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Note on the theory of temperature dependent resistors (**)

1 - Introduction

A thermistor is a particular electrical resistor whose resistivity ρ varies of many order of magnitude with the temperature u [7]. The constitutive relation $\rho = \rho(u)$ can have in practice very different properties. A typical example is given in Figure 1. The corresponding current/voltage characteristic $i = i(v)$ at constant temperatures is shown in Figure 2 as obtained by direct measurements. When the thermistor is in series with an ordinary resistor R and a fixed external voltage V, the straight line $Ri = V - v$ expressing Ohm's law, intersects, for certain values of the parameters, the curve $i = i(v)$ in three different points and the three corresponding states of equilibrium can be found experimentally. Goal of this paper is to check analytically, with examples, this possibility of multiple solutions and more generally to integrate the problem. The equations are well-known and over one hundred years old (see e.g. [3] and [6]), however for the sake of completeness they are briefly recalled below.

Let $\boldsymbol{J}$ be the current density and φ the electric potential. Assuming Ohm's law we have

$$\boldsymbol{J} = -\sigma \nabla \varphi \tag{1.1}$$

where σ is the temperature dependent electrical conductivity. Let $\boldsymbol{q}$ be the total

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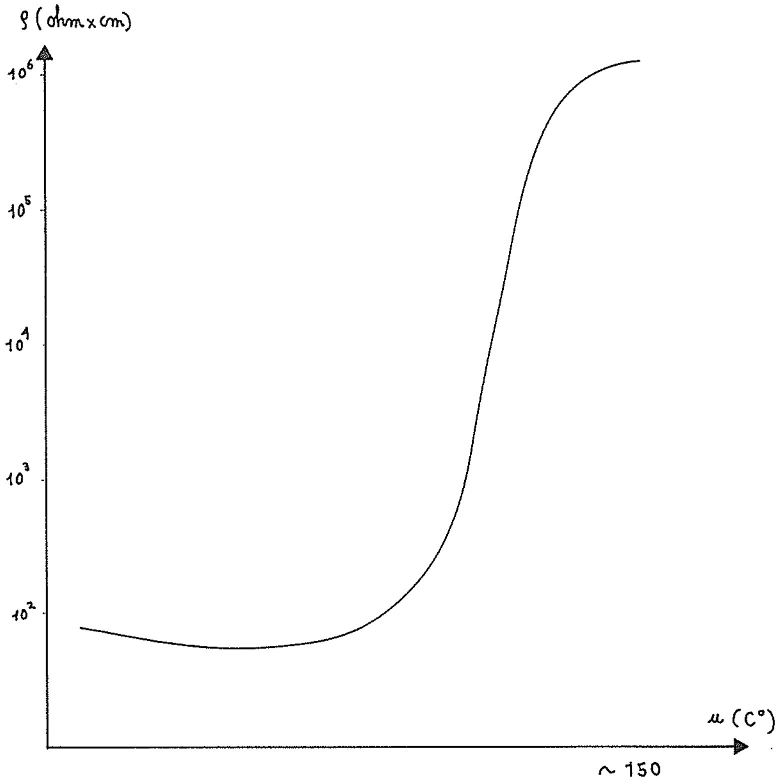

Fig. 1.

flow of energy (both thermal and electrical) and u the temperature, then

$$q = -\varkappa \nabla u + \varphi J \tag{1.2}$$

where $\varkappa$ is the (constant) thermal conductivity. By the conservation of charge and energy we have

$$\nabla \cdot J = 0 \qquad \nabla \cdot q = 0 \,. \tag{1.3}$$

Inserting (1.1) and (1.2) in (1.3) we obtain

$$\nabla \cdot (\sigma \nabla \varphi) = 0 \tag{1.4}$$

$$-\nabla \cdot (\varkappa \nabla u) = \nabla \cdot (\varphi\sigma \nabla \varphi) \,. \tag{1.5}$$

It follows from (1.4) and (1.5)

$$-\nabla \cdot (\varkappa \nabla u) = \sigma |\nabla \varphi|^2 \tag{1.6}$$

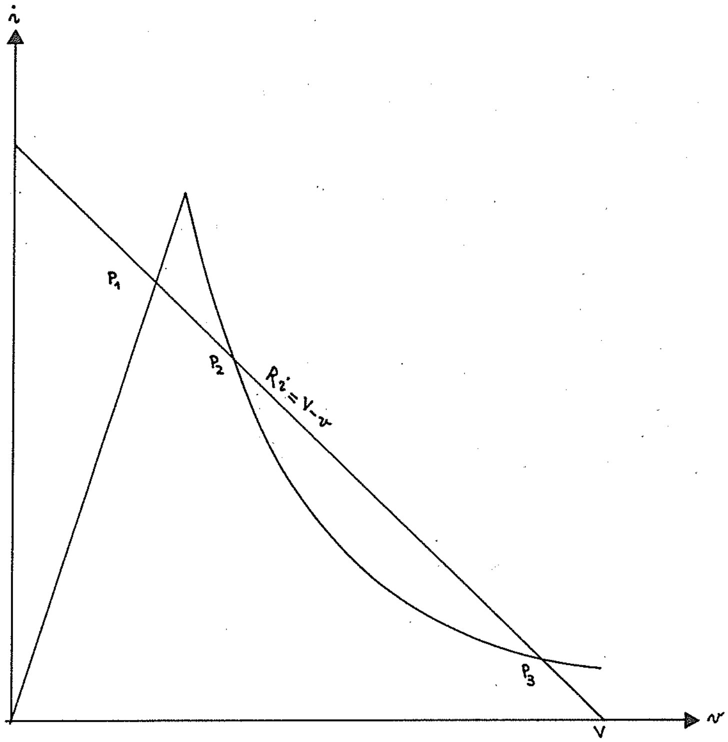

Fig. 2.

where the term in the right hand side represents the Joule heating. The nonlinear system (1.4) and (1.6) has been recently reconsidered. A. C. Fowler, I. Frigaard and S. D. Howison consider in [4] the parabolic version of the problem. The steady-state case is studied in [2] using a transformation which permits a considerable simplification, but is valid only for special boundary conditions.

In this paper we consider the one-dimensional version of (1.4) and (1.5) i.e.

$$(\sigma(u)\,\varphi')' = 0 \tag{1.7}$$

$$-\varkappa u'' = (\varphi\sigma(u)\,\varphi')'\,. \tag{1.8}$$

We show that the complete integration of (1.7), (1.8) with various boundary conditions, can be reduced to the search of the solutions of a trascendental equation under very general hypotheses on the conductivity law, which is allowed to be discontinuous. Three types of boundary conditions are considered: convection for the temperature and fixed voltage in 2, fixed temperature and potential depending on a external limiting resistor in 3, and Dirichlet conditions on both temperature and potential in 4. However, the method of integration, which is elementary in nature, applies to various other situations. For example it is possible to consider the boundary conditions of 2 for the temperature and those of 3 for the potential. Two examples of multiplicity of solutions are given in 2 and 3. Moreover an analytic expression for the current/voltage characteristic is derived in 4. Although the numerical output of this formula has not been compared with the experimental evidence, the qualitative features are remarkably similar. Various conditions of non-existence and uniqueness are given. The results are complete in the case of Dirichlet boundary conditions. Our functional setting will be very simple. We denote by $C_S^0[-L, L]$ ($C_S^0(\mathbb{R}^1)$) the class of functions which are continuous in $[-L, L]$, ($\mathbb{R}^1$) except for a finite number of points of discontinuity of the first kind. In these points the functions take on the value of the limit from the right. $C_S^1[-L, L]$ will be the set of functions which are continuous in $[-L, L]$ and whose derivative belongs to $C_S^0[-L, L]$.

2 - Problem Pb_1. Thermal convection

The boundary conditions studied in this section are

$$\varphi(-L) = 0 \qquad \varphi(L) = V \qquad V > 0 \tag{2.1}$$

$$u'(-L) = 0 \tag{2.2}$$

$$u'(L) = K(u_E - u(L)) \tag{2.3}$$

where K is a positive constant and u_E the given external temperature. Let $\rho(u)$ be the electrical resistivity i.e. $\rho(u) = 1/\sigma(u)$. Since there are cases of practical interest in which the graph of $\rho(u)$ is very steep, we assume $\rho(u) \in C_S^0(\mathbb{R})^1$ and on physical grounds

$$\rho(u) > 0\,. \tag{2.4}$$

To keep into account the possible discontinuities of $\rho(u)$, we give to problem (1.4), (1.5), (2.1), (2.2) and (2.3) the following integral formulation.

To find $\varphi(u) \in C_S^1[-L, L]$ *satisfying* (2.1) *and*

$$\int_{-L}^{L} \sigma(u)\, \varphi'\, v'\, \mathrm{d}x = 0 \tag{2.5}$$

for all $v \in C_S^1[-L, L]$, $v(-L) = v(L) = 0$, *and* $u(x) \in C_S^1[-L, L]$ *such that*

$$\lim_{x \to -L^-} u'(x) = 0 \qquad \lim_{x \to L^+} u'(x) = K(u(L) - u_E) \tag{2.6}$$

$$\int_{-L}^{L} \varkappa u' w' \, dx = -\int_{-L}^{L} \varphi\sigma(u)\varphi' w' \, dx \tag{2.7}$$

for all $w \in C_S^1[-L, L]$, $w(-L) = w(L) = 0$.

By the lemma of Du-Boys-Reymond [1] from (2.5) and (2.7) it follows immediately

$$\sigma(u)\varphi' = C_1 \tag{2.8}$$

$$\varkappa u' + \varphi\sigma(u)\varphi' = C_2 \,. \tag{2.9}$$

By (2.1) and (2.8) we have $C_1 > 0$. Moreover, from (2.8) and (2.9), we get

$$\varkappa u' + \varphi C_1 = C_2 \tag{2.10}$$

hence $u' \in C^0[-L, L]$ and (2.2) (2.3) make sense. From (2.2) and (2.10) we obtain $C_2 = 0$. Moreover by the one dimensional maximum principle, we have $V > \varphi > 0$ in $(-L, L)$; thus $u'(x) < 0$ in $(-L, L]$. It follows from (2.3) $u(x) \geqslant u(L) > u_E$. For this reason we need only to assume $\rho(u) \in C_S^0[u_E, \infty)$. Setting $x = L$ in (2.10) we obtain

$$C_1 = \frac{\varkappa K(u(L) - u_E)}{V} \,. \tag{2.11}$$

Put $v = \varphi w$ in (2.5) with $w \in C_S^1[-L, L]$, $w(-L) = w(L) = 0$. We get

$$\int_{-L}^{L} \sigma(u)\varphi'^2 w \, dx = -\int_{-L}^{L} \varphi\sigma(u)\varphi' w' \, dx \,.$$

Substituting in (2.7) and recalling (2.8) we obtain

$$\int_{-L}^{L} \varkappa u' w' \, dx = C_1^2 \int_{-L}^{L} \rho(u) w \, dx \,. \tag{2.12}$$

Using now an old trick of the calculus of variations, we define

$$\phi(x) = \int_0^x \rho(u(t)) \, dt \,.$$

Substituting in (2.12) and making an integration by parts, now possible, we infer

$$\int_{-L}^{L} (\varkappa u' + C_1^2 \phi)\, w'\, \mathrm{d}x = 0\,. \tag{2.13}$$

Thus we have, apart from a finite number of points,

$$-\varkappa u'' = C_1^2 \rho(u)\,. \tag{2.14}$$

Define

$$F(u) = \int_{u_E}^{u} \rho(t)\, \mathrm{d}t\,.$$

From (2.14) it follows

$$\varkappa u'^2 = 2C_1^2 [F(u(-L)) - F(u)]\,. \tag{2.15}$$

By (2.8) we get in $[-L, L]$

$$u' = -\sqrt{\frac{2}{\varkappa}}\, \sigma(u)\, \varphi' \sqrt{F(u(-L)) - F(u)} \tag{2.16}$$

and by (2.11)

$$u' = -\sqrt{2\varkappa}\, \frac{K(u(-L) - u_E)}{V} \sqrt{F(u(-L)) - F(u)}\,. \tag{2.17}$$

Put $u(L) = \alpha$ and $u(-L) = \beta$, $\beta > \alpha$. From (2.16) and (2.17) we arrive, by separation of variables, to the following system in the unknowns α and β

$$F(\beta) - F(\alpha) = \frac{V^2}{2\varkappa} \tag{2.18}$$

$$\int_{\alpha}^{\beta} \frac{\mathrm{d}u}{\sqrt{F(\beta) - F(u)}} = 2KL\sqrt{2\varkappa}\, \frac{\alpha - u_E}{V}\,. \tag{2.19}$$

Define

$$A = \frac{V^2}{2\varkappa} \qquad B = \frac{2KL\sqrt{2\varkappa}}{V} \qquad D = \int_{u_E}^{\infty} \rho(t)\, \mathrm{d}t$$

with D possibly equal to $+\infty$. Discussing the system (2.18), (2.19) we arrive to the following

Theorem 2.1. *Problem Pb_1 has a solution only if*

$$A < D. \tag{2.20}$$

If (2.20) *holds, the search of the solutions of Pb_1 is reduced to solving the equation*

$$H(\beta) = Bu_E \tag{2.21}$$

in $[\hat{\beta}, \infty)$, where $\hat{\beta} = F^{-1}(A)$ and

$$H(\beta) = BF^{-1}(F(\beta) - A) - \int_{F^{-1}(F(\beta)-A)}^{\beta} \frac{du}{\sqrt{F(\beta) - F(u)}}\,. \tag{2.22}$$

In particular if

$$\rho(t) \geqslant \rho_0 > 0 \tag{2.23}$$

then Pb_1 has at least one solution. Moreover, when in addition to (2.23), *we have $\rho(t) \in C^1[u_E, \infty)$ and*

$$\rho'(t) > 0 \tag{2.24}$$

the solution of Pb_1 is unique.

Proof. The necessity of condition (2.20) follows immediately from (2.18). If (2.20) holds, we can solve (2.18) with respect to α when $\beta \geqslant \hat{\beta}$. Then substituting in (2.19) we obtain (2.21). However, (2.20) is not sufficient to guarantee the existence of a solution for arbitrary u_E as the choice $\rho(t) = (1 + t^2)^{-1}$ shows. When condition (2.23) holds we have

$$H(\hat{\beta}) < Bu_E \qquad \lim_{\beta \to \infty} H(\beta) = +\infty$$

because in this case the integral in the right hand side of (2.22) remains bounded. Therefore (2.21) has at least one solution for every u_E. Finally, when (2.24) holds we have $H'(\beta) > 0$ and therefore uniqueness.

We note that in the elementary calculations involved in the proof of Theorem 2.1 is useful to make the substitution

$$\frac{1}{z} = \sqrt{F(\beta) - F(u)}$$

in the integral entering into the definition of $H(\beta)$. As an application of equation (2.21) we treat the important case of the metallic conduction.

Example 2.1. By the Wiedemann-Franz law we have $\rho(t) = at$, $a > 0$. Assumption (2.4) is not satisfied, but the theory equally applies if we assume $u_E > 0$, since $u(x) \geqslant u_E$.

Thus in this case the singularity of $\sigma(t)$ in $t = 0$ is irrelevant. Equation (2.21) can be solved and we find

$$\beta = \frac{V}{\sqrt{a\varkappa}\,\sin(2\xi L)}$$

where ξ is the unique solution in the interval $(0, \pi/4L)$ of the equation

$$\operatorname{cotg}(2L\xi) = \frac{\gamma + \xi}{K} \qquad \text{where} \qquad \gamma = \frac{K u_E \sqrt{a\varkappa}}{V}.$$

Problem Pb_1 can then be integrated explicitely and the solution is given by

$$\varphi(x) = \frac{V}{\sin(2\xi L)} \sin[\xi(x+L)]$$
$$u(x) = \frac{V}{\sqrt{a\varkappa}\,\sin(2\xi L)} \cos[\xi(x+L)].$$

In general one cannot expect uniqueness for problem Pb_1. This can be seen with the following

Example 2.2. Assume $\rho(t) = M$ if $t < 0$ and $\rho(t) = N$ if $t > 0$ with $M > (3/2)N$. Put for simplicity $\sqrt{2\varkappa} = 1$, $2L = 1$, $V = 1$ and $K = 1$. Let us take as relevant parameter $u_E \in \mathbb{R}^1$. The function $H(\beta)$ can be written down explicitely and is plotted in Figure 3. We find that there is only one solution if $u_E < 3/M$ or $u_E > 2/N$ and three solutions if $3/M \leqslant u_E \leqslant 2/N$.

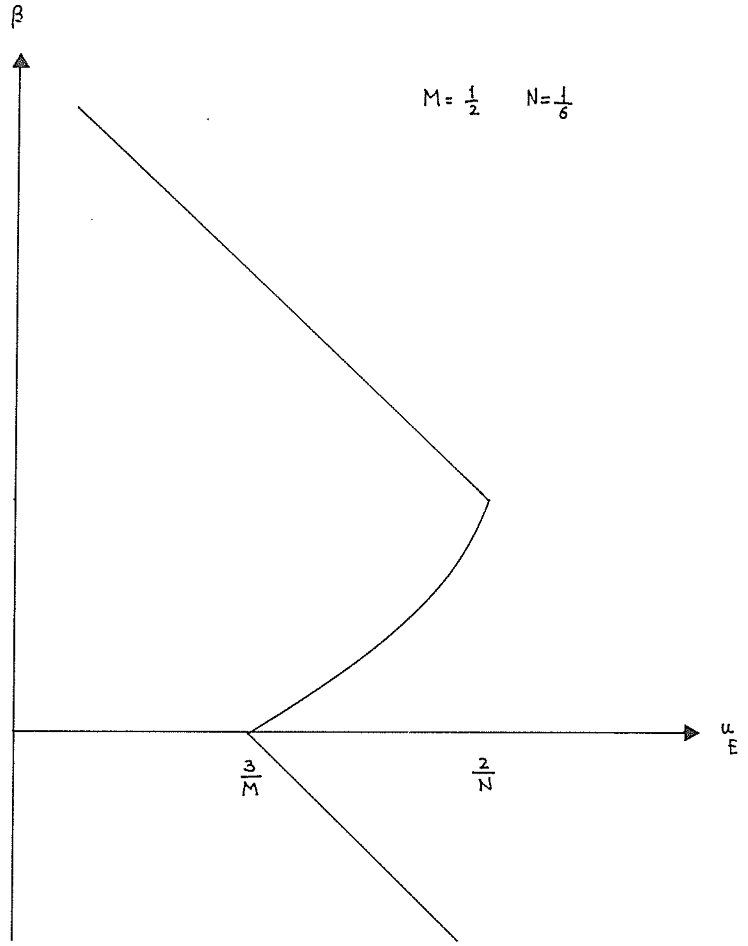

Fig. 3.

3 - Problem Pb_2. The thermistor in a current limiting circuit

The device is supposed in this section to be connected in series with a fixed resistor R and a difference of potential V. We shall therefore deal with the following boundary conditions

$$\varphi(-L) = 0 \qquad \varphi(L) = v \tag{3.1}$$

$$u(-L) = u(L) = \overline{u} \tag{3.2}$$

where $\overline{u}$ is a given constant and v, the voltage applied to the thermistor, is an unknown of the problem. By Ohm's law we have

$$Ri + v = V \tag{3.3}$$

where $i = \sigma(\overline{u})\,\varphi'(L)$. The one-dimensional maximum principle implies $u \geqslant \overline{u}$. Therefore we assume $\rho(t) \in C^0_S[\overline{u},\ \infty)$ and

$$\rho(t) > 0 \qquad \text{in } [\overline{u},\ \infty)\,. \tag{3.4}$$

An integral formulation can easily be given in order to consider the possible discontinuities of $\rho(t)$. Let $u(0) = \beta$ and define

$$F(t) = \int_{\overline{u}}^{t} \rho(\xi)\,\mathrm{d}\xi\,. \tag{3.5}$$

Proceeding along lines similar to those of **2** we obtain the following system of equations in the unknowns β and v

$$F(\beta) = \frac{v^2}{8\varkappa} \qquad \int_{\overline{u}}^{\beta} \frac{\mathrm{d}u}{\sqrt{F(\beta) - F(u)}} = \sqrt{\frac{2}{\varkappa}}\,\frac{(V - v)\,L}{R}\,.$$

Eliminating v we arrive to a single equation in β:

$$H(\beta) = \sqrt{2\varkappa}\,\frac{LV}{R} \tag{3.6}$$

where

$$H(\beta) = \int_{\overline{u}}^{\beta} \frac{\mathrm{d}u}{\sqrt{F(\beta) - F(u)}} + \frac{4L}{R}\sqrt{F(\beta)}\,. \tag{3.7}$$

A simple discussion of equation (3.6) leads to the following

Theorem 3.1. *If* (3.4) *holds, problem* Pb_2 *has at least one solution for every* V. *In addition when* $\rho(t) \in C^1[\overline{u},\ \infty)$, ρ' *is bounded and* $\rho' \leqslant 0$, *the solution is unique.*

Proof. We have $H(\overline{u}) = 0$. Moreover if

$$\int_{\overline{u}}^{\infty} \rho(t)\,\mathrm{d}t = +\infty$$

the second term in the right hand side of equation (3.7) diverges to $+\infty$. On the other hand when

$$\int_{\bar{u}}^{\infty} \rho(t)\, dt < +\infty$$

it is the first term in the right hand side of (3.7) which tends to $+\infty$. Hence there is at least one solution of equation (3.6) for every $V > 0$ and therefore of Pb_2. Under the assumptions of the second part of the theorem we can compute $H'(\beta)$. We find $H'(\beta) > 0$. This implies the uniqueness of the solution.

The case of the metallic conduction can be treated in a way similar to Example 2.1. Therefore we give only the following example of non-uniqueness which is directly related to the situation of multiple solutions described in the introduction.

Example 3.1. Assume for simplicity $\bar{u} = -1$ and $4L = R$. Moreover, to mimic roughly the temperature/resistivity characteristic given in Figure 1 we choose $\rho(t) = m$ if $-1 \leqslant t < 0$ and $\rho(t) = M$ when $u > 0$ with $m < M$. With a straightforward calculation we find $H(\beta)$ which is plotted in Figure 4. Hence we

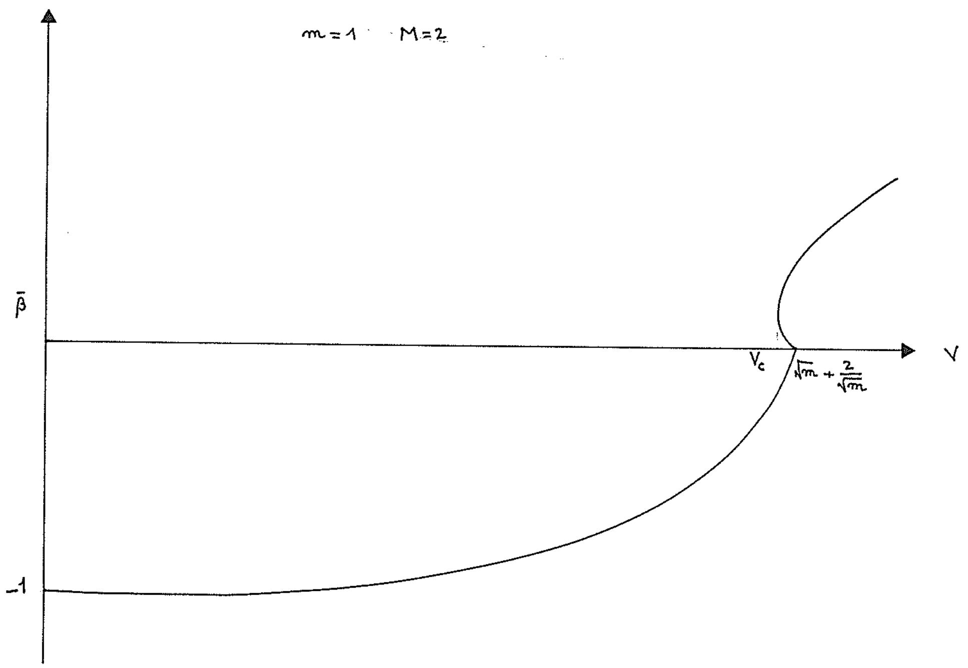

Fig. 4.

conclude that problem Pb_2 has exactly one solution if $0 \leqslant V < V_C$ or $V > \sqrt{m} + 2/\sqrt{m}$ and three solutions when $V_C \leqslant V \leqslant \sqrt{m} + 2/\sqrt{m}$ with

$$V_C = \sqrt{m}[(1 + \frac{2}{m})^2 - (\frac{2}{m} - \frac{2}{M})^2]^{1/2} .$$

We refer to the forthcoming paper [5] for a similar situation.

4 - Problem Pb_3. Dirichlet boundary conditions

In this section we add to equations (1.3) and (1.4) the boundary conditions

$$\varphi(-L) = 0 \qquad \varphi(L) = v \tag{4.1}$$

$$u(-L) = \bar{u} = u(L) \tag{4.2}$$

where v and $\bar{u}$ are now both given constants. We assume again $\rho(t) \in C_S^0[\bar{u}, \infty)$ and $\rho(t) > 0$. The possible discontinuities of the resistivity can be dealt with an integral formulation of the problem as in **2**. Integrating we arrive at

$$\sigma(u)\,\varphi' = C_1 \tag{4.3}$$

$$\varkappa u'^2 = 2C_1^2\,[F(u(-L)) - F(u)] \tag{4.4}$$

where $$F(t) = \int_{\bar{u}}^{t} \rho(\xi)\,\mathrm{d}\xi .$$

Let $\beta = u(0)$. By separation of variables in (4.4) we arrive, taking into account (4.3), to the equations

$$F(\beta) = \frac{v^2}{8\varkappa} \tag{4.5}$$

$$C_1 = \sqrt{\frac{\varkappa}{2}}\,\frac{1}{L}\int_{\bar{u}}^{\beta} \frac{\mathrm{d}u}{\sqrt{F(\beta) - F(u)}} . \tag{4.6}$$

By simple inspection of (4.5) we obtain

Theorem 4.1. *Let*

$$l = \int_{\bar{u}}^{\infty} \rho(t)\,\mathrm{d}t .$$

If $l=\infty$, then problem Pb_3 has for every $v \geqslant 0$ only one solution. When $l<\infty$, problem Pb_3 has no solution if $l \leqslant v^2/8\varkappa$ and exactly one solution if $l > v^2/8\varkappa$.

Given the physical meaning of the constant C_1, we obtain the following current/tension relation inserting in (4.6) the value of β given by (4.5)

$$i(v)=\sqrt{\frac{\varkappa}{2}}\,\frac{1}{L}\int_{\bar{u}}^{F^{-1}(v^2/8\varkappa)}\frac{du}{\sqrt{\dfrac{v^2}{8\varkappa}-F(u)}}\,. \tag{4.7}$$

We want now to show with an example that the shape of the empiric current/voltage graph given in Figure 2 and discussed in the introduction can be predicted by (4.7).

Example 4.1. To retain only the essential constants assume $8\varkappa=1$, $L=1/4$ and $\bar{u}=-1$. Moreover let us take as in Example 3.1 $\rho(t))m$ if $-1\leqslant t<0$ and $\rho(t)=M$ for $0<t$ with $0<m<M$. After a direct calculation we find

$$i(v)=\begin{cases}(2/m)\,v & \text{when } 0\leqslant v<\sqrt{m}\\ (2/m)\,v-(2/m-2/M)\sqrt{v^2-m} & \text{if } \sqrt{m}\leqslant v\,.\end{cases} \tag{4.8}$$

Final remark. Using the result of the above example we can discuss from a different point of view the three solutions situation of Example 3.1 corresponding to the case of the thermistor in series with an external resistor R and fixed applied voltage V. In fact the load straight line $i=(1/R)(V-v)$ has, for suitable values of the parameters, three points of intersection with the curve $i=i(v)$ given by (4.8). We note that solutions similar to those considered in this paper are presented in [8].

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Abstract

The equations of the steady state thermistor problem in one space dimension are integrated explicity with various boundary conditions. We obtain various results of existence, nonexistence, uniqueness and non-uniqueness of solution.

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