let now \( G'_h \) be the circle obtained from \( G \) when the complex Möbius transformation \( T_h \) (see (8)) is applied.

Substituting in (15) the relations

\[
z = T_h^{-1}(z') = \frac{-\overline{\rho}z' + e^{(-h+2g)i}z'}{e^{(h_2/2g)i}z' - \rho} \quad \overline{z} = \frac{-\overline{\rho}z' + e^{(h-2g)i}}{e^{(-h+2g)i}z' - \rho}
\]

the equation of \( G'_h \) is

\[
(\rho^2 + \overline{C}e^{(-h+2g)i} + Ce^{(h+2g)i} + 1)z'\overline{z} - (2\overline{\rho}e^{(h+2g)i} + \overline{C}\rho^2 + C e^{(h+2g)i})z' + (2\rho e^{(-h+2g)i} + C e^{(-h+2g)i} + C \rho e^{(h+2g)i} + 1 = 0.
\]

The center of \( G'_h \) is given by the following recurrence formula

\[
C'_h = \frac{2\overline{\rho} e^{(-h+2g)i} + \overline{C}e^{(-h+2g)i} + C\rho^2}{1 + C e^{(h+2g)i} + \overline{C}e^{(-h+2g)i} + \overline{\rho}^2}.
\]

In particular, let now \( G_k \) be a circle which contains a side of the fundamental region \( P_g \) of \( G_g \) (\( k \in \mathbb{Z}, -2g < k \leq 2g \)). The equation of \( G_k \) is

\[
z\overline{z} - \rho e^{(-h+2g)i}z - \rho e^{(h-2g)i}z + 1 = 0.
\]

Obviously \( G_k \) has center \( C_k = \rho e^{(h+2g)i} \) and radius \( \sqrt{\rho^2 - 1} \). Using (16), the center \( C_{h,k} \) of \( G'_{h,k} = T_h(G_k) \) is given by the following formula

\[
C'_{h,k} = \frac{\rho^3 e^{(h-2g)i} + 2\rho e^{(-h+2g)i} + \rho e^{(-h+2k-2g)i} + \rho e^{((h+2k-2g)i} + \rho^2}{1 + \rho^2 e^{(-h+k-2g)i} + \rho^2 e^{(h+k-2g)i} + \rho^2}.
\]
so that the equation $G_{h, k}'$ is
\[
(\rho^2 + \rho^2 e^{(-h + k)\pi/2g}i + \rho^2 e^{((h + k)\pi/2g)i} + 1) z' \bar{z}' \\
-(2\rho e^{(h\pi/2g)i} + \rho^2 e^{(-k\pi/2g)i} + \rho e^{(-k + 2h)\pi/2g}i) z' \\
-(2\rho e^{(-h\pi/2g)i} + \rho^2 e^{(k\pi/2g)i} + \rho e^{(-(k + 2h)\pi/2g)i}) \bar{z}' + 1 \\
+\rho^2 e^{(-(h + k)\pi/2g)i} + \rho^2 e^{((h + k)\pi/2g)i} + \rho^2 = 0.
\]

Relations (12)', (13)', (14), (16), (16)' allow us to study by recurrence the distribution of all vertices of the regular tessellation $\Pi_g$. 

References


Riassunto

Si descrive una semplice costruzione geometrica dei classici gruppi delle superfici. Come conseguenza, si determina l'equazione di una qualsiasi trasformazione di questi gruppi. Infine, si ottengono formule ricorrenti per rappresentare, mediante computer, le pavimentazioni regolari indotte dai suddetti gruppi.

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