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Biconformal cosymplectic manifolds (**) 

0 - Introduction

In the last twenty years many papers have been devoted to almost cosymplectic manifolds $M(\Omega, \gamma, \xi, g)$. These are $(2m + 1)$-dimensional manifolds $M$ endowed with a pseudo-Riemannian metric $g$, a vector field $\xi$ and a 2-form $\Omega$. If $\gamma$ denotes the 1-form associated to $\xi$ by the metric $g$ (we write $\gamma = b(\xi)$), then $\Omega^m \wedge \gamma \neq 0$ holds.

An interesting subclass of the almost cosymplectic manifolds are the manifolds with a conformal cosymplectic structure: in terms of the cohomology operator $d^\omega$ [8], defined by

$$(0.1) \quad d^\omega \alpha = d\alpha + \omega \wedge \alpha \quad d\omega = 0 \quad \text{for any} \ \alpha \in \Lambda^p(M)$$

they are distinguished by the additional relation

$$d^\omega \gamma = 0$$

for some 1-form $\omega$ (see [3]$_1$, [12]$_1$, [16]$_1$ and [16]$_2$). The best known examples of conformal cosymplectic manifolds are the Kenmotsu manifolds (or K-manifolds [11]).

In the present paper a biconformal cosymplectic manifold (abbr. B.C.) is defined to be a conformal cosymplectic manifold satisfying

$$d^\omega \gamma = 0 \quad d^{2\gamma} \Omega = 0$$

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where \( \lambda \in C^\infty M \) is given by \( w = d \log \lambda \). (Remember: \( d(2\lambda \eta) \) must vanish because of (0.1)).

In 2 we deal with some properties of Lie algebra defined by the B.C. cosymplectic structure. Since the horizontal distribution

\[
D_h := \{ Z \in \mathfrak{X} M : \eta(Z) = 0 \}
\]

is involutive, \( M \) is foliated by \((2m)\)-dimensional symplectic hypersurfaces \( M_h \) normal to \( \xi \). It is shown that \( \log \lambda \) is a Hamiltonian function for the symplectic form \( \Omega_h = \Omega|_{M_h} \).

In 3 we consider general quasi-Sasakian manifold \( M(\Phi, \Omega, \eta, \xi, g) \) [12] and prove that \( M \) is endowed with a B.C. cosymplectic structure if and only if the structure vector field \( \xi \) is contact quasi-concurrent [3] with horizontal and closed associated vector field \( W \in D_h \). If \( V \in D_h \) is any horizontal vector field, one has the formula

\[
\mathcal{L}_\xi b(V) = \rho b(V) + b[\xi, V] + g(V, W) \eta \quad \text{with} \quad \rho = \frac{\text{div} \xi}{m}.
\]

It is proved that any manifold \( M(\Phi, \Omega, \eta, \xi, g) \) is foliated by a totally geodesic 3-dimensional submanifold tangent to \( \xi \), \( W \) and \( \phi W \), and that \( g(W, W) \) is an isoparametric function [18]. It is also showed that compact manifolds \( M(\Phi, \Omega, \eta, \xi, g) \) or space-forms \( M(\Phi, \Omega, \eta, \xi, g) \) do not exist.

In 4 we outline some properties of the immersions \( x: M_h \to M \), \( y: M_I \to M \), where \( M_I \) is an invariant submanifold of \( M \), and \( z: M_A \to M \), where \( M_A \) is an anti-invariant submanifold of dimension \( m \).

1 - Preliminaries

Let \((M, g)\) be a Riemannian or pseudo-Riemannian \( C^\infty \)-manifold and let \( \nabla \) be the covariant differential operator defined by the metric tensor \( g \). We assume in the following that \( M \) is orientable and that the connection \( \nabla \) is symmetric.

Let \( \Gamma(TM) = \mathfrak{X} M \) and \( b: TM \to T^* M \) be the set of sections of the tangent bundle \( TM \) and the musical isomorphism [15] defined by \( g \), respectively.

Following [15], we denote by

\[
A^q(M, TM) = \Gamma \operatorname{Hom}(\Lambda^q TM, TM)
\]

the set of vector valued \( q \)-forms, \( q < \dim M \), and write for the exterior covariant
derivative operator with respect to $\nabla$

$$d^\nabla: A^q(M, TM) \to A^{q+1}(M, TM).$$

(Notice that in general $d^{\nabla^2} = d^\nabla \circ d^\nabla \neq 0$.)

If $p \in M$, then the vector valued 1-form $dp \in A^1(M, TM)$ stands for the soldering form of $M$. ($dp$ is also called the «line element». Since $\nabla$ is symmetric, one has $d^\nabla(dp) = 0$ [6].)

The cohomology operator $d^\omega$ was defined in Introduction as

$$d^\omega = d + e(\omega),$$

(cf. [8]) acting on $\Lambda M$, where $e(\omega)$ denotes the exterior product by the closed 1-form $\omega \in \Lambda^1 M$.

Clearly one has

$$d^\omega \circ d^\omega = 0.$$

Any form $u \in \Lambda M$ such that

$$d^\omega u = 0$$

is said to be $d^\omega$-closed, and $\omega$ is called the cohomology form (abbr. c.f.).

An exterior concurrent vector field is defined ([16], [14]) as a vector field $X \in \mathfrak{X}M$ for which the relation

$$d^\nabla(\nabla X) = \nabla^2 X = \pi \wedge dp \in A^2(M, TM)$$

holds for some $\pi \in \Lambda^1 M$.

If $X$ is a tangent vector field, then the 1-form $\pi$, which is called the concurrence form, is expressed by $\pi = f \, b(X)$, where $f \in C^\infty(M)$ is the conformal scalar associated with $X$.

A pair $(\alpha, \beta)$, where $\alpha$ is a $p$-form and $\beta$ is a $(p-1)$-form, defines a $p$-cocycle if and only if

$$d\beta = 0 \quad d^\omega \alpha = \Omega \wedge \beta$$

where $\Omega$ is $d^\omega$-closed. In order that $(\alpha, \beta)$ is an exact $p$-cocycle, it is necessary and sufficient that there exists a $(p-1)$-cochain $(\psi_2, \psi)$ (in the sense of the differential cohomology of Chevalley) such that

$$\alpha = -d^\omega \psi_2 + \Omega \wedge \psi_1 \quad \beta = d\psi_1.$$
2 - Biconformal cosymplectic manifolds

Let \( M(\Omega, \gamma, \xi, g) \) be a \((2m+1)\)-dimensional Riemannian \( C^\infty \)-manifold endowed with an almost cosymplectic structure \( 1 \times \text{Sp}(m; \mathbb{R}) \) in the broad sense:

Let \( \Omega \in \Lambda^2 M \), \( \gamma \in \Lambda^1 M \) and \( \xi = b^{-1} \gamma \in \mathfrak{X} M \) be the structure 2-form, the structure 1-form and the structure vector field of \( 1 \times \text{Sp}(m; \mathbb{R}) \) respectively.

The \((2m)\)-distribution \( D_h := \{ Z \in \mathfrak{X} M : \gamma(Z) = 0 \} \) annihilated by \( \gamma \) is called the horizontal distribution and any field \( Z \in \mathfrak{X} M \) on \( M \) may be written as

\[
Z = Z_h + \gamma(Z) \xi
\]

where \( Z_h \in D_h \) is the horizontal component of \( Z \).

If for any globally exact basic form \( w \in D_h^* = \{ \alpha \in \Lambda^1 M : \alpha(\xi) = 0 \} \), say

\[
w = d \log \lambda
\]

the structure forms \( \gamma \) and \( \Omega \) satisfy

\[
d^w \gamma = 0 \quad d^{2x} \Omega = 0
\]

we say that the pairing \( (\gamma, \Omega) \) defines a biconformal cosymplectic structure (abr. B.C.-structure). By (2.2) and (2.3) it follows that \( \gamma \) (resp. \( \Omega \)) is \( d^w \)-closed (resp. \( d^{2x} \)-closed), and \( w \) and \( 2\lambda \gamma \) will be called the cohomology forms associated with \( (\gamma, \Omega) \). Clearly one has

\[
d(\lambda \gamma) = 0
\]

and we agree to call \( \lambda \in C^\infty M \) the structure scalar associated with B.C.-structure. It is also easily seen from (2.3) that \( D_h \) defines a \((2m)\)-foliation and that the restriction \( \Omega|_{D_h} \) is a symplectic form. Referring to (1.6), we see that for some 1-form \( \varphi \), the pairing \( (\alpha, \beta) \), such that

\[
\alpha = - d^{2x} \varphi + \lambda \Omega \quad \beta = d\lambda
\]

defines an exact cocycle.

Let now \( Z_h \in D_H \) be any horizontal vector field. By (2.3) one gets

\[
(\mathcal{L}_{Z_h} + \iota(Z_h)) \gamma = 0
\]

and

\[
d^{2x} (i_{Z_h} \Omega) = \mathcal{L}_{Z_h} \Omega.
\]
Since by (1.2) one derives from (2.5)
\[ d^{2\gamma} (\mathcal{L}_Z \Omega) = 0 \]
one may say that any \( Z \in \mathfrak{X} \) is an \textit{infinitesimal conformal transformation} of \( \gamma \) and that the Lie derivative \( \mathcal{L}_Z \Omega \) is \( d^{2\gamma} \)-closed, as is \( \Omega \).

Since \( \gamma(\xi) = 1 \), it is also easily seen that one has
\[ \mathcal{L}_\xi \Omega = -2\lambda \Omega \tag{2.6} \]
which proves that the structure vector \( \xi \) is an infinitesimal conformal transformal of \( \Omega \).

Furthermore, let \( Z \in \mathfrak{X} \) be any vector field of \( M \), and let \( \mu: TM \to T^*M; \ Z \to i_Z \Omega \) be the bundle isomorphism defined by \( \Omega \). If \( Z \) is such that it satisfies
\[ d^{2\gamma} (\mu Z) = 0 \tag{2.7} \]
we agree to say that \( Z \) is a \( d^{2\gamma} \)-closed \textit{vector field}.

For any vector field \( Z \) satisfying (2.7), one finds after a short calculation and by reference to (2.3) that
\[ \mathcal{L}_Z \Omega = -2\lambda \gamma(Z) \Omega \tag{2.8} \]
i.e. \( Z \) is an infinitesimal conformal automorphism of \( \Omega \).

In a similar manner, any vector field \( Z \) such that
\[ d^w \gamma(Z) = d\gamma(Z) + \gamma(Z) w = 0 \tag{2.9} \]
will be defined as \textit{contact} \( d^w \)-closed.

From the first equation (2.3) one quickly gets
\[ \mathcal{L}_Z \gamma = -w(Z) \gamma \]
i.e. \( Z \) is an infinitesimal conformal transformation of the structure 1-form \( \gamma \).

Next taking the Lie derivative of \( \Omega \) with respect to \( Z \), one has by (2.3)
\[ \mathcal{L}_Z \Omega = d(\mu Z) - 2\lambda \gamma(Z) \Omega + 2\lambda \gamma \wedge (\mu Z) \tag{2.10} \]
and taking into account of (2.9), one derives by exterior differentiation that
\[ d^{2\gamma} (\mathcal{L}_Z \Omega) = 0 \tag{2.11} \]
Hence the Lie derivative of \( \mathcal{L}_Z \Omega \) is \( d^{2\gamma} \)-closed, as is \( \Omega \).
Let now $L$ be the (1.1)-operator defined by
\[ L: \alpha \mapsto \alpha \wedge \Omega \quad \alpha \in \Lambda^1 M \]
(see also [8]) and set
\[ L^q \alpha = \alpha_q = \alpha \wedge \Omega^q \in \Lambda^{2q+1} M. \]
If $Z_h \in D_h$ is any horizontal vector field of $M$, one derives from above after some calculation that
\[ d^{2q} (\mathcal{L}_{Z_h} \alpha_q) = 0. \]
Therefore one may say that the Lie derivative $\mathcal{L}_{Z_h}$ of any $(2q + 1)$-form $L^q \alpha = \alpha_q$ is $d^{2q}$-closed. Denote now by $M_h$ the leaf of the horizontal foliation $D_h$ (that is the hypersurface of $M$ normal to the structure vector field $\xi$). Clearly $M_h$ is a symplectic manifold having $\Omega_h = \Omega|_{M_h}$ as its structure 2-form. Let
\[ W = \mu^{-1} w \]
be the dual vector field of $w$ with respect to $\Omega_h$. (In order to simplify, we denote the elements induced by $\pi: M_h \to M$ by the same letters.) Since by (2.2) one has
\[ i_W \Omega_h = d \log \lambda \]
it follows that on $M_h$, $W$ is a symplectic vector field [15] and $\log \lambda$ is a Hamiltonian function on $M_h$.

Theorem 2.1. Let $M(\Omega, \eta, \lambda, \xi, g)$ be a $(2m + 1)$-dimensional biconformal cosymplectic $C^\infty$-manifold with structure tensor fields $(\Omega, \eta, \lambda, \xi)$ and let $D_h = \{ Z \in \mathfrak{X} M: \eta(Z) = 0 \}$ be the $(2m)$-foliation annihilated by the structure 1-form $\eta$.

One has the following properties:

(i) The structure vector field $\xi$ and any $d^{2q}$-closed vector field $Z \in \mathfrak{X} M$ are infinitesimal conformal transformations of $\Omega$.

(ii) If $Z_h \in D_h$ and $Z$ is any horizontal vector and any contact $d^w$-closed ($w = d \log \lambda$) vector field respectively, then the Lie derivatives $\mathcal{L}_{Z_h} \Omega$ and $\mathcal{L}_{Z} \Omega$ are $d^{2q}$-closed, as is the structure 2-form $\Omega$.

(iii) If $L^q: \lambda^1 M \to \lambda^{2q+1} M$; $L^q \alpha = \alpha \wedge \Omega^q$, then the Lie derivatives of all the $(2q + 1)$-forms $L^q \alpha$ with respect to any horizontal vector field $Z_h$ are $d^{2q}$-closed,
as is $\Omega$. Finally, if $\Omega_h = \Omega|_{D_h}$ (restriction of $\Omega$ on $D_h$) is the symplectic form, then log $\lambda$ is a Hamiltonian function of $\Omega_h$.

3 - B.C. quasi-Sasakian manifolds

Let $M(\Phi, \Omega, \eta, \xi, g)$ be a $(2m + 1)$-dimensional quasi-Sasakian $C^\infty$-manifold. As is known, the structure tensor fields $(\Phi, \Omega, \eta, \xi)$ satisfy

$$\begin{aligned}
\Phi^2 &= - \text{Id} + \eta \otimes \xi \\
\Phi \xi &= 0 \\
\eta(\xi) &= 1
\end{aligned}$$

(3.1)

$$\begin{aligned}
\eta(Z) &= g(Z, \xi) \\
g(\Phi Z, \Phi Z') &= g(Z, Z') - \eta(Z) \eta(Z') \\
\Omega(Z, Z') &= g(\Phi Z, Z') \Rightarrow i_Z \Omega = b(\Phi Z)
\end{aligned}$$

where $Z, Z' \in \mathfrak{X}M$ are any vector fields on $M$. By imposing different geometric properties on the structure vector field $\xi$, one obtains different types of quasi-Sasakian manifolds.

Referring to the concept of a contact quasi-concurrent vector field [3]_2 we shall assume in this paper that $\xi$ is such a geometrical vector field. In this case, following [3]_2, the covariant derivative $\nabla \xi$ of $\xi$ satisfies

$$\nabla \xi = - \lambda \, dp + \eta \otimes W + \lambda \xi \otimes \xi$$

(3.2)

where $\lambda \in C^\infty M$ is a conformal scalar and $W \in D_h$ is a horizontal vector field which is called the associated vector field of $\xi$.

Consider on $M$ a local field of $\Phi$-orthonormal frames [9]_1, denoted by

$$\mathcal{O}_\Phi = \text{vect} \{ e_a, \ e_{a^*} = \Phi e_a, \ e_0 = \xi | a = 1, \ldots, m; \ a^* = a + m \}$$

and let

$$\mathcal{O}_\Phi^\ast = \text{covect} \{ \omega^A | A = 1, \ldots, 2m \}$$

be the corresponding coframe. Cartan's structure equations written in index-free form, are then

$$\nabla e = \theta \otimes e \in A^1(M, TM)$$

(3.3)

$$d\omega = - \theta \wedge \omega$$

(3.4)

$$d\theta = - \theta \wedge \theta + \Theta$$

(3.5)

where $\theta \in \Lambda^1 M$ are the local connection forms in the tangent bundle $TM$ and $\Theta \in \Lambda^2 M$ are the curvature 2-forms on $M$. With respect to $\mathcal{O}_\Phi^\ast$, the soldering form
dp and the structure 2-form Ω are expressed by

\[(3.6)\]
\[dp = \omega^{a} \otimes e_{a} + \omega^{a^{*}} \otimes e_{a^{*}} + \gamma \otimes \xi \]

\[(3.7)\]
\[\Omega = \sum_{a} \omega^{a} \wedge \omega^{a^{*}}.\]

Setting

\[(3.8)\]
\[W = W^{a}e_{a}, \quad W^{a} \in \mathcal{C}^{\infty}_{M} \quad a \in \{a, a^{*}\}\]

one derives from (3.2) with the help of (3.3), (3.6) and (3.8) that

\[(3.9)\]
\[\theta = W^{a} \gamma - \lambda \omega^{a}.\]

Putting

\[(3.10)\]
\[w = b(W) = \sum_{a} W^{a} \omega^{a} \in \Lambda^{1}_{} M\]

we shall assume in addition that VW is self-adjoint [15], that is

\[(3.11)\]
\[dw = 0.\]

Now by the structure equations (3.4) and by exterior differentiation of the structure 1-form \(\eta(\eta = \omega^{0})\), one gets

\[(3.12)\]
\[d^{w}_{\eta} = 0\]

that is \(\eta\) is \(d^{w}\)-closed. Further, taking the exterior derivatives of (3.7), one gets from (3.1), (3.4) and (3.9) that

\[(3.13)\]
\[d^{2\phi}_{\omega} \Omega = 0\]

which shows that \(\Omega\) is \(d^{2\phi}_{\omega}\)-closed. Hence, going back to the equation (2.3), we can obtain from (3.12) and (3.13) that the quasi-Sasakian manifold \(M(\phi, \Omega, \eta, \xi, g)\) under consideration is endowed with a B.C.-cosymplectic structure. Omitting reference to the generating point \(p \in M\), one has by definition, that for any \(Z \in \mathcal{X}_{} M\)

\[
\text{div } Z = \text{tr } (\nabla Z) = \sum_{A} \omega^{A} (\nabla_{e_{A}} Z).
\]

Thus by (3.2) one quickly gets

\[(3.14)\]
\[\text{div } \xi = -2m\lambda.\]

Hence for any quasi-Sasakian manifold \(M(\phi, \Omega, \eta, \xi, g)\) with a structure vector \(\xi\)
satisfying (3.2), the structure scalar $\lambda$ represents, up to the factor $-2m$, the divergence of $\xi$.

We notice that one has

\begin{equation}
\nabla_{\phi Z} \xi = -\lambda \Phi Z \Rightarrow g(\nabla_{\phi Z} \xi, Z) = 0
\end{equation}

and for any horizontal vector fields $Z_h, Z'_h \in D_h$ the equation

\begin{equation}
g(\nabla_{Z_h} Z'_h, Z_h) + g(\nabla_{Z'_h} Z_h, Z'_h) = -2\lambda g(Z_h, Z'_h)
\end{equation}

holds. Following a known definition, it follows from (3.16) and (3.14) that the structure vector field $\xi$ is a horizontal conformal vector field (or a $D_h$-conformal vector field).

Let now $v \in \Lambda^1 M$ be any semi-basic 1-form (i.e. $v(\xi) = 0$), define $V := b^{-1}(v)$ and denote by $\sigma_h$ the volume element of $D_h$. Making use of (3.3) and (3.4), one finds by (3.2), (3.8) and (3.9) that

\begin{equation}
\mathcal{L}_\xi v = \varphi v + b[\xi, V] + g(V, W) \sigma_h
\end{equation}

where $[ , ]$ denotes the Lie bracket, $b[\xi, V]$ is the dual form of the vector field $[\xi, V]$ and

\begin{equation}
\varphi = \frac{\text{div} \xi}{m}.
\end{equation}

Further if $*: \Lambda^q T^* M \to \Lambda^{2m+1-q} T^* M$ denotes the star operator, one finds from (3.17) by a straightforward calculation that

\begin{equation}
\mathcal{L}_\xi * v = * \mathcal{L}_\xi v + \frac{2m-1}{2} \sigma_h * v + g(V, W) \sigma_h.
\end{equation}

It should be noticed that the above formulae are «mutatis mutandis» similar to those of T. Branson [2] for general conformal vector fields.

By (3.11), (3.12) and (3.13) one readily finds

\[ \mathcal{L}_\xi \Omega = 2\lambda \Omega \quad \text{and} \quad \mathcal{L}_\xi \eta = 0 \]

which shows that $\xi$ defines an infinitesimal conformal transformation of $\Omega$ and that $\eta$ is a relative integral invariant of $\xi$ [1]. Hence, by reference to [5], we agree to say that $\xi$ defines an almost biconformal vector field on $M(\Phi, \Omega, \eta, \xi, g)$.

Let us now go back to the equation (3.2) and let $Z \in \mathcal{X} M$ be any vector field on
$M$. Then the structure equations (3.1) are completed with the following structure equation

\[(\nabla \phi) Z = \nabla(\phi Z) - \phi \nabla Z\]

\[= \lambda \gamma(Z) \phi d\mu - \gamma(Z) \gamma \otimes \phi W + (\lambda b(\phi Z) - g(W, \phi Z) \gamma) \otimes \xi\]

which holds for any B.C. quasi-Sasakian manifold. It should be noticed, that by setting $Z = \xi$ in (3.20), one obtains (3.2) again.

Consider now the contact $\phi$-Lie differential operator

\[\partial_{\phi} : Z \rightarrow (\mathcal{L}_\xi \phi) Z.\]

As is known (see for example [7]), one has

\[(\mathcal{L}_\xi \phi) Z = [\xi, \phi Z] - \phi[\xi, Z].\]

Let us go back to the case under discussion and set $Z = W$ in (3.21). First of all, since $\nabla W$ is self-adjoint, one finds by (3.2) that

\[\nabla_{\xi} W = \xi \lambda W - g(W, W) \xi.\]

Next, by making use of equation (3.20) and (3.1), one gets

\[\nabla \phi W = \phi \nabla W + \lambda b(\phi W) \otimes \xi\]

and this implies

\[\nabla_{\xi} \phi W = \phi \nabla_{\xi} W.\]

But by (3.22) one has

\[\phi \nabla_{\xi} W = \xi \lambda \phi W.\]

Finally, by means of (3.15), one gets

\[(\mathcal{L}_\xi \phi) W = 0.\]

Hence one may say that the associated vector field $W$ of $\xi$ is contact $\phi$-invariant.

It also should be noticed that (3.2) and (3.23) imply

\[\mathcal{L}_\phi W = 2 \xi \phi W.\]

Hence, following a known definition [5], we may say that $\phi W$ admits an infinitesimal transformation of $\xi$. 


Denote now by $D = \{ \xi, \nabla W, \Phi W \}$ the 3-distribution defined by $\xi$, $\nabla W$, and $\Phi W$. Since by (3.10) and (3.11) one may write

\begin{equation}
\nabla W = \lambda \eta \otimes W + (\lambda \eta - g(W, W) \eta) \otimes \xi
\end{equation}

then if $X'$ and $X''$ are any vector fields of $D$, it follows from (3.2), (3.23) and (3.27), that one has

$$\nabla_{X'} X'' \in D.$$

According to a well-known proposition (see for example [13]), this proves that $D$ defines an auto-parallel (or totally geodesic) foliation. Therefore, we may say that any B.C. quasi-Sasakian manifold is foliated by totally geodesic 3-dimensional submanifolds tangent to $\xi$, $W$ and $\Phi W$.

Next by (3.27) one quickly gets at any point $p \in M$

\begin{equation}
\text{tr}(\nabla W) = \text{div} W = - g(W, W)
\end{equation}

\begin{equation}
\frac{1}{2} \text{d}g(W, W) = \lambda g(W, W) \eta.
\end{equation}

Recall now the general formula

$$\Delta \nu = - \text{div} \left( \text{grad} \nu \right) \quad \nu \in C^\infty M.$$

Then, by (2.2), (3.28) and (3.29), one derives

\begin{equation}
\Delta g(W, W) = -2(2m-1) \lambda^2 g(W, W)
\end{equation}

which shows that $g(W, W)$ is an eigenfunction of $\Delta$ and has $-2(2m-1) \lambda^2$ as the associated eigenvalue. Since this eigenvalue is negative, we conclude by reference to a known property (see for example [17]) that compact B.C. quasi-Sasakian manifold do not exist. Further since by (3.29) one has

\begin{equation}
\text{grad} g(W, W) = 2\lambda g(W, W) \xi \Rightarrow \| \text{grad} g(W, W) \|^2 = 4\lambda^2 g(W, W)^2
\end{equation}

it follows by reference to a known definiton that $g(W, W)$ is an isoparametric function (see for example [18] or [9]$_2$).

On the other hand, by (2.2), (2.3), (3.2) and (3.27), taking the second covariant differential of $\xi$ and $W$, one finds

\begin{equation}
\nabla^2 \xi = \lambda (\lambda \eta - w) \wedge dp + (\eta \wedge w) \otimes (W \cdot \lambda \xi)
\end{equation}

\begin{equation}
\nabla^2 W = \lambda (\lambda \eta - g(W, W) \eta) \wedge dp + (\eta \wedge w) \otimes (\lambda W \cdot g(W, W) \xi).
\end{equation}
Consider now the vector valued 1-form

\begin{equation}
F = \xi \wedge W = w \otimes \xi - \gamma \otimes W \in A^1(M, TM).
\end{equation}

Operating on \( F \) by \( d^\nu \) and taking into account (3.32) and (3.33), one finds

\[ d^\nu F = \nabla^2 \xi \wedge w - \nabla^2 W \wedge \eta = 2\lambda^2 (w \wedge \eta) \wedge dp. \]

Hence by reference to [14] one may say that \( F \) is a 2-exterior concurrent vector valued 1-form, having \( 2\lambda^2 w \wedge \eta \) as a concurrence 2-form.

Since a problem of current interest is the curvature problem, we shall make now the following consideration. Making use of equations (3.3) and (3.20), one finds the relations

\begin{equation}
\theta^b_\xi = \theta^b_\eta^*, \quad \theta^a_\eta^* = \theta^a_\xi^*,
\end{equation}

which are characteristic for quasi-Sasakian manifolds. Now with the help of the structure equations (3.5) one derives from (3.26)

\begin{equation}
\begin{aligned}
\theta^b_\xi + \lambda^2 \omega^a \wedge \omega^b + \lambda (i_W \omega^a \wedge \omega^b) \wedge \eta &= \theta^b_\eta^* + \lambda^2 \omega^a \wedge \omega^b + \lambda (i_W \omega^a \wedge \omega^b) \wedge \eta, \\
\theta^a_\eta^* + \lambda^2 \omega^a \wedge \omega^b + \lambda (i_W \omega^a \wedge \omega^b) \wedge \eta &= \Theta^a_\xi^* + \lambda^2 \omega^a \wedge \omega^b + \lambda (i_W \omega^a \wedge \omega^b) \wedge \eta.
\end{aligned}
\end{equation}

Since the characteristic equation for space-forms \( M(K) \) are

\[ \Theta^a_\xi = K \omega^A \wedge \omega^{(B)} \]

it follows from (3.27) that non-trivial quasi-Sasakian manifolds of constant curvature do not exist.

Let now \( R \) be the curvature tensor field on \( M(\Phi, \Omega, \eta, \xi, g) \). Then \( (R(Z, Z') \in \Gamma \text{ End } \Lambda M \). With the help of (3.20) and (3.27) one finds after some calculations

\begin{equation}
R(Z, Z') X + \Phi R(Z, Z') \Phi X
= (\eta(X) R(Z, Z') + g(X, R(Z, Z') \xi) \xi + \lambda^2 (\Phi Z' \wedge \Phi Z) \Phi X + \lambda^2 (Z' \wedge Z) X
+ \lambda g(X, \Phi W)(\eta(Z') \Phi Z - \eta(Z) \Phi Z') \eta(g(X, W) - \lambda \eta(X))(Z' Z) \xi + (-\eta(Z) g(X, Z')
+ \eta(Z') g(X, Z))(\lambda^2 \xi + \lambda W) + \lambda (-\eta(Z) g(\Phi X, Z') + \eta(Z') g(\Phi X, Z)) \Phi W.
\end{equation}

The following theorem combines all results obtained in this section.
Theorem 3.1. Let \( M(\Phi, \Omega, \gamma, \xi, g) \) be a \((2m+1)\)-dimensional quasi-Sasakian manifold and let \( D_h = \{ Z \in X_M; \gamma(Z) = 0 \} \) be the horizontal \((2m)\)-distribution annihilated by the structure 1-form \( \gamma \). Then the necessary and sufficient condition in order that \( M \) be endowed with a B.C.-structure, is the structure vector field \( \xi \) be contact quasi-concurrent with horizontal and closed associated vector field \( W \in D_h \). Any such manifold \( M(\Phi, \Omega, \gamma, \xi, g) \) is foliated by a totally geodesic 3-dimensional submanifold tangent to \( \xi, W, \) and \( \Phi W \). One has also the following properties:

(i) \( \xi \) is a \( D_h \)-conformal vector field and \( \text{div} \xi = -2m\lambda \), where \( \lambda(\text{d} \log \lambda = -b^{-1}(W)) \) is the structure scalar of the B.C.-structure.

(ii) \( g(W, W) \) is an eigenfunction of \( \Delta \) and it is an isoparametric function \((18)\).

(iii) If \( \nu \) is any semi-basic 1-form and \( V = b^{-1}\nu \) is its dual vector field, one has the following formulae

\[
\mathcal{L}_\xi \nu = \varphi \nu + b[\xi, V] + g(V, W) \gamma
\]

\[
\mathcal{L}_\xi * \nu = * \mathcal{L}_\xi \nu + \frac{2m-1}{2} \rho * \nu + g(V, W) \sigma_h
\]

where \( \varphi = \frac{\text{div} \xi}{m} \) and \( \sigma_h \) is the volume element of \( D_h \).

(iv) \( W \) is a contact \( \Phi \)-invariant vector field, and B.C. quasi-Sasakian manifolds of constant curvature do not exist.

4 - Submanifolds of B.C. quasi-Sasakian manifolds

We shall discuss in this section various striking properties of some submanifolds of the manifold \( M(\Phi, \Omega, \gamma, \xi, g) \) under discussion. First of all consider the immersion \( x:\ M_h \to M \), where \( M_h \) is the hypersurface normal to the structure vector field \( \xi \) (see 2). Because of (3.2), the second fundamental quadratic form associated with \( x \) is

\[
(4.1) \quad I = - \langle dp \nabla_\xi \rangle = i_g
\]

(we denote elements induced by \( x \) by the same letters), and the above equation proves that \( M_h \) is an umbilical hypersurface of \( M \). Since on \( M_h \) one has

\[
\nabla_\xi = -\lambda dp
\]
it follows at once from (2.2) that

\begin{equation}
\nabla^2 \xi = -\lambda w \wedge dp.
\end{equation}

So, referring to (1.4), one may say that \(\xi\) is a normal exterior recurrent vector field.

Let then \(\theta_0\) be the normal curvature 2-forms associated with \(x\). One derives at once from (4.2) that

\begin{equation}
\theta_0 = -\lambda w \wedge \omega^s
\end{equation}

which shows that all forms \(\theta_0\) are conformal to the induced value of the cohomology form \(w\).

Further by (3.36) it is easily seen that if \(M_h\) is a space-form of curvature \(K\), then necessarily \(K + \lambda^2 = 0\) that is \(M_h\) is an extrinsic hypersphere.

Moreover, if \(\langle L \rangle\) denoted the length of the second fundamental quadratic form of \(M_h\), then on \(M_h(-\lambda^2)\), \(\langle L \rangle\) is constant, and since the mean curvature vector of \(M_h(-\lambda^2)\) (i.e. \(\xi\)) is nowhere zero, it follows that the product submanifold \(M_h(-\lambda^2) \times M_h(-\lambda^2)\) is an \(\mathcal{H}\)-submanifold (see [4]) in \(M \times M\). It should be noticed that by virtue of (4.2), one has on \(M_h(-\lambda^2)\)

\[\nabla^2 \xi = 0\]

and this shows that the normal connection \(\nabla^1\) associated with \(x\): \(M_h(-\lambda^2) \rightarrow M(\phi, \Omega, \eta, \xi, g)\) is flat. Let now \(X \in \mathfrak{X}M_h\) be any exterior concurrent vector field on \(M_h\). Then it follows from (1.4) that

\begin{equation}
\nabla^2 X = f b(X) \wedge dp
\end{equation}

holds for some \(C^\infty M\).

On the other hand, on \(M_h\), the formula (3.37) moves to

\begin{equation}
R(Z, Z')X + \Phi R(Z, Z') \Phi X
= \lambda^2 (b(\Phi X) \wedge dp(Z, Z') + \lambda^2 (b(X) \wedge dp(Z, Z')).
\end{equation}

Then if \(X\) satisfies (4.4), one derives from (4.5) that \(f = \lambda^2\), and in this case one also has

\[\nabla^2 \Phi X = \lambda^2 b(\Phi X) \wedge dp\]

that is the property of exterior concurrency for \(X\) is invariant when applying \(\Phi\) to \(X\).
Let us now $M_I$ be an **invariant submanifold** of $M(\Phi, \Omega, \eta, \xi, g)$, that is $\xi$ is tangent to $M_I$ for any tangent vector field $Z$ to $M_I$. Assume that $M_I$ is of codimension 2 and is defined by

$$\omega^r = 0 \quad \omega^{r*} = 0$$

$$r = m + 1 - l \quad r^{*} = r + m.$$ 

Hence the soldering form $dp_I$ of $M_I$ is

$$dp_I = \omega^i \otimes e_i + \omega^{i*} \otimes e_{i*} + \eta \otimes \xi,$$

$$i = 1, \ldots, m - l \quad i^{*} = i + m.$$

(We denote the other elements induced by $y$: $M_I \rightarrow M$ by the same letters.) Consequently the **mean curvature vector valued** 2$(m - l)$-form $\mathcal{C} \in A^{2(m - l)}(M_I, TM_I)$ is

$$\mathcal{C} = \Sigma (-1)^{i-1} \omega^1 \wedge \ldots \wedge \omega^i \wedge \ldots \wedge \omega^{m-l} \wedge \omega^{1*} \wedge \ldots \wedge \omega^{m-l*} \wedge \eta \otimes e_I$$

$$+ \Sigma (-1)^{i*} \omega^1 \wedge \ldots \wedge \omega^{m-l} \wedge \omega^{1*} \wedge \ldots \wedge \omega^{m-l*} \wedge \eta \otimes e_{I*}$$

$$+ \omega^1 \wedge \ldots \wedge \omega^{m-l} \wedge \omega^{1*} \wedge \ldots \wedge \omega^{(m-l)*} \otimes \xi.$$ 

Applying the operator $d^\nabla$ to $\mathcal{C}$, one has

$$d^\nabla \mathcal{C} = (2(m - l) + 1) \sigma_l \otimes H$$

where $\sigma_I$ and $H$ are the volume element of $M_I$ and the **mean curvature vector field** associated with $y$: $M_I \rightarrow M$, respectively. With the help of (3.35) one gets $d^\nabla \mathcal{C} = 0 \Rightarrow H = 0$, which expresses that any $M_I$ is **minimal** in $M(\Phi, \Omega, \eta, \xi, g)$.


Finally consider the immersion $z$: $M_A \rightarrow M$, where $M_A$ is an **anti-invariant submanifold** of dimension $m$ of $M$ [4]. Then by definition $M_A$ is normal to $\xi$ and if $Z$ is any tangent vector field to $M_A$, then $\Phi Z$ is a normal vector field to $M_A$. If we assume that $M_A$ is defined by

$$\omega^{a*} = 0 \quad \eta = 0$$

then the equations (3.36) become

$$\theta_b^o + \lambda^2 \omega^b \wedge \omega^a = \theta_{b*}^o \quad \theta_{b*}^a = \theta_{a*}^b.$$ 

From above we derive that if $M_A$ is of constant curvature $K$ and the normal connection $\nabla^2$ is flat, then necessarily $K = -\lambda^2$ and $M_A$ is of hyperbolic type.
Theorem 4.1. Let $M(\Phi, \omega, \varepsilon, \xi, \eta)$ be any B.C. quasi-Sasakian manifold. We consider the following immersions: $x: M_h \rightarrow M$, $y: M_I \rightarrow M$, and $z: M_A \rightarrow M$ where $M_h$, $M_I$ and $M_A$ are the symplectic hypersurface normal to $\xi$, an invariant submanifold and an anti-invariant submanifold of dimension $m$, respectively.

One has the following properties:

(i) $M_h$ is an umbilical hypersurface, and if $M_h$ is a space-form, it is necessarily of hyperbolic type, $M(-\lambda^2)$. In this case $M(-\lambda^2)$ is an extrinsic hypersphere and the product $M(-\lambda^2) \times M(-\lambda^2)$ is an $\mathfrak{u}$-submanifold of $M \times M$. Further the conformal associated scalar with any exterior concurrent vector field $X$ on $M_h$ is $\lambda^2$, and the property of exterior concurrency for $X$ is invariant when applying $\Phi$ to $X$.

(ii) Any invariant submanifold $M_I$ of $M$ is minimal.

(iii) Any anti-invariant submanifold $M_A$ is of constant curvature and a flat normal connection is of hyperbolic type $M(-\lambda^2)$.

References


[9] V. V. GOLDBERG and R. ROSCA: [•] Biconformal vector fields on manifolds endo-
Let $M$ be Riemannian $(2m + 1)$-dimensional $C^\infty$-manifold endowed with a structure 2-form $\Omega$, a structure 1-form $\gamma$ and a structure vector field $\xi$ dual to $\gamma$. As a generalization of conformal cosymplectic manifolds, $M(\Omega, \gamma, \xi, g)$ is defined in the present paper as a "biconformal cosymplectic manifold" if both structure forms $\Omega$ and $\gamma$ are "cohomologically closed" (in the sense of F. Guedira and A. Lichnerowicz). With such a structure denoted by $B.C. \text{Sp}(2m + 1, R)$ is associated a closed 1-form $\omega$ and its dual vector $W$. Different properties of the $d^c$-cohomology and the Lie algebra on $M$ involving $\gamma$, $\nu$, $\Omega$, $W$ and $\xi$ are discussed. If $M_h$ is the hypersurface normal to $\xi$, the following salient properties are established: (a) If $M$ is a conformal cosymplectic manifold, then $M_h$ is a symplectic manifold. (b) If $M$ is a biconformal cosymplectic manifold, then $M_h$ is a conformal symplectic manifold.
As an application, we get the necessary and sufficient condition for a \((2m + 1)\)-dimensional quasi-Sasakian manifold \(M(\Phi, \Omega, \gamma, \xi, g)\) to be endowed with a B.C.-structure. Some striking properties of invariant and anti-invariant submanifolds of a B.C.-quasi-Sasakian manifold are discussed.

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