Riv. Mat. Univ. Parma (4) **17** (1991), 99-109

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Exponential stability of linear impulsive differential equations (**)

1 - Introduction

In relation to numerous applications in science and technology recently the theory of impulsive differential equations develops intensively (Lakshmikantham and Liu (1989); Lakshmikantham, Bainov and Simeonov (1989); Leela (1977); Milev and Bainov (to appear); Samoilenko and Perestyuk (1987); Simeonov and Bainov (1988)). In the present paper the notion of *exponential stability* for linear impulsive differential equations at fixed moments is made precise.

2 - Preliminaries

Let $t_0 < t_1 < \ldots < t_i < \ldots$, $\lim t_i = \infty$ as $i \to \infty$ be a given sequence of real numbers. Consider the linear impulsive differential equation (LIDE) at fixed moments

$$(1) \qquad \frac{dx}{dt} = A(t)\,x \qquad t \neq t_i \qquad x(t_i + 0) = B_i x(t_i) \qquad (i = 1, 2, \ldots)$$

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(**) The present investigation is supported by Ministery of Culture, Science and Education of People's Republic of Bulgaria under Grant 61.
MR classification: 34A37. – Ricevuto: 3-IX-1990.

where the $n \times n$ coefficient matrix $A(t)$ is piecewise continuous in the interval $[t_0, +\infty)$ with points of discontinuity of the first kind at $t = t_i$ $(i = 1, 2, \dots)$ and the impulse matrices B_i $(i = 1, 2, \dots)$ are constant. The underlying vector space is R^n or C^n.

The solutions $x(t)$ defined in the interval $[t_k + 0, +\infty)$ are continuously differentiable for $t \neq t_i$ with points of discontinuity of the first kind at $t = t_i$, $i > k$. Let us note that $x(t_i) := x(t_i - 0)$ $(i = 1, 2, \dots)$. The fundamental matrix $X(t, s)$ of LIDE (1) for $t \geqslant s$, $t \in [t_m + 0, t_m + 1]$, $s \in [t_{j-1} + 0, t_j]$, $m \geqslant j - 1$, admits the representation

$$X(t, s) = U(t)\, U^{-1}(t_m + 0)\, B_m\, U(t_m) \dots U^{-1}(t_j + 0)\, B_j\, U(t_j)\, U^{-1}(s) \tag{2}$$

where $U(t)$ is the fundamental matrix of the equation $\dfrac{\mathrm{d}x}{\mathrm{d}t} = A(t)\, x$. The fundamental matrix is invertible if and only if the impulse matrices B_i $(j \leqslant i \leqslant m)$ are nonsingular.

Def. 1. LIDE (1) is said to be: (a) *stable* if for any $\varepsilon > 0$ and for any $s \geqslant t_0$ there exists $\delta > 0$ such that for each solution x for which $|x(s)| < \delta$ the inequality $|x(t)| < \varepsilon$ holds for $t \geqslant s$; (b) *uniformly stable* if for any $\varepsilon > 0$ there exists $\delta > 0$ such for any $s \geqslant t_0$ and for each solution x for which $|x(s)| < \delta$ the inequality $|x(t)| < \varepsilon$ is valid for $t \geqslant s$.

Def. 2. LIDE (1) is said to be: (a) *asymptotically stable* if it is stable and, moreover, for any $s \geqslant t_0$ there exists $\eta = \eta(s) > 0$ and for any $\varepsilon > 0$ there exists $T > 0$ such that for each solution x for which $|x(s)| < \eta$ the inequality $|x(t)| < \varepsilon$ holds for $t \geqslant s + T$; (b) *uniformly asymptotically stable* if it is uniformly stable and, moreover, there exists $\eta > 0$ and for any $\varepsilon > 0$ there exists $T > 0$ such that for each solution x and for any $s \geqslant t_0$ for which $|x(s)| < \eta$ the inequality $|x(t)| < \varepsilon$ is valid for $t \geqslant s + T$.

Remark 1. All solutions of LIDE (1) are stable (uniformly stable, equiasymptotically stable or uniformly asymptotically stable) if and only if its zero solution enjoys the same property.

3 - Main results

Denote by L_k $(k = 0, 1, 2, \ldots)$ the linear space of solution $x(t)$ of LIDE (1) defined in the interval $[t_k + 0, \ +\infty)$. Let $e_j := \operatorname{col}(\delta_1^j, \ldots, \delta_n^j)$ where

$$\delta_i^j = \begin{matrix} 0 & \text{for } i \neq j \\ 1 & \text{for } i = j \end{matrix}$$

is Kronecker's symbol and col(...) stands for a *column vector.*

The solutions $x_j(t) = X(t, \ t_k + 0)\, e_j$ $(j = 1, 2, \ldots, n)$ are linearly independent as elements of the vector space L_k. We shall note that their restrictions to the interval $[t_{k+1} + 0, \ +\infty)$ as elements of the linear space L_{k+1} are linearly dependent if the impulse matrix B_{k+1} is singular. In this case both merging of solutions at the point $t_{k+1} + 0$ and noncontinuability to the left of some solutions of L_{k+1} are observed.

Each solution $x(t)$ with initial value $x(t_k + 0) = \operatorname{col}(\lambda_1, \ldots, \lambda_n)$ is a linear combination of the solutions $x_j(t)$ $(j = 1, 2, \ldots, n)$

$$x(t) = X(t, \ t_k + 0)\, x(t_k + 0) = \lambda_1 x_1(t) + \ldots + \lambda_n x_n(t), \tag{3}$$

i.e., L_k $(k = 0, 1, 2, \ldots)$ are n-dimensional linear spaces.

The classical Def. 1 and Def. 2 are valid for ordinary differential equations as well. For LIDE (1) the study of exponential stability is appropriate with the aim to take into account the specific character, of this class of ordinary differential equations.

Def. 3. LIDE (1) is sai to be: (a) *exponentially stable* if for any nonnegative integer k there exist positive constants α_k and N_k such that for each solution $x \in L_k$ the following inequality should hold

$$|x(t)| \leqslant N_k \mathrm{e}^{-\alpha_k t} |x(t_k + 0)| \qquad \text{for } t \geqslant t_k + 0; \tag{4}$$

(b) *uniformly exponentially stable* if there exist positive constants α and N such that for any nonnegative integer k and for each solution $x \in L_k$ the following inequality should be valid

$$|x(t)| \leqslant N \mathrm{e}^{-\alpha(t-s)} |x(s)| \qquad \text{for } t \geqslant s \geqslant t_k + 0; \tag{5}$$

(c) *weakly exponentially stable* (*weakly uniformly exponentially stable*) with respect to the space of solutions L_k if inequality (4) (inequality (5)) is valid only for the solutions $x \in L_k$, where k is a fixed number.

Remark 2. For LIDE (1) Def. 1 is equivalent to the following Def. 4 (Milev and Bainov (to appear) Propositions 1 and 2).

Def. 4. LIDE (1) is said to be: (a) *stable* if for any nonnegative integer k there exists a positive constant N_k such that for each solution $x \in L_k$ the following inequality should hold

$$(6) \qquad |x(t)| \leqslant N_k\,|x(t_k+0)| \qquad \text{for } t \geqslant t_k+0\,;$$

(b) *uniformly stable* if there exists a positive constant N such that for any nonnegative integer k and for each solution $x \in L_k$ the following inequality should hold

$$(7) \qquad |x(t)| \leqslant N|x(s)| \qquad \text{for } t \geqslant s \geqslant t_k+0\,.$$

Remark 3. A straighforward verification yields that for LIDE (1) exponential stability and uniform exponential stability implies uniform stability.

Proposition 1. *If* LIDE (1) *is exponentially stable, then it is asymptotically stable.*

Proof. Let $s \in [t_k+0,\ t_{k+1}]$. By the inequality of Gronwall-Bellman

$$|U(t_k+0)\,U^{-1}(s)| \leqslant \exp \int_{t_k}^{s} |A(\theta)|\,\mathrm{d}\theta\,.$$

Choose
$$\eta = N_k^{-1} \exp\,(-\alpha_k s - \int_{t_k}^{s} |A(\theta)|\mathrm{d}\theta)\,.$$

Let ε be an arbitrary positive number. Choose

$$T = \begin{matrix} -\alpha_k^{-1} \ln \varepsilon > 0 & \text{for } 0 < \varepsilon < 1 \\ 1 & \text{for } \varepsilon \geqslant 1\,. \end{matrix}$$

Then for each solution $x \in L_k$ and for any $t \geqslant s+T$ we have

$$|x(t)| \leqslant N_k\,\mathrm{e}^{-\alpha_k t}\,|x(t_k+0)| = N_k\,\mathrm{e}^{-\alpha_k(t-s)}\,\mathrm{e}^{\alpha_k s}\,|U(t_k+0)\,U^{-1}(s)\,x(s)|$$

$$< N_k\,\mathrm{e}^{(\alpha_k s + \int_{t_k}^{s} |A(\theta)|\mathrm{d}\theta)}\,\eta \mathrm{e}^{-\alpha_k T} \leqslant \varepsilon\,.$$

Hence LIDE (1) is equiasymptotically stable.

Remark 4. The inverse assertion is not true. We shall construct an example of LIDE which is equiasymptotically stable but is not exponentially stable.

Example 1. Let $t_k = k$ $(k = 1, 2, \ldots)$. Consider LIDE

$$(8) \qquad \frac{dx}{dt} = 0 \qquad t \neq k \qquad x(k+0) = \frac{k-1}{k}x(k-0) \qquad (k = 1, 2, \ldots).$$

The solution $x(t)$ can be written down in the form

$$x(t) = \frac{[s]}{[t]}x(s).$$

Note that $[k+0] = k$ and $[k-0] = k-1$. A straightforward verification shows that LIDE (8) is equiasymptotically stable but not exponentially stable since

$$\limsup_{t \to +\infty} \frac{1}{t} \ln |x(t)| = 0.$$

Proposition 2. Def. (2b) *is equivalent to* Def. (3b).

Proof. Let LIDE (1) be uniformly asymptotically stable. For any solution $x \neq 0$ and for any fixed s of the definition domain of x there exists a positive constant c such that $c|x(s)| = \eta/2$. Since cx is a solution too, then by Def. (2b) for any $t \geqslant T$ we have

$$c|x(t+s)| < \varepsilon = c2\varepsilon\eta^{-1}|x(s)| \quad \text{i.e.} \quad |x(t+s)| < 2\varepsilon\eta^{-1}|x(s)|.$$

Fix ε so that $2\varepsilon\eta^{-1} < e^{-1}$. Thus there exists a positive constant T such that for any solution x and for any s of the definition domain of x for $t \geqslant T$ the following inequality hold

$$(9) \qquad |x(t+s)| \leqslant e^{-1}|x(s)|.$$

Hence there exists a positive constant T such that for any nonnegative integer k and for any solution $x \in L_k$ for $s \geqslant t_k + 0$ and $t \geqslant T$ inequality (9) is valid. Let $t \geqslant T$ and let $t \in [mT, (m+1)T]$, where m is a positive integer. Since

$\frac{t}{m} \geqslant T$, then in view of (9)

$$|x(\frac{t}{m}+s)| \leqslant \mathrm{e}^{-1}|x(s)|$$

$$|x(2\frac{t}{m}+s)| \leqslant \mathrm{e}^{-1}|x(\frac{t}{m}+s)|$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$|x(m\frac{t}{m}+s)| \leqslant \mathrm{e}^{-1}|x(\frac{m-1}{m}t+s)|$$

i.e.
$$|x(t+s)| \leqslant \mathrm{e}^{-m}|x(s)| \leqslant \mathrm{e}^{-(t/T)+1}|x(s)|.$$

Set $\alpha = \frac{1}{T} > 0$ and obtain that for $t \geqslant T$

$$|x(t+s)| \leqslant \mathrm{e}\cdot\mathrm{e}^{-\alpha t}|x(s)|.$$

Let $t \in [0, T]$. Since LIDE (1) is uniformly stable, then by Def. (4b) there exists a positive constant $\tilde{N}$ such that

$$|x(t+s)| \leqslant \tilde{N}|x(s)| \leqslant \tilde{N}\mathrm{e}^{\alpha T}\mathrm{e}^{-\alpha t}|x(s)|.$$

Hence there exist positive constants $\alpha = \frac{1}{T}$ and $N = \max(\mathrm{e}, \tilde{N}\mathrm{e}^{\alpha T})$ such that for any nonnegative integer k and for any solution $x \in L_k$ the following inequality is valid

$$|x(t+s)| \leqslant N\mathrm{e}^{-\alpha t}|x(s)| \qquad \text{for } t \geqslant 0 \text{ and } s \geqslant t_k+0$$

i.e. LIDE (1) is uniformly exponentially stable.

The inverse assertion follows from inequality (5). Choose $\eta = \frac{1}{N}$, $T = -\alpha^{-1}\ln\varepsilon > 0$ for $0 < \varepsilon < 1$ or $T = 1$ for $\varepsilon \geqslant 1$ and obtain that

$$|x(t)| \leqslant N\mathrm{e}^{-\alpha(t-s)}|x(s)| < N\mathrm{e}^{-\alpha T}\eta \leqslant \varepsilon.$$

Proposition 3. *Let* LIDE (1) *be exponentially stable. There exists a positive constant α and for any positive integer k there exist positive constants $\tilde{N}_k$ such that for any solution $x \in L_k$ the following inequality should hold*

$$|x(t)| \leqslant \tilde{N}_k\mathrm{e}^{-\alpha t}|x(t_k+0)| \qquad \text{for } t \geqslant t_k+0.$$

Proof. Since LIDE (1) is exponentially stable, then by (4) for any positive integer k and for any solution $x \in L_k$ we have $\chi[x] \leqslant -\alpha_k$, where $\chi[x]$ stands for

Lyapunov's characteristic exponent

$$\chi[x] = \limsup_{t \to \infty} \frac{1}{t} \ln |x(t)|.$$

Since L_k is a finite dimensional linear space and for any solution $x \in L_k$ the representation (3) is valid, then

$$\chi[x] \leqslant \max_{1 \leqslant j \leqslant n} \chi[x_j] = -\beta_k \leqslant -\alpha_k < 0.$$

Denote by $\widetilde{x}_k$ $(k = 0, 1, 2, \ldots)$ a solution of L_k with the maximal characteristic exponent, i.e. $\chi[\widetilde{x}_k] = -\beta_k$. The restriction of the solution $\widetilde{x}_k(t)$ to the interval $[t_{k+1} + 0, +\infty)$ is an element of the space L_{k+1}, hence $-\beta_k \leqslant -\beta_{k+1}$.

If we suppose that there exist $n+1$ different exponents $-\beta_{k_0} < -\beta_{k_1} < \ldots < -\beta_{k_n} < 0$, then the restrictions of the solutions $\widetilde{x}_{k_0}, \widetilde{x}_{k_1}, \ldots, \widetilde{x}_{k_n}$ to the interval $[t_{k_n} + 0, +\infty)$ are elements of the n-dimensional linear space L_{k_n} and they should be linearly independent since they have different characteristic exponents. Hence among the exponents β_k $(k = 0, 1, 2, \ldots)$ there are at most n different and let $\max_{k=0,1,2,\ldots} \{-\beta_k\} = -\beta$.

For an arbitrary solution $x \in L_k$ the representation (3) is valid and since $\max_{1 \leqslant j \leqslant n} \chi[x_j] \leqslant -\beta$, then for any $\varepsilon \in (0, 1)$ there exists a positive constant N_k such that $|x_j(t)| \leqslant N_k^* \mathrm{e}^{-\beta(1-\varepsilon)t}$. Hence

$$|x(t)| \leqslant n|x(t_k + 0)| N_k^* \mathrm{e}^{-\beta(1-\varepsilon)t} = \widetilde{N}_k \mathrm{e}^{-\alpha t} |x(t_k + 0)|$$

where $\widetilde{N}_k = nN_k^*$ and $\alpha = \beta(1 - \varepsilon)$.

We shall show that there exist LIDE which are exponentially stable but not uniformly asymptotically stable.

Example 2. Let $t_k = \mathrm{e}^k$ $(k = 0, 1, 2, \ldots)$ and consider LIDE(10)

$$\text{(10)} \qquad \frac{\mathrm{d}x}{\mathrm{d}t} = (\{\ln t\} - \frac{1}{2})\, x \qquad t \neq t_k \qquad x(t_k + 0) = \mathrm{e}^{-tk} x(t_k) \qquad k = 1, 2, \ldots$$

where $\{y\} = y - [y]$ is the fractional part of the number y. We shall note that $\{k + 0\} = 0$ and $\{k - 0\} = 1$. For $t \geqslant s$ the solution is written down in the form

$$x(t) = x(s)\ \exp\{t\{\ln t\} - s\{\ln s\} - \frac{3}{2}(t - s)\}$$

and a straightforward verification yields that LIDE (10) is esponentially stable but not uniformly asymptotically stable.

Proposition 4. *If* LIDE (1) *is weakly exponentially stable (weakly uniformly exponentially stable) with respect to the space* L_k, *then* LIDE (1) *is weakly exponentially stable (weakly uniformly exponentially stable) with the same exponent with respect to the spaces* L_i, $0 \leqslant i \leqslant k$, *as well.*

Proof. By the inequality of Gronwall-Bellman for any $\tau_1, \tau_2 \in [t_{k-1}+0, t_k]$ the following inequality holds

$$|U(\tau_1)\, U^{-1}(\tau_2)| \leqslant \exp \int_{t_{k-1}}^{t_k} |A(\theta)|\, \mathrm{d}\theta = a_k\,.$$

Let LIDE (1) be weakly exponentially stable with respect to the space L_k. For any solution $x \in L_{k-1}$ its restriction to the interval $[t_k+0,\ +\infty)$ belongs to the space L_k and by Def. (3c) for any $t \geqslant t_k+0$ we have

$$|x(t)| \leqslant N_k\, \mathrm{e}^{-\alpha_k t}\, |x(t_k+0)| = N_k\, \mathrm{e}^{-\alpha_k t}\, |B_k\, U(t_k)\, U^{-1}(t_{k-1}+0)\, x(t_{k-1}+0)|$$

$$\leqslant N_k\, |B_k|\, a_k\, \mathrm{e}^{-\alpha_k t}\, |x(t_{k+1}+0)|\,.$$

If $t \in [t_{k-1}+0,\ t_k]$, then

$$|x(t)| = |U(t)\, U^{-1}(t_{k-1}+0)\, x(t_{k-1}+0)| \leqslant a_k\, |x(t_{k-1}+0)|$$

$$\leqslant a_k\, \mathrm{e}^{\alpha_k t_k}\, \mathrm{e}^{-\alpha_k t}\, |x(t_{k-1}+0)|\,.$$

Choosing $\alpha_{k-1} = \alpha_k$ and $N_{k-1} = \max(N_k\, |B_k|\, a_k,\ a_k\, \mathrm{e}^{\alpha_k t_k})$ we obtain that LIDE (1) is weakly exponentially stable with respect to the space L_{k-1} as well.

Now let LIDE (1) be weakly uniformly exponentially stable with respect to the space L_k. For any solution $x \in L_{k-1}$ its restriction to the interval $[t_k+0,\ +\infty)$ belongs to L_k and for $t \geqslant s \geqslant t_k+0$ inequality (5) is valid.

If $t_{k-1}+0 \leqslant s \leqslant t_k < t$, then

$$|x(t)| \leqslant N\, e^{-\alpha(t-t_k)}\, |x(t_k+0)| = N\, e^{\alpha(t-s)}\, e^{\alpha(t_k-s)}\, |B_k U(t_k) U^{-1}(s) x(s)|$$

$$\leqslant N |B_k|\, a_k\, e^{\alpha(t_k-t_{k-1})}\, e^{-\alpha(t-s)}\, |x(s)|\,.$$

If $t_{k-1}+0 \leqslant s \leqslant t \leqslant t_k$, then

$$|x(t)| = |U(t) U^{-1}(s) x(s)| \leqslant a_k\, e^{\alpha(t_k-t_{k-1})}\, e^{-\alpha(t-s)}\, |x(s)|\,.$$

Hence choosing

$$\tilde{N} = \max\left(N,\ a_k e^{\alpha(t_k-t_{k-1})},\ N|B_k|\, a_k e^{\alpha(t_k-t_{k-1})}\right)$$

we obtain that LIDE (1) is weakly uniformly exponentially stable with respect to the space L_{k-1} as well.

Proposition 5. *Let* LIDE (1) *be weakly exponentially stable (weakly uniformly exponentially stable) with respect to the space L_{k-1} and let the impulse matrix B_k be nonsingular. Then* LIDE (1) *is weakly exponentially stable (weakly uniformly exponentially stable) with the same exponent with respect to the space L_k as well.*

Proof. Since the impulse matrix B_k is nonsingular, then each solution of L_k is a restriction of a solution of L_{k-1}. Hence if LIDE(1) is weakly uniformly exponentially stable with respect to L_{k-1}, then it is weakly uniformly exponentially stable with respect to L_k as well.

Now let LIDE (1) be weakly exponentially stable with respect to L_{k-1}. Then for $t \geqslant t_k+0$ we have

$$|x(t)| \leqslant N_{k-1} e^{-\alpha_{k-1} t} |x(t_{k-1}+0)| = N_{k-1} e^{-\alpha_{k-1} t} |U(t_{k-1}) U^{-1}(t_k) B_k^{-1} x(t_k+0)|$$

$$\leqslant N_{k-1} |B_k^{-1}|\, e^{\int_{t_{k-1}}^{t_k} |A(\theta)|\, d\theta}\, e^{-\alpha_{k-1} t} |x(t_k+0)| = N_k e^{-\alpha_{k-1} t} |x(t_k+0)|$$

where $N_k = N_{k-1} |B_k^{-1}| \cdot \exp\left(\int_{t_{k-1}}^{t_k} |A(\theta)|\, d\theta\right)$.

Hence LIDE (1) is weakly exponentially stable with respect to the space L_k as well.

Proposition 6. *Let the impulse matrices B_i $(i = 0, 1, 2, \ldots)$ of* LIDE (1) *be nonsingular. If* LIDE (1) *is weakly exponentially stable (weakly uniformly expo-*

nentially stable) with respect to a fixed space L_k, then LIDE (1) *is exponentially stable (uniformly exponentially stable).*

Proof. Proposition 6 is a corollary of Proposition 4 and Proposition 5.

Remark 5. If the impulse matrix B_k is singular, then it is possible for LIDE (1) to be weakly uniformly exponentially stable with respect to L_k. We shall illustrate this by the following example.

Esample 3. Let $t_i = i$ $(i = 0, 1, 2, \dots)$ and consider LIDE (11)

$$\text{(11)} \qquad \frac{dx}{dt} = Ax \qquad t \neq t_i \qquad x(t_i + 0) = B_i x(t_i) \qquad (i = 1, 2, \dots)$$

where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad B_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \dots \quad B_i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{for } i \geq 2 .$$

A straightforward verification yields that LIDE (11) is weakly uniformly exponentially stable with respect to the space L_0 since the impulse at the moment t_1 crumples the «inconvenient» solutions. LIDE (11) is not weakly stable with respect to any of the spaces L_k $(k \geq 1)$ since on the intervals $[t_k + 0, +\infty)$ $(k = 1, 2, \dots)$, the problem coincides with the classical one and the matrix A has an eigenvalue greater than zero.

Remark 6. If in Example 3 we define the impulse matrices by the equality

$$B_i = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \text{ for } i = 10j + 1 \quad \text{and} \quad B_i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ for } i \neq 10j + 1, \; j = 0, 1, 2, \dots$$

then LIDE (11) becomes uniformly exponentially stable.

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Abstract

In the present paper the notion of exponential stability for linear impulsive differential equations at fixed moments is made precise.

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