Rings in which all proper ideals are isomorphic (**) 

Let $R$ be a nonzero ring with or without an identity. A nonzero ideal different from $R$ will be called proper ideal of $R$. A ring having no proper ideal will be called a weakly simple ring. Let $G = (R, +)$, the additive group of $R$. The 0-rank of $G$ will be called the 0-rank of $R$, which is the cardinality of a maximal independent subset of elements with infinite order in $G$. This terminology can be found in [2].

Def. Let $R$ be a ring. $R$ is called a PII-ring, briefly PII, if all proper ideals are isomorphic as rings.

Let $T_R = \{ x \in R : x \text{ has finite order in } G \}$ and let $B^r$ (resp. $B^l$) denote the right (resp. left) annihilator of ideal $B$ in $R$. It is well-known that $T_R$, $B^r$ and $B^l$ are ideals of $R$.

Lemma 1. Suppose that $R$ is PII, $A$ a proper ideal of $R$. Then:

1. $A$ has infinite characteristic iff $(A, +)$ is torsion free.
2. $A$ has finite characteristic iff $pA = 0$ holds, for some prime number $p$.

Proof. Let $T_A = A \cap T_R$. Then $T_A$ is an ideal of $R$. If $T_A \neq 0$, then $T_A \cong A$ implies $A = T_A$. If there are two nonzero, $x, y$ in $A$ such that $p^m x = 0$ and $q^n y = 0$, for distinct primes $p, q$ and positive integers $m, n$, then $\{ x \in A : p^m x = 0 \} \cong \{ y \in A : q^n y = 0 \}$. If $m \geq 2$, $p^m x = 0$ and $p^{m-1} x \neq 0$, then $\{ x \in A : p^2 x = 0 \} \cong \{ x \in A : px = 0 \}$. Thus we conclude that $pA = 0$, for some pri-

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me number \( p \). So \( A \) has infinite characteristic iff \( T_A = 0 \), i.e. \( (A, +) \) is torsion free.

**Lemma 2.** Suppose that \( R \) is PI\(_I\) and \( A \) is a proper ideal with \( A^2 \neq 0 \). Then:

1. \( A^n \neq 0 \) holds, for each positive integer \( n \).
2. \( A^n = A^{n+1} \) iff \( A^2 = A \).
3. If \( (A, +) \) is torsion free and \( R \) has finite 0-rank then \( A^2 = A \) and \( R \) is Artinian.
4. \( ^1B = B^r \) holds, for each proper ideal \( B \) of \( R \).

**Proof.**

1. Let \( A^n = 0 \) but \( A^{n-1} \neq 0 \). Then \((A^{n-1})^2 = 0 \). So \( A^2 = 0 \), which is a contradiction.
2. If \( A^n = A^{n+1} \), then \( A^n = (A^n)^2 \). So \( A^2 = A \).
3. Put \( A_0 = \bigcap_{n=1}^\infty nA \). Then \( (A_0, +) \) is divisible. Since \((nA)(nA) \subseteq n(nA)\) and \( nA \cong A \), for all \( n \), \( A^2 \subseteq A_0 \). Thus \( (A^n, +) \) is divisible since \( A^n = A_0 \). Suppose that

\[
A \supseteq A^2 \supseteq A^3 \supseteq \ldots
\]

Then by [2] (Theorem 4.1.3)

\[
(A, +) = (A^2, +) \oplus K_1 \quad K_1 \neq 0
\]

\[
= (A^3, +) \oplus K_2 \oplus K_1 \quad K_2 \neq 0
\]

\[
\ldots \ldots
\]

this contradicts the finite 0-rank of \( R \). The same contradiction arises if \( A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \supseteq \ldots \) is a descending chain of proper ideals of \( R \). Hence \( R \) is Artinian.

4. Let \( B^r \neq 0 \). Since \((B^r \cdot B)^2 = 0 \) and \( B^r \cdot B \neq A \), \( B^r \cdot B = 0 \) and \( B^r \subseteq A \). Similarly, \( ^1B \subseteq B^r \). So \( ^1B = B^r \).

**Theorem 1.** Suppose that \( R \) is PI\(_I\), \( T_R = 0 \) and \( A \) is a proper ideal with \( A^2 \neq 0 \). Then the following statements are equivalent:

1. \( A \) is PI\(_I\) as a ring.
2. \( A \) is a hereditarily idempotent ring, i.e. \( I^2 = I \) holds for each ideal \( I \) of \( A \).
3. \( A^2 = A \).
4. Each ideal of \( A \) is an ideal of \( R \).
Proof. (1) ⇒ (2). Suppose that $A$ is PI, $B$ is a proper ideal of $A$ and $\overline{B}$ is the ideal generated by $B$ in $R$. Then by Lemma 2, $\overline{B}^3 \neq 0$ and $\overline{B}^3 \subset B$. So $B \equiv \overline{B}^3 \equiv A$.


(2) ⇒ (3). Obviously.

(3) ⇒ (4). Let $B$ be an ideal of $A$. Then $\overline{B}^2 = \overline{B}$ since $\overline{B} \equiv A$ and $B \subset \overline{B} = \overline{B}^2 \subset B$, i.e. $B = \overline{B}$ is an ideal of $R$.

(4) ⇒ (1). Obviously.

Theorem 2. Suppose that $R$ is PI, $T_R = 0$ and $A$ is a proper ideal of $R$ with $A^2 \neq 0$. If $R$ has finite 0-rank then the following statements are equivalent for all proper ideals $B$ of $R$:

1. $B^r \neq 0$;
2. $R = B \oplus B^r$, where $B$ and $B^r$ are weakly simple rings.

Proof. (2) ⇒ (1). Obviously.

(1) ⇒ (2). Since $(B \cap B^r)^2 = 0$ and $A^2 \neq 0$, $B \cap B^r = 0$. Suppose that $R \neq B \oplus B^r$. Then $B \equiv B \oplus B^r$. By Lemma 2 (3) and Theorem 1, there are nonzero ideals $C, D$ of $R$ such that $B = C \oplus D$. Since $C \equiv B \equiv D$, there are nonzero ideals $E, F, G, H$ of $R$ such that $C = E \oplus F$ and $D = G \oplus H$. Continuing in this way, we obtain a lot of nonzero ideals $B_1, B_2, B_3, \ldots$ such that $B = B_1 \oplus B_2 \oplus B_3 \oplus \ldots$, which contradicts the finite 0-rank of $R$. Similarly, if $K$ is a proper ideal of $B^r$, then $K$ is a proper ideal of $R$. Thus $B \equiv B \oplus K$, which leads to the same contradiction.

By [1] (Theorem 3.8 and Cor. 3.9), for the ideal $B$ described in Theorem 2, we have the following

Corollary. $R, B$ as above. Then:

1. If $R$ has an identity, then $B^r \neq 0$ iff $R = B \oplus B^r$ where $B, B^r$ are simple.
2. If $R$ is commutative, then $B^r \neq 0$ iff $R = B \oplus B^r$, where $B, B^r$ are fields.

We now pay attention to the case that $R$ is commutative.

Theorem 3. Suppose that $R$ is a commutative PI-ring, $T_R = 0$ and $A$ is a proper ideal of $R$. Then:

1. If $A^2 = 0$, then $R$ is either a null or an infinite cyclic group or a local ring in which the maximal ideal consists of all elements $x$ with $x^2 = 0$ of $R$. 

(2) If $A^2 \neq 0$, then for each proper ideal $K$ of $R$, 

$$(R, +) = (K, +) \oplus K_1$$

where $K_1$ is a subgroup of $(R, +)$. In particular, $R = R^2$.

(3) If $A^2 \neq 0$ and $R$ has finite 0-rank, then $R$ is a direct sum of two fields.

Proof. (1) Let $A^2 = 0$. Then $B^2 = 0$ holds for each proper ideal $B$ of $R$.

Case 1. $R^2 = 0$. In this case, every subring of $R$ is an ideal of $R$. For any $0 \neq a \in R$, let $Z[a] = \{na : n \in Z\}$, where $Z$ denotes the set of all integers. Then $Z[a]$ is an ideal of $R$, in particular, a null ring. If $Z[a] \neq R$, then $Z[a] \cong A$. Assume that $R \neq nR$, for some positive integer $n$. Then $nR \cong Z[na]$. So $R \cong Z[a]$. Assume that $R = nR$, for all integers $n$. Then, by [2] (Theorem 4.1.5) $(R, +) \cong \Sigma \oplus Q$, where $Q$ is the rational numbers additive group. Clearly, $Q$ has a proper subgroup $B$ that is not cyclic, e.g.

$$\frac{1}{p_1^2} \not\in B = \bigcup_{i=1}^{\infty} \left\{ \frac{1}{p_1p_2\ldots p_i} \right\}$$

where $p_1, p_2, p_3 \ldots$ are all distinct prime numbers. Thus $R$ has a proper subring that is not a null ring on an infinite cyclic group, this is a contradiction. So $R = Z[a]$, i.e. $R$ is a null ring on an infinite cyclic group.

Case 2. $R^2 \neq 0$. We shall prove that $R$ has an identity 1. In fact, since $R$ is commutative, there is an element $a$ in $R$ such that $a^2 \neq 0$. Thus $R = Za + Ra$, where $Z$ is the set of integers. If $Ra$ is a proper ideal of $R$, then $(Ra)^2 = 0$ since $Ra \cong A$. So $2Za + Ra$ is a proper ideal of $R$, $a^2 = 0$ follows from $(2a)^2 = 0$. Hence $R = Ra$. Let $a = xa$. Then $(x^2 - x)a = 0$. Note that $((Ra)^r)^2 = 0$. If we put $t = x^2 - x$, then: (i) $x^2 = x$ if $t = 0$; (ii) $x - 2xt + t$ is a nonzero idempotent element if $t \neq 0$.

In either case, we obtain a nonzero idempotent element $e$ in $R$. Thus $R = Re \oplus R(1 - e)$, where $R(1 - e) = \{y - ye : y \in R\}$. So $R(1 - e) = 0$ since $R^2 \neq 0$. $e = 1$ is an identity of $R$.

Now, put $B = \{x \in R : x^2 = 0\}$. To show that $B$ is an ideal, it is enough to show that $B$ is a subgroup of $(R, +)$. Suppose that $x + y \not\in B$ for some nonzero $x, y \in B$. As in the case for $a$, $R = R(x + y)$. So we have a $z$ in $R$ such that $z(x + y) = 1$. Thus $1 - zy \in (Rx)^r$ and $(1 - zy)^2 = 0$. So $2zy = 1$ and $y = 2zy^2 = 0$ contrary to $y \neq 0$.

(2) From the proof of Lemma 2, $(A, +)$ is divisible, $(K, +)$ is divisible, for each proper ideal $K$ of $R$. By [2] (Theorem 4.1.3), $(R, +) = (K, +) \oplus K_1$, where $K_1$ is a
nonzero subgroup of \((R, +)\). In particular, take \(K = R^2\). If \(R \neq R^2\), then for each proper subgroup \(H\) of \(K_1\), \(R^2 \oplus H\) is a proper ideal of \(R\). So \(H\) is divisible. Note that \(K_1\) is also torsion free. Thus we can take a proper subgroup \(H \cong Z\), which is a contradiction because \(Z\) is not divisible.

(3) From the corollary of Theorem 2, it is enough to prove that there is a proper ideal \(B\) in \(R\) such that \(B^\alpha \neq 0\). Suppose that \((Ra)^\alpha = 0\) for all \(a \neq 0 \in R\). Let \(a \neq 0 \in R\). By Lemma 2, \(a^nR = a^{n+1}R\) holds for some \(n\). Thus \(R = aR\), whence \(R\) is a field, which contradicts the fact that \(A\) is a proper ideal of \(R\).

Example 1. Let \(F\) be a field. Let \(R = \{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in F \}\). Then \(R\) is a local ring with a unique proper ideal \(A = \{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} : b \in F \}\).

The author is unable to give an example of \(R\) described in Theorem 3 (2). But some properties of such rings are obtained.

Theorem 4. Suppose that \(R\) is a commutative \(\Pi\)-ring. \(T_R = 0\) and \(A\) is a proper ideal with \(A^2 \neq 0\). Then:

1. \(y \in Ry\) for all \(y \in R\).
2. If \((Ry)^\alpha = 0\), for some \(y \neq 0 \in R\), then \(R\) has an identity.

Proof. Note that \((A, +)\) is divisible (see the proof of Lemma 2 (3)). Firstly we shall prove that \(x \in Rx\), for all proper ideals \(K\) and \(x \in K\).

Let \(x \neq 0 \in K\) and let \(I = Zx + Rx\). Then \(I\) is a proper ideal of \(R\), moreover \((I/Rx, +)\) is cyclic (with generator \(x + Rx\)). But, \((I, +)\) is divisible and so is \((Rx, +)\). Consequently, if \(x\) is not in \(Rx\), then \((I/Rx, +) \cong \Sigma \oplus Q\) by [2] (Theorem 4.1.5), which is a contradiction. So \(I = Rx\) i.e. \(x \in Rx\).

Now suppose that \(y \notin Ry\), for some \(y \in R\). It is clear that \(Ry \neq 0\). Because, if \(Ry = 0\) then the ideal generated by \(y\) in \(R\) is either \(R\) or isomorphic to \(A\), which gives \(A^2 = 0\). According to the above fact, \(R = nZy + Ry\), for all integers \(n > 1\). Thus \(y = mny - ry\) for \(m \in Z\) and \(r \in R\), consequently \(R = (nm - 1)Zy + Ry = Ry\) which is a contradiction.

If \((Ry)^\alpha = 0\) for some \(y \neq 0 \in R\), then there is a nonzero idempotent element \(x\) in \(R\) such that \(y = xy\). If \(R(1 - x) = \{ r - rx : r \in R \} \neq 0\), then \(R = Rx \oplus R(1 - x)\). By Theorem 1, \(Rx\) and \(R(1 - x)\) are \(\Pi\), by [2] (Cor. 3.9), \(Rx\) and \(R(1 - x)\) are fields. Thus \(R\) has an identity.
Theorem 5. Suppose that $R$ is a commutative $\pi$-ring and $A$ is a proper ideal. If $T_R \neq 0$ then $R = T_R$. Moreover $p^2R = 0$, for some prime $p$.

Proof. If $R \neq T_R$, then $pT_R = 0$ by Lemma 1, for some prime $p$. If $R^2 = 0$, then we can take an $a \in R$ but $a \notin T_R$. So $Z[a] \cap T_R = 0$, whence $Z[a] \cong T_R$, which to leads to $Z[a] \subseteq T_R$, a contradiction. If $R^2 \neq 0$ and $R \neq T_R$, then we can take an $a \in R$ but $a \notin T_R$, and so $na \notin T_R$ for all integers $n \neq 0$. If $R \neq aR$ then $aR \subsetneq T_R$ since $aR \cong T_R$, and so $R = Za + T_R$. Thus $R \neq 2Za + T_R$ and $2Aa \subsetneq T_R$, whence $a \in T_R$, this is a contradiction. Hence $R = aR$, for all $a \in R$ but $a \notin T_R$. Let $x \in R$ such that $a = ax$. So $x^2 - x \in (Ra)^r$.

(i) $A^2 = 0$: it is clear that $(Ra)^r \neq R$ since $a = ax^2$. So $((Ra)^r)^2 = 0$. Thus we obtain a nonzero idempotent element $e$ in $R$, consequently $e$ is an identity since $R^2 \neq 0$.

(ii) $A^2 \neq 0$: $((Ra)^r)^2 = ((R \cap (Ra)^r)^2 = (Ra \cap (Ra)^r)^2 = 0$. Thus $(Ra)^r = 0$, whence we still have an identity $e$ in $R$.

Now let $e$ be an identity in $R$. Then for all $a \in R$ but $a \notin T_R$, $a$ is invertible. In particular $a = pe$ is invertible. Thus $peT_R = pT_R = 0$, which to leads to $T_R = 0$, a contradiction! As in the proof of Lemma 1, $p^2R = 0$ holds.

Example 2. $R = Z_p^2 = (\bar{1}, \bar{2}, \bar{3}, \ldots, \bar{p^2})$, the ring of integers modulo $p^2$, is an example of a ring described in Theorem 5.

Example 3. $R = R_1 \oplus R_2$, where $(R_1, +) \cong (Z_p, +)$ and $R_i^2 = 0$ ($i = 1, 2$).

Theorem 6. Suppose that $R$ is a commutative $\pi$-ring, $R^2 \neq 0$ and $p^2R = 0$ but $pR \neq 0$, for some prime $p$. Then:

(1) $R$ has an identity.

(2) $R$ is a local ring with a unique maximal ideal $B$ consisting of all elements $x$ that $px = 0$ in $R$.

Proof. (1) Let $x \in R$ but $px \neq 0$. Then $R = Zx + Rx$. If $x \notin Rx$ then $R^2 = (Zx + Rx)^2 = Rx \equiv pR$ and $px^2 = 0$. Thus $A = \{0, px, 2px, \ldots, (p - 1)px\}$ is a proper ideal of $R$ and $Rx \equiv A$. If $Zx \cap Rx = 0$ then $A \oplus Rx \equiv Rx$ which is impossible because $Rx$ has just $p$ elements. Thus, let $0 \neq nx \in Rx$. Then $n = pm$ for some integer $0 < m < p$. So $R = Zmx + Rx = Zmx + A = Zmx \equiv Z_p^m$ and $R$ has
an identity, which contradicts $x \notin Rx$. If $x \in Rx$ then $x = ax$ thus $a^2 - a \in (Rx)''$. Since $((Rx)'')^2 = 0$, as in the proof of Theorem 3 (1), there is an identity in $R$.

(2) By (1), for all $x \in R$ but $px \neq 0$, $R = Rx$. So $x$ is invertible, and then $B$ is a unique maximal ideal of $R$.

References


Abstract

This paper is concerned primarily with rings having the property that all proper ideals are isomorphic as rings.

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