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Biconformal cosymplectic manifolds (**)

0 - Introduction

In the last twenty years many papers have been devoted to *almost cosymplectic manifolds* $M(\Omega, \eta, \xi, g)$. These are $(2m+1)$-dimensional manifolds M endowed with a pseudo-Riemannian metric g, a vector field ξ and a 2-form Ω. If η denotes the 1-form associated to ξ by the metric g (we write $\eta = \flat(\xi)$), then $\Omega^m \wedge \eta \neq 0$ holds.

An interesting subclass of the almost cosymplectic manifolds are the manifolds with a *conformal cosymplectic structure*: in terms of the cohomology operator d^ω [8], defined by

$$(0.1) \qquad d^\omega \alpha = d\alpha + \omega \wedge \alpha \qquad d\omega = 0 \qquad \text{for any } \alpha \in \Lambda^p(M)$$

they are distinguished by the additional relation

$$d^w \eta = 0$$

for some 1-form w (see $[3]_1$, $[12]_1$, $[16]_1$ and $[16]_2$). The best known examples of conformal cosymplectic manifolds are the *Kenmotsu manifolds* (or *K-manifolds* [11]).

In the present paper a *biconformal cosymplectic manifold* (abbr. *B.C.*) is defined to be a conformal cosymplectic manifold satisfying

$$d^w \eta = 0 \qquad d^{2\lambda\eta} \Omega = 0$$

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where $\lambda \in C^{\infty} M$ is given by $w = \mathrm{d} \log \lambda$. (Remember: $\mathrm{d}(2\lambda\eta)$ must vanish because of (0.1)).

In 2 we deal with some properties of Lie algebra defined by the B.C. cosymplectic structure. Since the *horizontal distribution*

$$D_h := \{Z \in \mathcal{X} M: \eta(Z) = 0\}$$

is involutive, M is foliated by $(2m)$-dimensional symplectic hypersurfaces M_h normal to ξ. It is shown that $\log \lambda$ is a Hamiltonian function for the symplectic form $\Omega_h = \Omega|_{M_h}$.

In 3 we consider general quasi-Sasakian manifold $M(\Phi, \Omega, \eta, \xi, g)$ [12]$_2$ and prove that M is endowed with a B.C. cosymplectic structure if and only if the structure vector field ξ is contact quasi-concurrent [3]$_2$ with horizontal and closed associated vector field $W \in D_h$. If $V \in D_h$ is any horizontal vector field, one has the formula

$$\mathcal{L}_\xi \flat(V) = \rho \flat(V) + \flat[\xi, V] + g(V, W)\eta \qquad \text{with } \rho = \frac{\operatorname{div} \xi}{m}.$$

It is proved that any manifold $M(\Phi, \Omega, \eta, \xi, g)$ is foliated by a totally geodesic 3-dimensional submanifold tangent to ξ, W and ΦW, and that $g(W, W)$ is an isoparametric function [18]. It is also showed that compact manifolds $M(\Phi, \Omega, \eta, \xi, g)$ or space-forms $M(\Phi, \Omega, \eta, \xi, g)$ do not exist.

In 4 we outline some properties of the immersions $x: M_h \to M$, $y: M_I \to M$, where M_I is an invariant submanifold of M, and $z: M_A \to M$, where M_A is an anti-invariant submanifold of dimension m.

1 - Preliminaries

Let (M, g) be a Riemannian or pseudo-Riemannian C^{∞}-manifold and let ∇ be the covariant differential operator defined by the metric tensor g. We assume in the following that M is orientable and that the connection ∇ is symmetric.

Let $\Gamma(TM) = \mathcal{X} M$ and $\flat: TM \to T^* M$ be the set of sections of the tangent bundle TM and the *musical isomorphism* [15] defined by g, respectively.

Following [15], we denote by

$$A^q(M, TM) = \Gamma \operatorname{Hom}(\Lambda^q TM, TM)$$

the set of vector valued q-forms, $q < \dim M$, and write for the exterior covariant derivative operator with respect to ∇

$$d^{\nabla}: A^{q}(M, TM) \rightarrow A^{q+1}(M, TM).$$

(Notice that in general $d^{\nabla^2} = d^{\nabla} \circ d^{\nabla} \neq 0$.)

If $p \in M$, then the vector valued 1-form $dp \in A^1(M, TM)$ stands for the soldering form of M. (dp is also called the «line element». Since ∇ is symmetric, one has $d^{\nabla}(dp) = 0$ [**6**].)

The cohomology operator d^{ω} was defined in Introduction as

$$d^{\omega} = d + e(\omega), \tag{1.1}$$

(cf. [8]) acting on ΛM, where $e(\omega)$ denotes the exterior product by the closed 1-form $\omega \in \Lambda^1 M$.

Clearly one has

$$d^{\omega} \circ d^{\omega} = 0. \tag{1.2}$$

Any form $u \in \Lambda M$ such that

$$d^{\omega} u = 0 \tag{1.3}$$

is said to d^{ω}-*closed*, and ω is called the *cohomology form* (abbr. c.f.).

An *exterior concurrent vector field* is defined ([**16**]$_2$, [**14**]) as a vector field $X \in \mathcal{X}M$ for which the relation

$$d^{\nabla}(\nabla X) = \nabla^2 X = \pi \wedge dp \in A^2(M, TM) \tag{1.4}$$

holds for some $\pi \in \Lambda^1 M$.

If X is a tangent vector field, then the 1-form π, which is called the *concurrence form*, is expressed by $\pi = f\, \flat(X)$, where $f \in C^{\infty}(M)$ is the *conformal scalar* associated with X.

A pair (α, β), where α is a p-form and β is a $(p-1)$-form, defines a p-*cocycle* if and only if

$$d\beta = 0 \qquad d^{\omega}\alpha = \Omega \wedge \beta \tag{1.5}$$

where Ω is d^{ω}-closed. In order that (α, β) is an *exact* p-cocycle, it is necessary and sufficient that there exists a $(p-1)$-cochain (ψ_2, ψ_1) (in the sense of the differential cohomology of Chevalley) such that

$$\alpha = -d^{\omega}\psi_2 + \Omega \wedge \psi_1 \qquad \beta = d\psi_1. \tag{1.6}$$

2 - Biconformal cosymplectic manifolds

Let $M(\Omega,\ \eta,\ \xi,\ g)$ be a $(2m+1)$-dimensional Riemannian C^∞-manifold endowed with an *almost cosymplectic structure* $1\times \mathrm{Sp}(m; \boldsymbol{R})$ in the broad sense: Let $\Omega\in \Lambda^2 M$, $\eta \in \Lambda^1 M$ and $\xi = \flat^{-1}\eta \in \mathcal{X}M$ be the structure 2-form, the structure 1-form and the structure vector field of $1\times \mathrm{Sp}(m; \boldsymbol{R})$ respectively.

The $(2m)$-distribution $D_h := \{Z\in \mathcal{X}M\colon\ \eta(Z)=0\}$ annihilated by η is called the *horizontal distribution* and any field $Z\in\mathcal{X}M$ on M may be written as

$$Z = Z_h + \eta(Z)\,\xi \tag{2.1}$$

where $Z_h \in D_h$ is the *horizontal component* of Z.

If for any globally exact *basic form* $w\in D_h^* = \{\alpha\in\Lambda^1 M\colon\ \alpha(\xi)=0\}$, say

$$w = \mathrm{d}\log\lambda \tag{2.2}$$

the structure forms η and Ω satisfy

$$\mathrm{d}^w\eta = 0 \qquad \mathrm{d}^{2\lambda\eta}\Omega = 0 \tag{2.3}$$

we say that the pairing $(\eta,\ \Omega)$ defines a *biconformal cosymplectic structure* (abr. *B.C.-structure*). By (2.2) and (2.3) it follows that η (resp. Ω) is d^w-closed (resp. $\mathrm{d}^{2\lambda\eta}$-closed), and w and $2\lambda\eta$ will be called the *cohomology forms* associated with $(\eta,\ \Omega)$. Clearly one has

$$\mathrm{d}(\lambda\eta) = 0$$

and we agree to call $\lambda\in C^\infty M$ the *structure scalar* associated with B.C.-structure. It is also easily seen from (2.3) that D_h defines a $(2m)$-*foliation* and that the restriction $\Omega_h = \Omega|_{D_h}$ is a *symplectic form*. Referring to (1.6), we see that for some 1-form φ, the pairing $(\alpha,\ \beta)$, such that

$$\alpha = -\,\mathrm{d}^{2\lambda\eta}\varphi + \lambda\Omega \qquad \beta = \mathrm{d}\lambda$$

defines an exact cocycle.

Let now $Z_h\in D_H$ be any horizontal vector field. By (2.3) one gets

$$(\mathcal{L}_{Z_h} + w(Z_h))\,\eta = 0 \tag{2.4}$$

and

$$\mathrm{d}^{2\lambda\eta}(i_{Z_h}\Omega) = \mathcal{L}_{Z_h}\Omega\,. \tag{2.5}$$

Since by (1.2) one derives from (2.5)

$$d^{2\lambda\eta}(\mathcal{L}_{Z_h}\Omega) = 0$$

one may say that any $Z_h \in D_h$ is an *infinitesimal conformal transformation* of η and that the Lie derivative $\mathcal{L}_{Z_h}\Omega$ is $d^{2\lambda\eta}$-closed, as is Ω.

Since $\eta(\xi) = 1$, it is also easily seen that one has

$$\mathcal{L}_\xi \Omega = -2\lambda\Omega \tag{2.6}$$

which proves that the structure vector ξ is an infinitesimal conformal transformal of Ω.

Furthermore, let $Z \in \mathcal{X}M$ be any vector field of M, and let μ: $TM \rightarrow T^*M$; $Z \rightarrow i_Z\Omega$ be the bundle isomorphism defined by Ω. If Z is such that it satisfies

$$d^{2\lambda\eta}(\mu Z) = 0 \tag{2.7}$$

we agree to say that Z is a $d_\Omega^{2\lambda\eta}$-*closed vector field.*

For any vector field Z satisfying (2.7), one finds after a short calculation and by reference to (2.3) that

$$\mathcal{L}_Z\Omega = -2\lambda\eta(Z)\Omega \tag{2.8}$$

i.e. Z is an infinitesimal conformal automorphism of Ω.

In a similar manner, any vector field Z such that

$$d^w\eta(Z) = d\eta(Z) + \eta(Z)w = 0 \tag{2.9}$$

will be defined as *contact* d^w-*closed.*

From the first equation (2.3) one quickly gets

$$\mathcal{L}_Z\eta = -w(Z)\eta$$

i.e. Z is an infinitesimal conformal transformation of the structure 1-form η.

Next taking the Lie derivative of Ω with respect to Z, one has by (2.3)

$$\mathcal{L}_Z\Omega = d(\mu Z) - 2\lambda\eta(Z)\Omega + 2\lambda\eta \wedge (\mu Z) \tag{2.10}$$

and taking into account of (2.9), one derives by exterior differentiation that

$$d^{2\lambda\eta}(\mathcal{L}_Z\Omega) = 0. \tag{2.11}$$

Hence the Lie derivative of $\mathcal{L}_Z\Omega$ is $d^{2\lambda\eta}$-closed, as is Ω.

Let now L be the (1.1)-operator defined by

$$L\colon \alpha \to \alpha \wedge \Omega \qquad \alpha \in \Lambda^1 M$$

(see also [8]) and set

$$L^q \alpha = \alpha_q = \alpha \wedge \Omega^q \in \Lambda^{2q+1} M .$$

If $Z_h \in D_h$ is any horizontal vector field of M, one derives from above after some calculation that

$$\mathrm{d}^{2\lambda\eta}(\mathfrak{L}_{Z_h} \alpha_q) = 0 .$$

Therefore one may say that the Lie derivative $\mathfrak{L}_{Z_h}$ of any $(2q+1)$-form $L^q \alpha = \alpha_q$ is $\mathrm{d}^{2\lambda\eta}$-closed. Denote now by M_h the *leaf* of the horizontal foliation D_h (that is the hypersurface of M normal to the structure vector field ξ). Clearly M_h is a symplectic manifold having $\Omega_h = \Omega|_{M_h}$ as its structure 2-form. Let

$$W = \mu^{-1} w$$

be the dual vector field of w with respect to Ω_h. (In order to simplify, we denote the elements induced by $x\colon M_h \to M$ by the same letters.) Since by (2.2) one has

$$i_W \Omega_h = \mathrm{d} \log \lambda$$

it follows that on M_h, W is a *symplectic vector field* [15] and $\log \lambda$ is a *Hamiltonian function* on M_h.

Theorem 2.1. *Let $M(\Omega, \eta, \lambda, \xi, g)$ be a $(2m+1)$-dimensional biconformal cosymplectic C^∞-manifold with structure tensor fields $(\Omega, \eta, \lambda, \xi)$ and let $D_h = \{Z \in \mathcal{X}M\colon \eta(Z) = 0\}$ be the $(2m)$-foliation annihilated by the structure 1-form η.*

One has the following properties:

(i) *The structure vector field ξ and any $d_\Omega^{2\lambda\eta}$-closed vector field $Z \in \mathcal{X}M$ are infinitesimal conformal transformations of Ω.*

(ii) *If $Z_h \in D_h$ and Z is any horizontal vector and any contact d^w-closed ($w = \mathrm{d}\log\lambda$) vector field respectively, then the Lie derivatives $\mathfrak{L}_{Z_h}\Omega$ and $\mathfrak{L}_Z \Omega$ are $\mathrm{d}^{2\lambda\eta}$-closed, as is the structure 2-form Ω.*

(iii) *If $L^q\colon \lambda^1 M \to \lambda^{2q+1} M$; $L^q \alpha = \alpha \wedge \Omega^q$, then the Lie derivatives of all the $(2q+1)$-forms $L^q\alpha$ with respect to any horizontal vector field Z_h are $\mathrm{d}^{2\lambda\eta}$-closed,* *as is* Ω. *Finally, if* $\Omega_h = \Omega|_{D_h}$ *(restriction of* Ω *on* D_h*) is the symplectic form, then* $\log\lambda$ *is a Hamiltonian function of* Ω_h.

3 - B.C. quasi-Sasakian manifolds

Let $M(\Phi, \Omega, \eta, \xi, g)$ be a $(2m+1)$-dimensional quasi-Sasakian C^∞-manifold. As is known, the structure tensor fields $(\Phi, \Omega, \eta, \xi)$ satisfy

$$\begin{aligned}&\Phi^2 = -\,\mathrm{Id} + \eta\otimes\xi \qquad \Phi\xi = 0 \qquad \eta(\xi) = 1\\ &\eta(Z) = g(Z, \xi) \qquad g(\Phi Z, \Phi Z') = g(Z, Z') - \eta(Z)\,\eta(Z')\\ &\Omega(Z, Z') = g(\Phi Z, Z') \Rightarrow i_Z\Omega = \flat(\Phi Z)\end{aligned} \tag{3.1}$$

where $Z, Z' \in \mathcal{X}M$ are any vector fields on M. By imposing different geometric properties on the structure vector field ξ, one obtains different types of quasi-Sasakian manifolds.

Referring to the concept of a *contact quasi-concurrent vector field* $[3]_2$ we shall assume in this paper that ξ is such a geometrical vector field. In this case, following $[3]_2$, the covariant derivative $\nabla\xi$ of ξ satisfies

$$\nabla\xi = -\,\lambda\,\mathrm{d}p + \eta\otimes W + \lambda\eta\otimes\xi \tag{3.2}$$

where $\lambda \in C^\infty M$ is a conformal scalar and $W \in D_h$ is a horizontal vector field which is called the *associated vector field* of ξ.

Consider on M a local field of Φ-orthonormal frames $[9]_1$, denoted by

$$\mathcal{O}_\Phi = \mathrm{vect}\,\{e_a,\ e_{a^*} = \Phi e_a,\ e_0 = \xi | a = 1, \ldots, m;\ a^* = a + m\}$$

and let

$$\mathcal{O}_\Phi^* = \mathrm{covect}\,\{\omega^A\,|A = 1, \ldots, 2m\}$$

be the corresponding coframe. Cartan's structure equations written in index-free form, are then

$$\nabla e = \theta\otimes e \in A^1(M, TM) \tag{3.3}$$

$$\mathrm{d}\omega = -\,\theta\wedge\omega \tag{3.4}$$

$$\mathrm{d}\theta = -\,\theta\wedge\theta + \Theta \tag{3.5}$$

where $\theta \in \Lambda^1 M$ are the local connection forms in the tangent bundle TM and $\Theta \in \Lambda^2 M$ are the curvature 2-forms on M. With respect to $\mathcal{O}_\Phi^*$, the soldering form

$\mathrm{d}p$ and the structure 2-form Ω are expressed by

(3.6) $$\mathrm{d}p = \omega^a \otimes e_a + \omega^{a^*} \otimes e_{a^*} + \eta \otimes \xi$$

(3.7) $$\Omega = \sum_a \omega^a \wedge \omega^{a^*} .$$

Setting

(3.8) $$W = W^\alpha e_\alpha \qquad W^\alpha \in C^\infty M \qquad \alpha \in \{a,\ a^*\}$$

one derives from (3.2) with the help of (3.3), (3.6) and (3.8) that

(3.9) $$\theta = W^\alpha \eta - \lambda \omega^\alpha .$$

Putting

(3.10) $$w = \flat(W) = \sum_\alpha W^\alpha \omega^\alpha \in \Lambda^1 M$$

we shall assume in addition that ∇W is self-adjoint [15], that is

(3.11) $$\mathrm{d}w = 0 .$$

Now by the structure equations (3.4) and by exterior differentiation of the structure 1-form $\eta (\eta = \omega^0)$, one gets

(3.12) $$\mathrm{d}^w \eta = 0$$

that is η is d^w-closed. Further, taking the exterior derivatives of (3.7), one gets from (3.1), (3.4) and (3.9) that

(3.13) $$\mathrm{d}^{2\lambda\eta} \Omega = 0$$

which shows that Ω is $\mathrm{d}^{2\lambda\eta}$-closed. Hence, going back to the equation (2.3), we can obtain from (3.12) and (3.13) that the quasi-Sasakian manifold $M(\Phi,\ \Omega,\ \eta,\ \xi,\ g)$ under consideration is endowed with a B.C.-cosymplectic structure. Omitting reference to the generating point $p \in M$, one has by definition, that for any $Z \in \mathcal{X} M$

$$\mathrm{div}\ Z = \mathrm{tr}(\nabla Z) = \sum_A \omega^A (\nabla_{e_A} Z) .$$

Thus by (3.2) one quickly gets

(3.14) $$\mathrm{div}\ \xi = -2m\lambda .$$

Hence for any quasi-Sasakian manifold $M(\Phi,\ \Omega,\ \eta,\ \xi,\ g)$ with a structure vector ξ

satisfying (3.2), the structure scalar λ represents, up to the factor $-2m$, the divergence of ξ.

We notice that one has

$$\nabla_{\Phi Z}\xi = -\lambda\Phi Z \Rightarrow g(\nabla_{\Phi Z}\xi,\ Z) = 0 \tag{3.15}$$

and for any horizontal vector fields $Z_h,\ Z'_h \in D_h$ the equation

$$g(\nabla_{Z_h} Z'_h,\ Z_h) + g(\nabla_{Z_{h'}} Z'_h,\ Z'_h) = -2\lambda g(Z_h,\ Z'_h) \tag{3.16}$$

holds. Following a known definition, it follows from (3.16) and (3.14) that the structure vector field ξ is a *horizontal conformal vector field* (or a *D_h-conformal vector field*).

Let now $v \in \Lambda^1 M$ be any *semi-basic* 1-form (i.e. $v(\xi) = 0$), define $V := \flat^{-1}(v)$ and denote by σ_h the volume element of D_h. Making use of (3.3) and (3.4), one finds by (3.2), (3.8) and (3.9) that

$$\mathcal{L}_\xi v = \rho v + \flat[\xi,\ V] + g(V, W)\eta \tag{3.17}$$

where [,] denotes the Lie bracket, $\flat[\xi,\ V]$ is the dual form of the vector field $[\xi,\ V]$ and

$$\rho = \frac{\operatorname{div}\xi}{m}. \tag{3.18}$$

Further if $*\colon \Lambda^q T^*M \to \Lambda^{2m+1-q}T^*M$ denotes the *star operator*, one finds from (3.17) by a straightforward calculation that

$$\mathcal{L}_\xi * v = * \mathcal{L}_\xi v + \frac{2m-1}{2}\rho * v + g(V,\ W)\sigma_h. \tag{3.19}$$

It should be noticed that the above formulae are «mutatis mutandis» similar to those of T. Branson [2] for general conformal vector fields.

By (3.11), (3.12) and (3.13) one readily finds

$$\mathcal{L}_\xi\Omega = 2\lambda\Omega \qquad d(\mathcal{L}_\xi\eta) = 0$$

which shows that ξ defines an infinitesimal conformal transformation of Ω and that η is a *relative integral invariant of* ξ [1]. Hence, by reference to [5], we agree to say that ξ defines an *almost biconformal vector fields* on $M(\Phi,\ \Omega,\ \eta,\ \xi,\ g)$.

Let us now go back to the equation (3.2) and let $Z \in \mathcal{X}M$ be any vector field on

M. Then the structure equations (3.1) are completed with the following structure equation

$$(3.20)\qquad (\nabla\Phi)Z = \nabla(\Phi Z) - \Phi\nabla Z$$

$$= \lambda\eta(Z)\,\Phi\,dp - \eta(Z)\,\eta\otimes\Phi W + (\lambda\mathfrak{b}\,(\Phi Z) - g(W,\ \Phi Z)\,\eta)\otimes\xi$$

which holds for any B.C. quasi-Sasakian manifold. It should be noticed, that by setting $Z=\xi$ in (3.20), one obtains (3.2) again.

Consider now the *contact Φ-Lie differential operator*

$$\mathcal{D}_\Phi:\ Z\to(\mathcal{L}_\xi\Phi)Z.$$

As is known (see for example [7]), one has

$$(3.21)\qquad (\mathcal{L}_\xi\Phi)Z = [\xi,\ \Phi Z] - \Phi[\xi,\ Z].$$

Let us go back to the case under discussion and set $Z=W$ in (3.21). First of all, since ∇W is self-adjoint, one finds by (3.2) that

$$(3.22)\qquad \nabla_\xi W = \xi\lambda W - g(W,\ W)\,\xi.$$

Next, by making use of equation (3.20) and (3.1), one gets

$$(3.23)\qquad \nabla\Phi W = \Phi\nabla W + \lambda\mathfrak{b}\,(\Phi W)\otimes\xi$$

and this implies

$$(3.24)\qquad \nabla_\xi\Phi W = \Phi\nabla_\xi W.$$

But by (3.22) one has

$$(3.25)\qquad \Phi\nabla_\xi W = \xi\lambda\Phi W.$$

Finally, by means of (3.15), one gets

$$(\mathcal{L}_\xi\Phi)W = 0.$$

Hence one may say that the associated vector field W of ξ is *contact Φ-invariant.*

It also should be noticed that (3.2) and (3.23) imply

$$(3.26)\qquad \mathcal{L}\Phi W = 2\lambda\Phi W.$$

Hence, following a known definition [5], we may say that ΦW admits an infinitesimal transformation of ξ.

Denote now by $D = \{\xi, \nabla V, \Phi W\}$ the 3-distribution defined by ξ, ∇V, and ΦW. Since by (3.10) and (3.11) one may write

$$\nabla W = \lambda\eta \otimes W + (\lambda w - g(W, W)\eta) \otimes \xi \qquad w = \flat^{-1}(W) \tag{3.27}$$

then if X' and X'' are any vector fields of D, it follows from (3.2), (3.23) and (3.27), that one has

$$\nabla_{X''} X' \in D.$$

According to a well-known proposition (see for example [13]), this proves that D defines an *auto-parallel* (or *totally geodesic*) foliation. Therefore, we may say that any B.C. quasi-Sasakian manifold is foliated by totally geodesic 3-dimensional submanifolds tangent to ξ, W and ΦW.

Next by (3.27) one quickly gets at any point $p \in M$

$$\operatorname{tr}(\nabla W) = \operatorname{div} W = -g(W, W) \tag{3.28}$$

$$\frac{1}{2}\, dg(W, W) = \lambda g(W, W)\eta. \tag{3.29}$$

Recall now the general formula

$$\Delta\nu = -\operatorname{div}(\operatorname{grad} \nu) \qquad \nu \in C^\infty M.$$

Then, by (2.2), (3.28) and (3.29), one derives

$$\Delta g(W, W) = -2(2m-1)\lambda^2 g(W, W) \tag{3.30}$$

which shows that $g(W, W)$ is an *eigenfunction* of Δ and has $-2(2m-1)\lambda^2$ as the *associated eigenvalue*. Since this eigenvalue is negative, we conclude by reference to a known property (see for example [17]) that compact B.C. quasi-Sasakian manifold do not exist. Further since by (3.29) one has

$$\operatorname{grad} g(W, W) = 2\lambda g(W, W)\xi \Rightarrow \|\operatorname{grad} g(W, W)\|^2 = 4\lambda^2 g(W, W)^2 \tag{3.31}$$

it follows by reference to a known definiton that $g(W, W)$ is an *isoparametric* function (see for example [18] or $[9]_2$).

On the other hand, by (2.2), (2.3), (3.2) and (3.27), taking the second covariant differential of ξ and W, one finds

$$\nabla^2 \xi = \lambda(\lambda\eta - w) \wedge dp + (\eta \wedge w) \otimes (W \cdot \lambda\xi) \tag{3.32}$$

$$\nabla^2 W = \lambda(\lambda w - g(W, W)\eta) \wedge dp + (\eta \wedge w) \otimes (\lambda W \cdot g(W, W)\xi). \tag{3.33}$$

Consider now the vector valued 1-form

$$F = \xi \wedge W = w \otimes \xi - \eta \otimes W \in A^1(M,\ TM). \tag{3.34}$$

Operating on F by d^{∇^2} and taking into account (3.32) and (3.33), one finds

$$\mathrm{d}^{\nabla^2} F = \nabla^2 \xi \wedge w - \nabla^2 W \wedge \eta = 2\lambda^2 (w \wedge \eta) \wedge \mathrm{d}p.$$

Hence by reference to [14] one may say that F is a 2-exterior concurrent vector valued 1-form, having $2\lambda^2 w \wedge \eta$ as a concurrence 2-form.

Since a problem of current interest is the curvature problem, we shall make now the following consideration. Making use of equations (3.3) and (3.20), one finds the relations

$$\theta^a_b = \theta^{a^*}_{b^*} \qquad \theta^a_{b^*} = \theta^b_{a^*} \tag{3.35}$$

which are characteristic for quasi-Sasakian manifolds. Now with the help of the structure equations (3.5) one derives from (3.26)

$$\begin{aligned} &\Theta^a_b + \lambda^2 \omega^a \wedge \omega^b + \lambda(i_W \omega^a \wedge \omega^b) \wedge \eta = \Theta^{a^*}_{b^*} + \lambda^2 \omega^{a^*} \wedge \omega^{b^*} + \lambda(i_W \omega^{a^*} \wedge \omega^{b^*}) \wedge \eta \\ &\Theta^a_{b^*} + \lambda^2 \omega^a \wedge \omega^{b^*} + \lambda(i_W \omega^a \wedge \omega^{b^*}) \wedge \eta = \Theta^b_{a^*} + \lambda^2 \omega^b \wedge \omega^{a^*} + \lambda(i_W \omega^b \wedge \omega^*) \wedge \eta. \end{aligned} \tag{3.36}$$

Since the characteristic equation for space-forms $M(K)$ are

$$\Theta^A_B = K\omega^A \wedge \omega^B$$

it follows from (3.27) that non-trivial quasi-Sasakian manifolds of constant curvature do not exist.

Let now R be the curvature tensor field on $M(\Phi,\ \Omega,\ \eta,\ \xi,\ g)$. Then $(R(Z,\ Z') \in \Gamma\ \mathrm{End}\ \Lambda M$. With the help of (3.20) and (3.27) one finds afther some calculations

$$R(Z,\ Z')X + \Phi R(Z,\ Z')\Phi X \tag{3.37}$$

$$= (\eta(X)R(Z,\ Z') + g(X,\ R(Z,\ Z')\xi)\xi + \lambda^2(\Phi Z' \wedge \Phi Z)\Phi X + \lambda^2(Z' \wedge Z)X$$

$$+\lambda g(X,\ \Phi W)(\eta(Z')\Phi Z - \eta(Z)\Phi Z')\lambda(g(X,\ W) - \lambda\eta(X))(Z' Z)\xi + (-\eta(Z)g(X,\ Z')$$

$$+\eta(Z')g(X,\ Z))(\lambda^2\xi + \lambda W) + \lambda(-\eta(Z)g(\Phi X,\ Z') + \eta(Z')g(\Phi X,\ Z))\Phi W.$$

The following theorem combines all results obtained in this section.

Theorem 3.1. *Let $M(\Phi, \Omega, \eta, \xi, g)$ be a $(2m+1)$-dimensional quasi-Sasakian manifold and let $D_h = \{Z \in \mathcal{X}M;\ \eta(Z) = 0\}$ be the horizontal $(2m)$-distribution annihilated by the structure 1-form η. Then the necessary and sufficient condition in order that M be endowed with a B.C.-structure, is the structure vector field ξ be contact quasi-concurrent with horizontal and closed associated vector field $W \in D_h$. Any such manifold $M(\Phi, \Omega, \eta, \xi, g)$ is foliated by a totally geodesic 3-dimensional submanifold tangent to ξ, W, and ΦW. One has also the following properties:*

(i) *ξ is a D_h-conformal vector field and* $\operatorname{div}\xi = -2m\lambda$, *where $\lambda(\mathrm{d}\log\lambda = -\flat^{-1}(W))$ is the structure scalar of the B.C.-structure.*

(ii) *$g(W, W)$ is an eigenfunction of Δ and it is an isoparametric function* [18].

(iii) *If v is any semi-basic 1-form and $V = \flat^{-1}v$ is its dual vector field, one has the following formulae*

$$\mathcal{L}_\xi v = \rho v + \flat[\xi, V] + g(V, W)\eta \qquad \mathcal{L}_\xi * v = * \mathcal{L}_\xi v + \frac{2m-1}{2}\rho * v + g(V, W)\sigma_h$$

where $\rho = \dfrac{\operatorname{div}\xi}{m}$ *and σ_h is the volume element of D_h.*

(iv) *W is a contact Φ-invariant vector field, and B.C. quasi-Sasakian manifolds of constant curvature do not exist.*

4 - Submanifolds of B.C. quasi-Sasakian manifolds

We shall discuss in this section various striking properties of some submanifolds of the manifold $M(\Phi, \Omega, \eta, \xi, g)$ under discussion.

First of all consider the immersion x: $M_h \to M$, where M_h is the hypersurface normal to the structure vector field ξ (see 2). Because of (3.2), the second fundamental quadratic form associated with x is

$$\mathrm{II} = -\langle \mathrm{d}p\, \nabla\xi \rangle = \lambda g \tag{4.1}$$

(we denote elements induced by x by the same letters), and the above equation proves that M_h is an *umbilical* hypersurface of M. Since on M_h one has

$$\nabla\xi = -\lambda\, \mathrm{d}p$$

it follows at once from (2.2) that

$$\mathrm{d}^{\nabla}(\nabla\xi) = \nabla^2\xi = -\lambda w \wedge \mathrm{d}p. \tag{4.2}$$

So, referring to (1.4), one may say that ξ is a normal exterior recurrent vector field.

Let then Θ_0^α be the normal curvature 2-forms associated with x. One derives at once from (4.2) that

$$\Theta_0^\alpha = -\lambda w \wedge \omega^\alpha \tag{4.3}$$

which shows that all forms Θ_0^α are conformal to the induced value of the cohomology form w.

Further by (3.36) it is easily seen that if M_h is a space-form of curvature K, then necessarily $K+\lambda^2=0$ that is M_h is an *extrinsic hypersphere.*

Moreover, if $\langle l \rangle$ denoted the length of the second fundamental quadratic form of M_h, then on $M_h(-\lambda^2)$, $\langle l \rangle$ is constant, and since the mean curvature vector of $M_h(-\lambda^2)$ (i.e. ξ) is nowhere zero, it follows that the product submanifold $M_h(-\lambda^2)\times M_h(-\lambda^2)$ is an $\mathcal{U}$-*submanifold* (see [4]) in $M\times M$. It should be noticed that by virtue of (4.2), one has on $M_h(-\lambda^2)$

$$\nabla^2\xi = 0$$

and this shows that the normal connection $\nabla^\perp$ associated with x: $M_h(-\lambda^2) \to M(\Phi,\ \Omega,\ \eta,\ \xi,\ g)$ is *flat.* Let now $X \in \mathcal{X}M_h$ be any exterior concurrent vector field on M_h. Then it follows from (1.4) that

$$\nabla^2 X = f\,\mathfrak{b}(X)\wedge \mathrm{d}p \tag{4.4}$$

holds for some $C^\infty M$.

On the other hand, on M_h, the formula (3.37) moves to

$$R(Z,\ Z')X + \Phi R(Z,\ Z')\Phi X \tag{4.5}$$

$$= \lambda^2(\mathfrak{b}(\Phi X)\wedge \Phi\ \mathrm{d}p)(Z,\ Z') + \lambda^2(\mathfrak{b}(X)\wedge \mathrm{d}p(Z,\ Z').$$

Then if X satisfies (4.4), one derives from (4.5) that $f=\lambda^2$, and in this case one also has

$$\nabla^2\Phi X = \lambda^2\,\mathfrak{b}(\Phi X)\wedge \mathrm{d}p$$

that is the property of exterior concurrency for X is invariant when applying Φ to X.

Let us now M_I be an *invariant submanifold* of $M(\Phi,\ \Omega,\ \eta,\ \xi,\ g)$, that is ξ is tangent to M_I for any tangent vector field Z to M_I. Assume that M_I is of codimension 2 and is defined by

(4.6)
$$\omega^r = 0 \qquad \omega^{r^*} = 0$$
$$r = m + 1 - l \qquad r^* = r + m.$$

Hence the soldering form dp_I of M_I is

(4.7)
$$dp_I = \omega^i \otimes e_i + \omega^{i^*} \otimes e_{i^*} + \eta \otimes \xi$$
$$i = 1,\ \ldots,\ m - l \qquad i^* = i + m.$$

(We denote the other elements induced by y: $M_I \to M$ by the same letters.) Consequently the *mean curvature vector valued* $2(m-l)$*-form* $\mathcal{H} \in A^{2(m-l)}(M_I,\ TM_I)$ is

(4.8)
$$\mathcal{H} = \Sigma(-1)^{i-1}\omega^1 \wedge \ldots \wedge \hat{\omega}^i \wedge \ldots \wedge \omega^{m-l} \wedge \omega^{1^*} \wedge \ldots \omega^{(m-l)^*} \wedge \eta \otimes e_I$$
$$+\Sigma(-1)^{i^*-1}\omega^1 \wedge \ldots \wedge \omega^{m-l} \wedge \omega^{1^*} \wedge \ldots \wedge \hat{\omega}^{i^*} \wedge \ldots \wedge \omega^{(m-l)^*} \wedge \eta \otimes e_{I^*}$$
$$+\omega^1 \wedge \ldots \wedge \omega^{m-l} \wedge \omega^{1^*} \wedge \ldots \wedge \omega^{(m-l)^*} \otimes \xi.$$

Applying the operator d^∇ to $\mathcal{H}$, one has

(4.9)
$$d^\nabla \mathcal{H} = (2(m-l)+1)\,\sigma_I \otimes H$$

where σ_I and H are the volume element of M_I and the *mean curvature vector field* associated with y: $M_I \to M$, respectively. With the help of (3.35) one gets $d^\nabla \mathcal{H} = 0 \Rightarrow H = 0$, which expresses that any M_I is *minimal* in $M(\Phi,\ \Omega,\ \eta,\ \xi,\ g)$. (See [**11**] for K-manifolds and [**19**]$_2$ for Sasakian manifolds.)

Finally consider the immersion z: $M_A \to M$, where M_A is an *anti-invariant* submanifold of dimension m of M [4]. Then by definition M_A is normal to ξ and if Z is any tangent vector field to M_A, then ΦZ is a normal vector field to M_A. If we assume that M_A is defined by

$$\omega^{a^*} = 0 \qquad \eta = 0$$

then the equations (3.36) become

(4.11)
$$\Theta^a_b + \lambda^2 \omega^a \wedge \omega^b = \Theta^{a^*}_{b^*} \qquad \Theta^a_{b^*} = \Theta^b_{a^*}.$$

From above we derive that if M_A is of constant curvature K and the normal connection ∇^2 is flat, then necessarily $K = -\lambda^2$ and M_A is of hyperbolic type.

Theorem 4.1. *Let $M(\Phi, \Omega, \eta, \xi, g)$ be any B.C. quasi-Sasakian manifold. We consider the following immersions: $x: M_h \rightarrow M$, $y: M_I \rightarrow M$, and $z: M_A \rightarrow M$ where M_h, M_I and M_A are the symplectic hypersurface normal to ξ, an invariant submanifold and an anti-invariant submanifold of dimension m, respectively.*

One has the following properties:

(i) *M_h is an umbilical hypersurface, and if M_h is a space-form, it is necessarily of hyperbolic type, $M(-\lambda^2)$. In this case $M(-\lambda^2)$. is an extrinsic hypersphere and the product $M(-\lambda^2) \times M(-\lambda^2)$ is an $\mathcal{U}$-submanifold of $M \times M$. Further the conformal associated scalar with any exterior concurrent vector field X on M_h is λ^2, and the property of exterior concurrency for X is invariant when applying Φ to X.*

(ii) *Any invariant submanifold M_I of M is minimal.*

(iii) *Any anti-invariant submanifold M_A is of constant curvature and a flat normal connection is of hyperbolic type $M(-\lambda^2)$.*

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Abstract

Let M be Riemannian $(2m+1)$*-dimensional* C^∞*-manifold endowed with a structure 2-form* Ω*, a structure 1-form* η *and a structure vector field* ξ *dual to* η*. As a generalization of conformal cosymplectic manifolds,* $M(\Omega, \eta, \xi, g)$ *is defined in the present paper as a «biconformal cosymplectic manifold» if both structure forms* Ω *and* η *are «cohomologically closed» (in the sense of F. Guedira and A. Lichnerowitz). With such a structure denoted by B.C.* $\mathrm{Sp}(2m+1, \boldsymbol{R})$ *is associated a closed 1-form w and its dual vector W. Different properties of the* d^ω*-cohomology and the Lie algebra on M involving* η*,* u*,* Ω*,* W *and* ξ *are discussed. If* M_h *is the hypersurface normal to* ξ*, the following salient properties are established:* (a) *If M is a conformal cosymplectic manifold, then* M_h *is a symplectic manifold.* (b) *If M is a biconformal cosymplectic manifold, then* M_h *is a conformal symplectic manifold.*

As an application, we get the necessary and sufficient condition for a $(2m+1)$*-dimensional quasi-Sasakian manifold* $M(\Phi, \Omega, \eta, \xi, g)$ *to be endowed with a B.C.-structure. Some striking properties of invariant and anti-invariant submanifolds of a B.C.-quasi-Sasakian manifold are discussed.*

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