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ALBERTO CAVICCHIOLI (*)

Möbius transformations and surface groups (**)

1 - Fuchsian groups

For the theory of Fuchsian groups we refer to [2], [3], [4], [5]. Now we recall some definitions and notations to make the reading clear. Given the complex plane $\boldsymbol{C}$, we set $\overline{\boldsymbol{C}} = \boldsymbol{C} \cup \{\infty\}$ (the 2-sphere). By Δ, $\partial\Delta$ we denote the *open unit disc* and the *unit circle* in $\boldsymbol{C}$ respectively.

The *hyperbolic plane* is the pair (Δ, δ), where is the metric on Δ derived from the differential

$$ds = \frac{2|dz|}{1-|z|^2} \qquad z \in \Delta .$$

The *lines* of the hyperbolic plane are the half circles or (euclidean) half lines orthogonal to $\partial\Delta$. Furthermore the hyperbolic angle is just the euclidean angle.

Given two distinct points z_1, $z_2 \in \Delta$, let z_3, z_4 be the limit points on $\partial\Delta$ of the unique hyperbolic line through z_1 and z_2. If the points follow in the order z_1, z_2, z_3, z_4, then the distance $\delta(z_1, z_2)$ is given by the formula

$$\delta(z_1, z_2) = \log[z_4, z_3, z_1, z_2]$$

where $[z_4, z_3, z_1, z_2]$ represents the cross-ratio of these points.

(*) Indirizzo: Istituto di Matematica, Università della Basilicata, Via N. Sauro 85, I-85100 Potenza.

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Let G be a group of homeomorphisms of Δ onto itself. We say that G *acts discontinuously on* Δ if, for every compact set $K \subset \Delta$, $\Psi(K) \cap K$ is empty except for a finite number of $\Psi \in G$.

Two configurations C, C' in Δ (points, curves, regions, etc.) are said to be *congruent with respect to* G if there exists an element $\Psi \in G$ such that $\Psi(C) = C'$. A *fundamental region* of G is a closed simply connected subset $F \subset \Delta$ such that Δ is the union of the images $\Psi(F)$, $\Psi \in G$, and any point common to $\Psi_1(F)$ and $\Psi_2(F)$ (where $\Psi_1, \Psi_2 \in G$, $\Psi_1 \neq \Psi_2$) lies on the boundary of both. All the congruent sets $\Psi(F)$, $\Psi \in G$, give a *tessellation* Π_G of Δ. Any curve common to two distinct sets $\Psi_1(F)$ and $\Psi_2(F)$ is said to be a *side* of Π_G. The *vertices* of Π_G are the end-points of sides of Π_G. Finally the *quotient space* Δ/G is defined as the set of orbits $Gz = \{\Psi(z)/\Psi \in G\}$ of points $z \in \Delta$.

A *linear fractional transformation* is a map $f\colon \overline{C} \to \overline{C}$ of the form

$$f(z) = \frac{az+b}{cz+d}$$

where $a, b, c, d \in C$ and $ad - bc \neq 0$. The number $ad - bc$ is called the *determinant* of f, written $\det(f)$. Obviously $\det(f)$ can always be assumed equal to 1 if the numerator and denominator of the fraction are divided by $\pm\sqrt{ad-bc}$. The map f is said to be a *complex Möbius transformation* if $b = \overline{c}$ and $d = \overline{a}$, where $\overline{x}$ is the conjugate of $x \in C$. In the last case, f carries Δ and $\partial\Delta$ into itself respectively. The group of all complex Möbius transformations is denoted by $\mathcal{M}$.

A *Fuchsian group* is a subgroup G of $\mathcal{M}$ which acts discontinuously on Δ.

Let now Σ_g be the closed connected orientable surface of genus g ($g \geqslant 2$). It is well-known that Σ_g can be obtained as the quotient space Δ/G_g, G_g being a Fuchsian group, named the *surface group* (see [1], p. 58; [3], p. 200). The group G_g is isomorphic to the fundamental group of Σ_g. Furthermore G_g induces on Δ a tessellation Π_g whose congruent tiles are regular $4g$-gonal regions with interior angles equal to $\pi/2g$.

In the present paper, we describe a simple geometric construction of the surface group G_g presented in terms of $2g$ complex Möbius transformations as generators and of one relation among them. The relation is directly computed by using the equations of the given generators. Our construction allows us to write the equation of any transformation of the group G_g. Furthermore we give a recurrence formula to obtain all the vertices of the tessellation Π_g. It seems that these facts have, as yet, not been listed in the literature.

2 - Geometric constructions

Given two points $X,\ Y \in \boldsymbol{C}$ the symbol $\overline{XY}$ represents the euclidean distance between X and Y. In an euclidean triangle XYZ of $\boldsymbol{C}$, we denote by $X\hat{Y}Z$ the angle at Y. If X is a point of the unit circle $\partial\Delta$, we denote by X^* its antipodal point. Obviously $X^{**} = X$.

Let now $a,\ b$ be two real numbers such that $a < 0$, $b > 0$ and $a^2 + b^2 < 1$. We consider the following distinct points of Δ

$$A = a + ib \qquad A' = -a + ib \qquad B = a - ib \qquad B' = -a - ib .$$

The unique complex Möbius transformation T, carrying $A,\ B,\ 1$ into $A',\ B',\ 1$ respectively, has the equation

$$T(z) = \frac{\alpha z + \beta}{\beta z + \alpha} \qquad z \in \Delta$$

$$\alpha,\ \beta \in \boldsymbol{R} \ \text{(real numbers)} \qquad \alpha^2 - \beta^2 = 1$$

$$\alpha = \frac{a^2 + b^2 + 1}{\sqrt{(a^2 + b^2 + 1)^2 - 4a^2}} \qquad \beta = \frac{-2a}{\sqrt{(a^2 + b^2 + 1)^2 - 4a^2}} .$$

Let $\mathcal{C}_1$ be the euclidean circle in $\boldsymbol{C}$ with center $C_1 = -\rho \in \boldsymbol{R}$ $(\rho > 1)$ and radius $\sqrt{\rho^2 - 1}$, i.e. the equation of $\mathcal{C}_1$ is

$$z\bar{z} + \rho z + \rho\bar{z} + 1 = 0 .$$

For a fixed integer $g\,(g \geq 2)$, let $\mathcal{C}_2$ be the euclidean circle in $\boldsymbol{C}$ with center

$$C_2 = -\rho \cos(\pi/2g) - \rho i \sin(\pi/2g)$$

and radius $\sqrt{\rho^2 - 1}$, i.e. the equation of $\mathcal{C}_2$ is

$$z\bar{z} + \rho\, e^{-(\pi/2g)\,i} z + \rho\, e^{(\pi/2g)\,i} \bar{z} + 1 = 0 .$$

Obviously $\mathcal{C}_1$, $\mathcal{C}_2$ are orthogonal to the unit circle $\partial\Delta$ $(z\bar{z} = 1)$. Now we assume that the point B belongs to the intersection $\mathcal{C}_1 \cap \mathcal{C}_2$. As a direct consequence, we can express $\rho, a,\ b$ as functions of g.

If $\mathcal{O}$ is the origin of $\boldsymbol{C}$, let $\mathcal{O}B$ be the straight half line in $\boldsymbol{C}$ beginning at $\mathcal{O}$ and passing through B. We denote by H the intersection point between $\mathcal{O}B$ and the straight line with ends C_1 and C_2. The triangles $C_1\mathcal{O}H$, C_1BH, $C_2\mathcal{O}H$ and C_2BH have a right angle at H. We compare the triangles $C_1\mathcal{O}H$ and C_1BH

which have the side C_1H in common. Then we have

$$\overline{C_1H} = \sqrt{\rho^2 - 1}\cos(\pi/4g) \tag{1}$$

in the triangle C_1BH and

$$\overline{C_1H} = \rho \sin(\pi/4g) \tag{2}$$

in the triangle $C_1\mathcal{O}H$. Equating (1) and (2) and using the formula $\cos(\pi/2g) = \cos^2(\pi/4g) - \sin^2(\pi/4g)$, we can determine ρ uniquely as a function of the fixed integer g, i.e.

$$\rho = \frac{\cos(\pi/4g)}{\sqrt{\cos(\pi/2g)}}\,. \tag{3}$$

By simple geometric arguments, the following relations are also true:

$$\begin{aligned} &C_1\hat{\mathcal{O}}H = C_2\hat{\mathcal{O}}H = H\hat{C}_1B = H\hat{C}_2B = \pi/4g \\ &C_1\hat{\mathcal{O}}C_2 = \pi/2g \qquad C_1\hat{B}C_2 = \pi - \pi/2g \\ &H\hat{C}_1\mathcal{O} = H\hat{C}_2\mathcal{O} = \pi/2 - \pi/4g \end{aligned} \tag{/}$$

$$\begin{aligned} &\overline{\mathcal{O}H} = \rho\cos(\pi/4g) \qquad \overline{HB} = \sqrt{\rho^2-1}\sin(\pi/4g) \\ &\overline{\mathcal{O}B} = \overline{\mathcal{O}H} - \overline{HB} = \sqrt{\cos(\pi/2g)} \\ &\overline{C_1B} = \overline{C_2B} = \text{radius of } \mathcal{C}_1(\mathcal{C}_2) = \frac{\sin(\pi/4g)}{\sqrt{\cos(\pi/2g)}}\,. \end{aligned} \tag{//}$$

This implies that

$$a = -\overline{\mathcal{O}B}\cos(\pi/4g) = -\sqrt{\cos(\pi/2g)}\cos(\pi/4g) \tag{4}$$

$$b = \overline{\mathcal{O}B}\sin(\pi/4g) = +\sqrt{\cos(\pi/2g)}\sin(\pi/4g)\,. \tag{5}$$

If K is the intersection point between $\mathcal{C}_1$ and the real axes, i.e.

$$K = -\sqrt{\frac{\cos(\pi/4g) - \sin(\pi/4g)}{\cos(\pi/4g) + \sin(\pi/4g)}} = -\sqrt{\operatorname{cotg}(\pi/4g + \pi/4)}$$

it follows that $T(K) = -K$. Moreover T carries each point $z = \gamma + i\varepsilon \in \mathcal{C}_1$ into its symmetric $z' = -\gamma + i\varepsilon \in \mathscr{C}_1'$ with respect to the imaginary axes, $\mathscr{C}_1'$ being the circle of equation $z\bar{z} - \rho z - \rho\bar{z} + 1 = 0$ (see fig. 1).

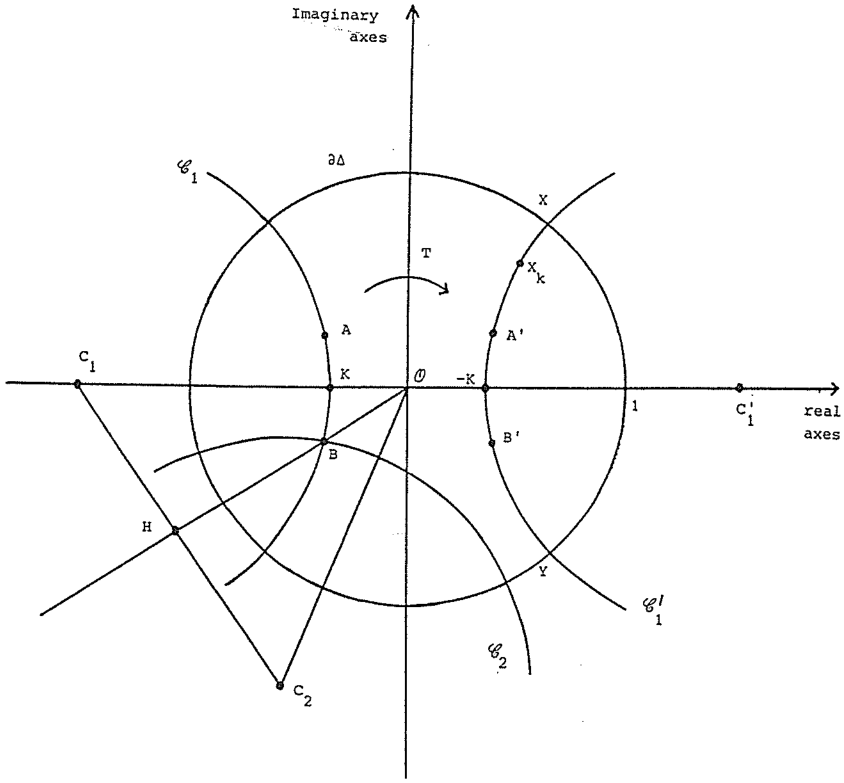

Fig. 1.

Using (4) and (5), we also have

$$\alpha/\beta = \frac{a^2+b^2+1}{-2a} = \frac{1+\cos(\pi/2g)}{2\sqrt{\cos(\pi/2g)}\,\cos(\pi/4g)} = \rho$$

since $1+\cos(\pi/2g) = 2\cos^2(\pi/4g)$. Thus we can write the equation of T as follows

$$T(z) = \frac{\rho z + 1}{z + \rho} \qquad z \in \Delta\,.$$

Let now P_g be the regular $4g$-gonal region in Δ whose internal angles are equal to $\pi/2g$ and whose vertices are the following

$$B_k = -\sqrt{\cos(\pi/2g)}\,\cos\left(\frac{(2k+1)}{4g}\pi\right) - \mathrm{i}\sqrt{\cos(\pi/2g)}\,\sin\left(\frac{(2k+1)}{4g}\pi\right) \tag{6}$$

$$(k \in Z,\ \ -2g < k \leqslant 2g).$$

Obviously $B_{-1} = A$, $B_0 = B$, $B_{2g-1} = B'$ and $B_{2g} = A'$. The $2g$ complex Möbius transformations (also named *translations*) of Δ, each of them carrying one face of P_g into its opposite face along lines joining the mid-points of pairs of opposite sides, are

$$T_h = R_{-h\pi/2g} T R_{h\pi/2g} \qquad (h = 1,\ 2,\ \ldots,\ 2g) \tag{7}$$

where $R_\Theta(z) = \mathrm{e}^{\Theta \mathrm{i}} z$ is the rotation around the origin $\mathcal{O}$ by the oriented angle Θ. Here we adopt the convention that UV means «apply first V, then U». Thus we have

$$T_h(z) = \frac{\rho z + \mathrm{e}^{(-h\pi/2g)\,\mathrm{i}}}{\mathrm{e}^{(h\pi/2g)\,\mathrm{i}} z + \rho} \qquad z \in \Delta \qquad \det(T_h) = \rho^2 - 1. \tag{8}$$

The surface group G_g is the subgroup of $\mathscr{M}$ generated by $T_1,\ T_2,\ \ldots,\ T_{2g}$. The regular $4g$-gonal region P_g is a fundamental region of G_g. The set of all the images $S(P_g)$, $S \in G_g$, gives a regular tessellation Π_g of type $(4g,\ 4g)$ in Δ (also compare [**1**], p.52). There are exactly $4g$ tiles $S(P_g)$, $S \in G_g$, at each vertex of Π_g. The area of P_g is $4(g-1)\pi$. The quotient space $\Delta/G_g \simeq P_g/G_g$ is the closed connected orientable surface Σ_g of genus g, which is triangulated by a complex with one vertex, $2g$ edges and one face. Furthermore G_g is the fundamental group of Σ_g. The relations of G_g are obtained by running around each vertex of P_g. By simmetry we have exactly one relation which is studied in the next section.

The transformation T_h $(h = 1,\ 2,\ \ldots,\ 2g)$ has two fixed points: $\xi_{1,h} = \mathrm{e}^{(-h\pi/2g)\,\mathrm{i}}$, $\xi_{2,h} = -\,\mathrm{e}^{(-h\pi/2g)\,\mathrm{i}}$ such that $\xi_{j,h} \in \partial\Delta$ $(j = 1,\ 2)$ and $\xi_{1,h}^* = \xi_{2,h}$. Thus the equation of T_h may be written as follows

$$\frac{z' - \xi_{1,h}}{z' - \xi_{2,h}} = K(h)\,\frac{z - \xi_{1,h}}{z - \xi_{2,h}}$$

where

$$z' = T_h(z) \qquad K(h) = \frac{\rho - e^{(h\pi/2g)\,i}\,\xi_{1,h}}{\rho - e^{(h\pi/2g)\,i}\,\xi_{2,h}}$$

is the *multiplier* of T_h (see [2], p. 15). The value of $K(h)$ determines the character of the transformation, i.e. T_h is a *hyperbolic transformation* since $K(h) = (\rho - 1)/(\rho + 1)$ is real (see [2], p. 18). Note that the transformation T_h^n (T_h repeated n times) has equation

$$\frac{z' - \xi_{1,h}}{z' - \xi_{2,h}} = \left(\frac{\rho - 1}{\rho + 1}\right)^n \frac{z - \xi_{1,h}}{z - \xi_{2,h}}$$

whence

$$T_h^n(z) = \frac{[(1+\rho)^n + (\rho - 1)^n]\,z + (1+\rho)^n\, e^{(-h\pi/2g)\,i} - (\rho - 1)^n\, e^{(-h\pi/2g)\,i}}{[(1+\rho)^n\, e^{(h\pi/2g)\,i} - (\rho - 1)^n\, e^{(h\pi/2g)\,i}]\,z + (1+\rho)^n + (\rho - 1)^n}.$$

Obviously $\lim_n T_h^n = \text{identity}$ since $\lim_n K^n(h) = 0$.

The *isometric circle* I_h (see [2], p. 23) of T_h (i.e. the complete locus of points in the neighbourhood of which lengths and areas are unaltered in magnitude) is given by the following equation

$$I_h\colon \left|\frac{e^{(h\pi/2g)\,i}}{\rho^2 - 1}\,z + \frac{\rho}{\rho^2 - 1}\right| = 1$$

and therefore

$$I_h\colon z\bar{z} + \rho\, e^{(h\pi/2g)\,i}\,z + \rho\, e^{(-h\pi/2g)\,i}\,\bar{z} + 1 = 0.$$

Thus I_h is the circle with center $-\rho\, e^{(-h\pi/2g)\,i}$ and radius $\rho^2 - 1$. The isometric circle I'_h of the inverse transformation T_h^{-1} is given by

$$I'_h\colon \left|\frac{e^{(h\pi/2g)\,i}}{\rho^2 - 1}\,z - \frac{\rho}{\rho^2 - 1}\right| = 1$$

whence

$$I'_h\colon z\bar{z} - \rho\, e^{(h\pi/2g)\,i}\,z - \rho\, e^{(-h\pi/2g)\,i}\,\bar{z} + 1 = 0.$$

Thus I'_h has center $\rho\, e^{(-h\pi/2g)\,i}$ and radius $\rho^2 - 1$. The equation of the straight line through the centers of the isometric circles is

$$e^{(h\pi/2g)\,i} z - e^{(-h\pi/2g)\,i} \bar{z} = 0. \tag{9}$$

Let L be the straight line orthogonal to the line (9) and containing the mid-point between the centers of I_h and I_h'. Then the equation of L is $e^{(h\pi/2g)\,i} z + e^{(-h\pi/2g)\,i} \bar{z} = 0$. Let J_{I_h} and J_L be the inversions (see [2], p. 10) in I_h and L respectively, i.e.

$$J_{I_h}(z) = \frac{-\rho\, e^{(-h\pi/2g)\,i} \bar{z} - 1}{\bar{z} + \rho\, e^{(h\pi/2g)\,i}} \qquad J_L(z) = -\, e^{(-h\pi/g)\,i} \bar{z}.$$

Then the transformation T_h is just the composition $J_L J_{I_h}$, i.e. it is equivalent to an inversion in I_h followed by a reflection in L.

3 - The relation

The relation of the group G_g is the following

$$T_1 T_2^{-1} \dots T_{2g-1} T_{2g}^{-1} T_1^{-1} T_2 \dots T_{2g-1}^{-1} T_{2g} = I \tag{10}$$

where I is the identity of Δ. Here we directly compute (10) by using the equations of the generators $T_1, T_2, \dots, T_{2g}$ of G_g (see (7), (8)). Substituting in the left side of (10)

$$T_h = R_{-h\pi/2g} T R_{h\pi/2g} \qquad T^{-1} = R_\pi T R_\pi$$

we obtain

$$R_{-\pi/2g} T R_{\pi/2g} R_{-2\pi/2g} R_\pi\, T R_\pi\, R_{2\pi/2g} \dots R_{-\frac{2g-1}{2g}\pi} R_\pi T R_\pi R_{\frac{2g-1}{2g}\pi} R_{-\frac{2g}{2g}\pi} T R_{\frac{2g}{2g}\pi}$$

$$= R_{-\pi/2g} (T R_{\pi-\pi/2g})^{4g} R_{\pi/2g}\,.$$

The fixed points of the linear fractional transformation

$$S(z) = (T R_{\pi-\pi/2g})(z) = \frac{-\rho\, e^{(-\pi/2g)\,i} z + 1}{-e^{(-\pi/2g)\,i} z + \rho}$$ are

$$\eta_1 = \frac{1}{\sqrt{\cos(\pi/2g)}}\, e^{(\pi/4g)\,i} \qquad \eta_2 = \sqrt{\cos(\pi/2g)}\, e^{(\pi/4g)\,i}.$$

Thus the multiplier K_S of S is given by

$$K_S = \frac{\rho - \eta_1}{\rho - \eta_2} = \frac{\rho - \sqrt{\cos(\pi/2g)}\, e^{(\pi/4g)\,i}}{\rho - \dfrac{1}{\sqrt{\cos(\pi/2g)}}\, e^{(\pi/4g)\,i}} = e^{(\pi/2g)\,i}$$

whence S is an *elliptic transformation* (see [2], p. 19). The transformation S^{4g} (S repeated $4g$ times) has equation

$$\frac{z' - \eta_1}{z' - \eta_2} = K_S^{4g} \frac{z - \eta_1}{z - \eta_2} \qquad z' = S^{4g}(z)$$

so that $S^{4g} = I$ (identity) since $K_S^{4g} = e^{2\pi} = 1$. This proves the formula (10).

The isometric circle I_S of S is given by

$$I_S: \left| \frac{-e^{(-\pi/2g)\,i} z + \rho}{(-\rho^2 + 1)\, e^{(-\pi/2g)\,i}} \right| = 1$$

whence

$$I_S: z\bar{z} - \rho\, e^{(-\pi/2g)\,i} z - \rho\, e^{(\pi/2g)\,i} \bar{z} + 1 = 0 .$$

Thus I_S is the circle with center $\rho\, e^{(\pi/2g)\,i}$ and radius $\rho^2 - 1$. Let now J_{I_S} and J_l be the inversions in I_S and l (straight line of equation $e^{(-\pi/2g)\,i} z + e^{(\pi/2g)\,i} \bar{z} = 0$) respectively, i.e.

$$J_{I_S}(z) = \frac{\rho\, e^{(\pi/2g)\,i} \bar{z} - 1}{\bar{z} - \rho\, e^{(-\pi/2g)\,i}} \qquad J_l(z) = \bar{z}\, e^{(\pi/2g)\,i} .$$

Then the elliptic transformation S is equivalent to an inversion in I_S followed by a reflection in l, i.e. $S = J_l J_{I_S}$.

4 - The composition law

In order to obtain the equation of an arbitrary transformation of the group G_g, we shall now consider the successive performance of the complex Möbius transformations $T_1, T_2, \ldots, T_{2g}$. First we introduce some notations to simplify the writing of the formulae. We use the symbol $[a_1;\ a_2]$ to represent the most general transformation

$$z' = \frac{a_1 z + \overline{a_2}}{a_2 z + \overline{a_1}} \qquad |a_1|^2 - |a_2|^2 \neq 0$$

which carries $\partial\Delta$ and Δ onto itself respectively. Thus, for each $h = 1, 2, \ldots, 2g$, we set

$$T_h = [\rho;\ e^{(h\pi/2g)\,i}]$$

Let now C_t^n $(t \leq n)$ be the set of all the simple combinations of n elements taken t to t. We use the notation $\sum_{q_1, q_2, \ldots, q_t}^{C, n}$ to represent a sum taken over all the simple combinations $(q_1, q_2, \ldots, q_t) \in C_t^n$, where $q_1 < q_2 < \ldots < q_t$. Then we have

$$T_{h_1, h_2}(z) = T_{h_1} T_{h_2}(z) = \frac{(\rho^2 + e^{((h_2 - h_1)\pi/2g)\,i})\, z + \rho\, e^{(-h_1\pi/2g)\,i} + \rho\, e^{(-h_2\pi/2g)\,i}}{(\rho\, e^{(h_1\pi/2g)\,i} + \rho\, e^{(h_2\pi/2g)\,i})\, z + e^{((h_1 - h_2)\pi/2g)\,i} + \rho^2}$$

whence

$$T_{h_1, h_2} = [\rho^2 + e^{((h_2 - h_1)\pi/2g)\,i};\ \rho(e^{(h_1\pi/2g)\,i} + e^{(h_2\pi/2g)\,i})].$$

Going on like this, we easily obtain

$$T_{h_1, \ldots, h_{2p}} = [\rho^{2p} + \rho^{2p-2}\Big(\sum_{s, r}^{C, 2p} e^{((h_r - h_s)\pi/2g)\,i}\Big) + \ldots + \rho^0\Big(\sum_{q_1, \ldots, q_{2p}}^{C, 2p} e^{((-h_{q_1} + h_{q_2} - \ldots + h_{q_{2p}})\pi/2g)\,i}\Big);$$

$$\rho^{2p-1}\Big(\sum_{r}^{C, 2p} e^{(h_r\pi/2g)\,i}\Big) + \rho^{2p-3}\Big(\sum_{s, r, t}^{C, 2p} e^{((h_s - h_r + h_t)\pi/2g)\,i}\Big)$$

$$+ \ldots + \rho\Big(\sum_{q_1, \ldots, q_{2p-1}}^{C, 2p} e^{((h_{q_1} - h_{q_2} + \ldots + h_{q_{2p-1}})\pi/2g)\,i}\Big)].$$

This also yields the formula for the odd case since $T_{h_1, \ldots, h_{2p+1}} = T_{h_1, \ldots, h_{2p}} T_{h_{2p+1}}$.

5 - The tessellation Π_g

In this section we give a recurrence formula to obtain all the vertices of the regular tessellation Π_g induced on Δ by the group G_g. Let us consider the vertices B_{2g-1} and B_{2g} of P_g, i.e.

$$B_{2g-1} = \sqrt{\cos(\pi/2g)}\cos(\pi/4g) - \mathrm{i}\sqrt{\cos(\pi/2g)}\sin(\pi/4g)$$

$$B_{2g} = \sqrt{\cos(\pi/2g)}\cos(\pi/4g) + \mathrm{i}\sqrt{\cos(\pi/2g)}\sin(\pi/4g).$$

Obviously B_{2g-1} and B_{2g} belong to the circle $\mathscr{C}_1'$ whose equation is (see fig. 1)

$$z\bar{z} - \rho z - \rho\bar{z} + 1 = 0.$$

If x is the real part of z and iy is its imaginary part, then $\mathscr{C}_1'$ is also represented by the equation

$$x^2 + y^2 - 2\rho x + 1 = 0. \tag{11}$$

Let now X, Y be the points in which the circle $\mathscr{C}_1'$ intersects the unit circle $\partial\Delta$ ($z\bar{z} = 1$), i.e.

$$X = \frac{\sqrt{\cos(\pi/2g)}}{\cos(\pi/4g)} + \mathrm{i}\,\mathrm{tg}(\pi/4g) \qquad Y = \frac{\sqrt{\cos(\pi/2g)}}{\cos(\pi/4g)} - \mathrm{i}\,\mathrm{tg}(\pi/4g).$$

Now we calculate the value of the cross-ratio

$$[X, Y, B_{2g}, B_{2g-1}] = \frac{B_{2g} - X}{B_{2g} - Y} \cdot \frac{B_{2g-1} - Y}{B_{2g-1} - X}.$$

We have

$$[X, Y, B_{2g}, B_{2g-1}]$$

$$= \frac{\left[\sqrt{\cos(\pi/2g)}\cos(\pi/4g) - \dfrac{\sqrt{\cos(\pi/2g)}}{\cos(\pi/4g)}\right]^2 + \left[\sqrt{\cos(\pi/2g)}\sin(\pi/4g) - \mathrm{tg}(\pi/4g)\right]^2}{\left[\sqrt{\cos(\pi/2g)}\cos(\pi/4g) - \dfrac{\sqrt{\cos(\pi/2g)}}{\cos(\pi/4g)}\right]^2 + \left[\sqrt{\cos(\pi/2g)}\sin(\pi/4g) + \mathrm{tg}(\pi/4g)\right]^2}$$

$$= \frac{\cos(\pi/4g) - \sqrt{\cos(\pi/2g)}}{\cos(\pi/4g) + \sqrt{\cos(\pi/2g)}} = \frac{\rho - 1}{\rho + 1}$$

since $\cos(\pi/2g) + 1 = 2\cos^2(\pi/4g)$ and ρ satisfies (3).

Let now X_k be the point of the circle $\mathscr{C}_1'$ such that

$$\delta(X_k, B_{2g}) = k\,\delta(B_{2g-1}, B_{2g})$$

where $\delta(A, B)$ denotes the hyperbolic distance between the points $A, B \in \Delta$. Since the Möbius transformations of the surface group G_g preserve the hyperbolic distance, an arbitrary vertex of $\Pi_g \cap \mathscr{C}_1'$ must be a point X_k for a suitable integer k. Then we have

$$\delta(X_k, B_{2g}) = \log[X, Y, X_k, B_{2g}] = k\,\delta(B_{2g-1}, B_{2g})$$

$$= k \log[X, Y, B_{2g}, B_{2g-1}] = \log[X, Y, B_{2g}, B_{2g-1}]^k$$

whence
$$[X, Y, X_k, B_{2g}] = [X, Y, B_{2g}, B_{2g-1}]^k = (\frac{\rho-1}{\rho+1})^k$$

$$\frac{X_k - X}{X_k - Y} = (\frac{\rho-1}{\rho+1})^k \frac{B_{2g}-X}{B_{2g}-Y}.$$

By an algebraic calculation and making use of some trigonometric formulae, we can obtain the point $X_k = x_k + \mathrm{i}\, y_k$ expressed in terms of ρ and k. The real part x_k of X_k is given by the following formula

$$x_k = \frac{(\rho-1)^{2k+1} + (\rho+1)^{2k+1}}{\rho[(\rho-1)^{2k+1} + (\rho+1)^{2k+1}] + 2(\rho+1)^{k+1}(\rho-1)^{k+1}}. \tag{12}$$

Since X_k belongs to $\mathscr{C}_1'$, the pair (x_k, y_k) satisfies (11). Thus we find easily that

$$y_k = \frac{\sqrt{\rho^2-1}\,[(\rho+1)^{2k+1} - (\rho-1)^{2k+1}]}{\rho[(\rho-1)^{2k+1} + (\rho+1)^{2k+1}] + 2(\rho+1)^{k+1}(\rho-1)^{k+1}}. \tag{13}$$

Putting $\sigma = (\rho-1)/(\rho+1)$, it follows that $\rho = (1+\sigma)/(1-\sigma)$, $\rho - 1 = (2\sigma)/(1-\sigma)$, $\rho + 1 = 2/(1-\sigma)$ and $\sqrt{\rho^2-1} = (2\sqrt{\sigma})/(1-\sigma)$. Substituting these relations in (14) and (15), we obtain

$$x_k = \frac{(1-\sigma)(1+\sigma^{2k+1})}{(1+\sigma)(1+\sigma^{2k+1}) + 4\sigma^{k+1}} \tag{12$'$}$$

$$y_k = \frac{2\sqrt{\sigma}\,(1-\sigma^{2k+1})}{(1+\sigma)(1+\sigma^{2k+1}) + 4\sigma^{k+1}} \tag{13$'$}$$

$$\sigma = \frac{\cos(\pi/4g) - \sqrt{\cos(\pi/2g)}}{\cos(\pi/4g) + \sqrt{\cos(\pi/2g)}}. \tag{14}$$

Now we give a recurrence formula to obtain all the centers of the circles orthogonal to $\partial\Delta$ and containing sides of Π_g.

Let $\mathscr{C}$ be a circle which is orthogonal to the unit circle $\partial\Delta$. The equation of $\mathscr{C}$ has the form

$$z\bar{z} - C\bar{z} - \bar{C}z + 1 = 0 \tag{15}$$

where C is a complex number. This circle has center C and radius $\sqrt{|C|^2 - 1}$. Since we shall be interested only in real circles, we shall require that $|C|^2 > 1$. Let now $\mathscr{C}'_h$ be the circle obtained from $\mathscr{C}$ when the complex Möbius transformation T_h (see (8)) is applied.

Substituting in (15) the relations

$$z = T_h^{-1}(z') = \frac{-\rho z' + e^{(-h\pi/2g)\,i}}{e^{(h\pi/2g)\,i} z' - \rho} \qquad \bar{z} = \frac{-\rho \bar{z}' + e^{(h\pi/2g)\,i}}{e^{(-h\pi/2g)\,i} \bar{z}' - \rho}$$

the equation of $\mathscr{C}'_h$ is

$$(\rho^2 + \bar{C}\rho\, e^{(-h\pi/2g)\,i} + C\rho\, e^{(h\pi/2g)\,i} + 1)\, z'\bar{z} - (2\rho\, e^{(h\pi/2g)\,i} + \bar{C}\rho^2 + C\, e^{(h\pi/2g)\,i})\, z'$$

$$- (2\rho\, e^{(-h\pi/2g)\,i} + C\rho^2 + \bar{C}\, e^{(-h\pi/g)\,i})\, \bar{z}' + \rho^2 + \bar{C}\rho\, e^{(-h\pi/2g)\,i} + C\rho\, e^{(h\pi/2g)\,i} + 1 = 0.$$

The center of $\mathscr{C}'_h$ is given by the following recurrence formula

$$C'_h = \frac{2\rho\, e^{(-h\pi/2g)\,i} + \bar{C}\, e^{(-h\pi/g)\,i} + C\rho^2}{1 + \bar{C}\rho\, e^{(-h\pi/2g)\,i} + C\rho\, e^{(h\pi/2g)\,i} + \rho^2}. \tag{16}$$

In particular, let now $\mathscr{C}_k$ be a circle which contains a side of the fundamental region P_g of G_g ($k \in Z$, $-2g < k \leqslant 2g$). The equation of $\mathscr{C}_k$ is

$$z\bar{z} - \rho\, e^{(-k\pi/2g)\,i} z - \rho\, e^{(k\pi/2g)\,i} \bar{z} + 1 = 0.$$

Obviously $\mathscr{C}_k$ has center $C_k = \rho\, e^{(k\pi/2g)\,i}$ and radius $\sqrt{\rho^2 - 1}$. Using (16), the center $C'_{h,k}$ of $\mathscr{C}'_{h,k} = T_h(\mathscr{C}_k)$ is given by the following formula

$$C'_{h,k} = \frac{\rho^3\, e^{(k\pi/2g)\,i} + 2\rho\, e^{(-h\pi/2g)\,i} + \rho\, e^{(-(k+2h)\pi/2g)\,i}}{1 + \rho^2\, e^{(-(h+k)\pi/2g)\,i} + \rho^2\, e^{(-(h+k)\pi/2g)\,i} + \rho^2} \tag{16$'$}$$

so that the equation $\mathscr{C}'_{h,\,k}$ is

$$(\rho^2 + \rho^2\, e^{(-(h+k)\pi/2g)\,i} + \rho^2\, e^{((h+k)\pi/2g)\,i} + 1)\, z'\, \bar{z}'$$

$$-(2\rho\, e^{(h\pi/2g)\,i} + \rho^3\, e^{(-k\pi/2g)\,i} + \rho\, e^{((-k+2h)\pi/2g)\,i})\, z'$$

$$-(2\rho\, e^{(-h\pi/2g)\,i} + \rho^3\, e^{(k\pi/2g)\,i} + \rho\, e^{(-(k+2h)\pi/2g)\,i})\, \bar{z}' + 1$$

$$+\rho^2\, e^{(-(k+h)\pi/2g)\,i} + \rho^2\, e^{((h+k)\pi/2g)\,i} + \rho^2 = 0.$$

Relations (12)′, (13)′, (14), (16), (16)′ allow us to study by recurrence the distribution of all vertices of the regular tessellation Π_g.

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Riassunto

Si descrive una semplice costruzione geometrica dei classici gruppi delle superfici. Come conseguenza, si determina l'equazione di una qualsiasi trasformazione di questi gruppi. Infine, si ottengono formule ricorrenti per rappresentare, mediante computer, le pavimentazioni regolari indotte dai suddetti gruppi.

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