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The space of spheres and conformal geometry (**)

In memory of G. I. KATZ

0 – The aim of this paper is to develop some integrogeometrical ideology and apply it to Riemannian Geometry.

We study the standard conformally flat manifolds $\mathbb{R}^n$ and S^n with fixed confomal diffeomorphism τ: $\mathbb{R}^n \cup \{\infty\} \to S^n$. The fact that automorphism group of the last structure (on S^n) is the finite-dimensional Lie group (no matter whether $n=2$ or $n>2$, because we mention only global automorphisms) shows clearly that it preserve actually much more rigid structure than the conformal structure. Indeed this is so: it preserves the space of spheres. The last manifold carries an invariant pseudo-Riemannian metric, and, consequently, an invariant density form. It turns out that both two invariant tensors give rise to some conformally-invariant constructions on the different submanifolds of $\mathbb{R}^n$ and S^n. We refer to [6] and [13] for other applications.

Let Σ be the manifold of all Euclidean $(n-1)$-spheres in $\mathbb{R}^n$ of positive radii and $\tilde{\Sigma}$ be the manifold of all $(n-1)$-spheres in S^n supplied with the spherical metric. Topologically Σ is homeomorphic to $\mathbb{R}^n \times R_+$ and $\tilde{\Sigma}$ is homeomorphic to the unit disk subbundle of the canonical line bundle over $\mathbb{R}P^n$, so homotopically equivalent to $\mathbb{R}P^n$. By means of τ, Σ is imbedded in $\tilde{\Sigma}$. Let G_n be the group of conformal automorphisms of S^n (i.e., the Möbius group), $G_n \approx O(n+1,\ 1)$. We'll also consider elements of G_n to be mappings from $\mathbb{R}^n$ to $\mathbb{R}^n$, defined everywhere except no more than one point. Of course, G_n acts also on $\tilde{\Sigma}$ and this action is tran-

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sitive. Fix $\sigma \in \tilde{\Sigma}$; the stabilizer H of σ in $\tilde{\Sigma}$ acts conformally on σ, itself, so that we have an exact sequence $1 \to \mathbb{Z}_2 \to H \to G_{n-1} \to 1$, in particular, H is semisimple and locally isomorphic to $O(n, 1)$. For $\tilde{\Sigma}$ to carry G_n-invariant pseudo-Riemannian metric there is a necessary and sufficient conditions that tangent space $T_\sigma \tilde{\Sigma}$ carries H-invariant pseudometric. For $T_\sigma \tilde{\Sigma} \approx \mathfrak{g}_n/\mathfrak{h}$ and the action of the Lie algebra $\mathfrak{h}$ on $\mathfrak{g}_n/\mathfrak{h}$ is the ad-action, the role of such a metric can be given to the Cartan-Killing metric if it's restriction on $\mathfrak{h}$ turns out to be non-degenerate. Actually the restriction of the Cartan-Killing metric of $\mathfrak{g}$ on $\mathfrak{h}$ is non-zero and proportional to its own Cartan-Killing metric of $\mathfrak{h}$-this can be computed directly or we can use the fact that all ad-invariant metrics on the simple Lie algebra are proportional to each other. Thus, we can formulate

Proposition 1. *There exists a G_n-invariant nondegenerate pseudo-Riemannian metric $g_{\tilde{\Sigma}}$ on $\tilde{\Sigma}$.*

For a more direct description of this metric see in [1] where G_n is identified with the orthogonal group of the quadratic form Q of signature $(n+1,\ 1)$ and $\tilde{\Sigma}$ with the coset space $\{Q(x) = 1\}/\mathbb{Z}_2$.

Now let us introduce an explicit formula for the restriction of $g_{\tilde{\Sigma}}$ on Σ. For this purpose consider the coordinate system on Σ: $\sigma \mapsto$ (coordinates of the center; radius). We will denote by $\pi_1,\ \pi_2$ two projections from $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ on $\mathbb{R}^n$ and $\mathbb{R}$ respectively.

Proposition 2. *Every G_n-invariant pseudometric on Σ is proportional to*

$$1/R^2\Big(\sum_{k=1}^{n} \mathrm{d}x_k^2 - \mathrm{d}R^2\Big) = g_\Sigma .$$

Proof. Use G_n-invariance, or see [2].

Remark. It seems hard to express the metric $g_{\tilde{\Sigma}}$ in terms of the spherical geometry on S^n.

Corollary. *Every G_n-invariant differential form of the highest dimension $(n+1)$ on Σ is proportional to*

$$\frac{1}{R^{n+1}}(\mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n \wedge \mathrm{d}R) = \omega_\Sigma .$$

Using these two tensors we will introduce some conformally invariant constructions. Namely, we will show in Theorem 1 and Proposition 3 that every closed set B in S^n determines some Finsler metric in $S^n \sim B$ such that if $g \in G_n$ leaves B invariant then $g|_{S^n \sim B}$ is ab isometry. For instance, it follows that if $G \subset G_n$ is a Lie subgroup, then the (open) union $M \subset S^n$ of the main type orbits carries a G-invariant Finsler metric. The same is true about the complement of the limit set of a Kleinian group.

We show that a higher-dimensional conformal analogue of the Schwartz lemma holds on this situation, that is, if B is a closed oriented hypersurface in S^n and M is one of the two components of $S^n \sim B$, then every conformal endomorphism of M is a contraction of the Finsler metric constructed (Theorem 3).

We then introduce conformal invariants of a hypersurface in $\mathbb{R}^n$. First we show that if γ: $[0, b] \to \mathbb{R}^2$ is a smooth curve, then the integral of «conformal length»

$$\int_\gamma \sqrt{\left|\frac{dK}{dl}\right|}\, dl$$

is invariant under the action of $G_2 \approx O(3, 1)$ on $\mathbb{R}^2$ (Theorem 5). We deduce from this an $SL_2(\mathbb{R})$-invariance of the Virasoro cocycle and Schwartzian (compare with [8], [12]). Then we show that if N is a hypersurface in $\mathbb{R}^n$, A is the second fundamental symmetric $(1-1)$ tensor on N, and $H = \frac{1}{n} \operatorname{Tr} A$ is the mean curvature, then metric $((A-HE)^2 \cdot, \cdot)$ is conformally invariant. This explains the nature of some known conformal invariants of Blaschke, White and Hsiung-Levko. From this, we construct some realizations of unitar ($n=2$) and Banach ($n \geqslant 2$) representation of $O(n+1, 1)$ in the spaces of tensors on S^n. They are in a sense «glued» to the natural representation of $O(n+1, 1)$ in the space $\mathbb{R}^{n+1, 1}$ and seem to be «quantization» of the isometric action of $O(n+1, 1)$ in the hyperbolic space H^{n+1}, «glued» to the conformal action in S^n.

Another application of our conformal invariants is the Efimov-type theorem, assuring the decay of curvature of almost everywhere non-convex hypersurface in the half-space.

This article is dedicated to the memory of my teacher Georgii I. Katz.

1 – Before going any further, let's make a technical.

Def. Let A be a subset of $\mathbb{R}^n$ (resp. S^n). Let $\sigma \in \Sigma$ (resp. $\tilde{\Sigma}$) be a sphere. We'll say that σ *divides* A, if A has a nonempty intersection with both two connected components of $\mathbb{R}^n \sim \sigma$ (resp. $S^n \sim \sigma$).

From now on fix an orientation on $\mathbb{R}^n$ and S^n. One of our basic tools will be the next *C-invariant.*

Def. Let A and B be subsets of $\mathbb{R}^n$ (resp. S^n) such that $\overline{A} \cap \overline{B} = \emptyset$. Let $\Sigma(A, B)$ be a set of all spheres σ which divide both A and B. Then $C(A, B)$ is defined by formula

$$C(A, B) = \left| \int_{\Sigma(A, B)} \omega_\Sigma \right|$$

and the same goes for S^n.

Lemma 1. (a) *If A and B are subsets of $\mathbb{R}^n$ and A or B is compact then* $C(A, B) < \infty$. (b) *If A and B are subsets of S^n and $\overline{A} \cap \overline{B} = \emptyset$ then always* $C(A, B) < \infty$.

Remark. It's possible for $A, B \subset \mathbb{R}^n$ that $\overline{A} \cap \overline{B} = \emptyset$ in $\mathbb{R}^n$ but $\overline{\tau(A)} \cap \overline{\tau(B)} \neq \emptyset$ in S^n (namely, both A and B are unbounded in $\mathbb{R}^n$ and therefore $\overline{\tau(A)} \cap \overline{\tau(B)} \ni \tau(\infty)$).

Proof. (a) Assume that A is compact. Let ρ be equal to $\rho(A, B)$ in euclidean metric. If $\sigma \in \Sigma(A, B)$, $x(\sigma)$ and $R(\sigma)$ it's center and radius then evidently $R(\sigma) \geqslant \rho/2$ and either $\rho(x(\sigma), A) \leqslant \rho/2$ or $\rho(x(\sigma), A) \geqslant \rho/2$ and $\rho(x(\sigma), A) + \operatorname{diam} A > R(\sigma) > \rho(x(\sigma), A)$ so

$$C(A, B) \leqslant \int_{\rho(x, A) \leqslant \rho/2} \mathrm{d}x \int_{\rho/2}^{\infty} \frac{\mathrm{d}R}{R^{n+1}} + \int_{\rho(x, A) > \rho/2} \mathrm{d}x \int_{\rho(x, A)}^{\rho(x, A) + \operatorname{diam} A} \frac{\mathrm{d}R}{R^{n+1}}$$

$$= \frac{1}{n} \left(\frac{2}{\rho}\right)^n V_n\left(\{x | \rho(x, A) \leqslant \frac{\rho}{2}\}\right) + n \int_{\rho(x, A) > \rho/2} \mathrm{d}x \left(\frac{1}{\rho^n(x, A)} - \frac{1}{(\rho(x, A) + \operatorname{diam} A)^n}\right).$$

When $x \to \infty$ then $\dfrac{1}{\rho^n(x, A)} - \dfrac{1}{(\rho(x, A) + \operatorname{diam} A)^n} \sim \dfrac{n \operatorname{diam} A}{\rho^{n+1}(x, A)}$ hence the last integral is finite (use polar coordinates in $\mathbb{R}^n$).

(b) We may assume that $\tau(\infty) \in \overline{B}$ and use τ^{-1} to reduce this case to (a). This is possible because of conformal invariance of C-invariant (see Lemma 2).

Remark. Let $r(\sigma)$ be the radius of $\sigma \in \tilde{\Sigma}$ in the spherical metric on S^n, then $\{\sigma | r(\sigma) > r > 0\}$ is compact in $\tilde{\Sigma}$; this is probably the best explanation of (b).

Lemma 2. *Let A and B be as in Lemma* 1. *Let $g \in G_n$; for the case of $\mathbb{R}^n$ assume that g has no pole on A and B* , i.e., $g^{-1}(\infty) \overline{\in} A \cup B$. *Then $C(A, B) = C(gA, gB)$.*

Proof is obvious—all constructions are invariant.

Example If $\# A = \# B = 2$, say $A = \{x, y\}$ and $B = \{2, w\}$ then we obtain the conformal invariant of the four points. It's easy to show that $C(\{x, y\}, \{z, w\}) \neq$ const even in the case $n = 2$. So, a formula must exist expressing our invariant through the cross-ratio.

If we fix A we'll obtain a set function $B \mapsto C(A, B)$, $\overline{B} \subset \mathbb{R}^n \sim \overline{A}$ (resp. $\overline{B} \subset S^n \sim \overline{A}$). In no case is this a measure, but rather a «length» or «perimeter». More precisely, the following statements hold.

Theorem 1. (On Finsler metric existence). *Let $B \subset \mathbb{R}^n$, is a smooth closed hypersurface, $p \in \mathbb{R}^n \sim B$, γ: $[-b, b] \to \mathbb{R}^n \sim B$ is a C^2-curve, $\gamma(0) = p$. Then:*

(a) *There exists a limit,* $\lim_{t \to 0}(1/2t)\, C(\gamma([-t, t]), B) = C_B(\gamma)$.

(b) *This limit depends only on the tangent vector* $\dot{\gamma}(p) = \gamma_*(\frac{\mathrm{d}}{\mathrm{d}t})_{t=0}$ *so* $C_B(\gamma) = C_B(\dot{\gamma}(\rho))$.

(c) *The map C_B: $T(\mathbb{R}^n \sim B) \to \mathbb{R}_+ \cup \{0\}$ defines a Lipschitz coefficient Finsler metric on $\mathbb{R}^n \sim B$* (i.e. *$C_B(\cdot)$ is a Lipschitz function on $T(\mathbb{R}^n \sim B)$*).

(d) *If $U \subset \mathbb{R}^n \sim B$ is an open set, $g \in G_n$ has no pole in $U \cup B$ then $g|_U$ is an isometry between (U, C_B) and (gU, C_{gB}).*

Proof. Let $\psi(p, B)$ be the subset in $\mathbb{R}^n$ consisting of such x, that the sphere $(x, \rho(x, p))$ divides B. Let $\lambda(p, B)$ be the set of spheres in S^n (i.e. hyperplanes included), consisting p and touching B at some point. Then evidently $\lambda(p, B)$ is a closed submanifold in $\tilde{\Sigma}$, diffeomorphic to B, hence its codimension is 2. We have $\psi(p, B) \subseteq \pi_1(\Sigma((\gamma([-t, t]), B))$ except maybe the subsest of $\psi(p, B)$ of measure 0, consisting of such q, that $p - q$ is orthogonal to $\dot{\gamma}(p)$. On the other hand, let $\sigma \in \Sigma(\gamma([-t, t]), B) \sim \pi_1^{-1}(\psi(p, B))$. Then σ divides $\gamma([-t, t])$ and divides B but the sphere $(x, \rho(x, p))$ does not divide B (here x is the center of σ). It follows immediately that for some s, $|s| < t$, the sphere $(x, \rho(x, \gamma(s)))$ touches B. We can easily estimate $\omega_{\tilde{\Sigma}}$-measure of the set of spheres, touching B and containing some point in $\gamma([-t, t])$. Namely, consider a smooth map β: $B \times [-t, t] \to \tilde{\Sigma}$ defined in the following way: $\beta(b, s)$ is the unique sphere touching B at b and containing $\gamma(S)$. As $\beta(B \times \{0\}) = \lambda(p, B)$ we see from the tube's volume formula that $\omega_{\tilde{\Sigma}}(\beta(B \times [-t, t])) = O(t^2)$. So, studying our limit, we can consider only such spheres, whose center is in $\psi(p, B)$. Next, given $x \in \psi(p, B)$ we want to study the fiber over x, i.e. the set $\pi_2(\pi_1^{-1}\{x\} \cap \Sigma(\gamma([-t, t]), B))$. Let $\varepsilon(x)$ be the maximal positive real number, such that for all $R \in [\rho(x, p) - \varepsilon, \rho(x, p) + \varepsilon]$ the sphere (x, R) divides B. Let $M = \sup_{-b<s<b} |\dot{\gamma}(s)|$. Then if $\varepsilon(x) > Mt$, then $(x, R) \in \Sigma(\gamma([-t, t]), B)$ iff (x, R) divides $\gamma([-t, t])$, i.e. $\min_{y \in \gamma([-t, t])} \rho(x, y) < R < \max_{y \in \gamma([-t, t])} \rho(x, y)$. As γ is C^2, one computes easily the right side to be $|x-p| + \dfrac{t}{|x-p|}|(x-p, \dot{\gamma}(p))| + C_1(x)t^2$ and the left side $|x-p| - \dfrac{t}{|x-p|}|(x-p, \dot{\gamma}(p))| + C_2(x)t^2$, where $C_1(x)$ and $C_2(x)$ are bounded continuous functions. Denote $\psi_t(p, B)$ the set $\psi(p, B) \cap \{x\colon \varepsilon(x) > Mt\}$ and by $\Sigma_t \subset \Sigma$ the set $\pi_1^{-1}(\psi_t(p, B)) \cap \Sigma(\gamma([-t, t]), B)$. By Fubini's theorem we have

$$\Big|\int_{\Sigma_t} \omega_\Sigma\Big| = \int_{\psi_1} \mathrm{d}x \int_{\lambda_1}^{\lambda_2} \frac{\mathrm{d}R}{R^{n+1}} = \int_{\psi_1} \frac{2t|(x-p, \dot{\gamma}(p)|}{|x-p|^{n+2}}(1 + C_3(x, t)t)\,\mathrm{d}x \qquad \text{where}$$

$$\psi_1 = \psi_1(p, B) \qquad \lambda_1 = |x-p| - \frac{t}{|x-p|}|x-p, \dot{\gamma}(p))| + C_2(x)t^2$$

$$\lambda_2 = |x-p| + \frac{t}{|x-p|}|x-p, \dot{\gamma}(p))| + C_1(x)t^2$$

for some continuous bounded $C_3(x, t)$. Suppose $\sigma \in (\Sigma(\gamma([-t, t]), B) \cap \pi_1^{-1}(\psi(p, B))) \sim \Sigma_t$. It means that $\varepsilon(x) < Mt$, where $\sigma = (x, R)$ and $|R - \rho(x, p)| < Mt$. Hence σ lies in $2Mt$-neighbourhood of $\lambda(p, B)$, so as before, the ω_Σ-measure of the last set is $O(t^2)$. It means that we actually can study the

$$\lim_{t\to 0} \int_{\psi_t(p, B)} \frac{|(x-p), \dot{\gamma}(p)|}{|x-p|^{n+2}}(1 + C_3(x, t)t)\,\mathrm{d}x.$$

However, $\bigcup_{t>0} \psi_t(p, B) = \psi(p, B)$ and $C_3(x, t)$ is bounded which gives immediately that this limit exist and is equal to

$$(*) \qquad \int_{\psi(p, B)} \frac{|(x-p, \dot{\gamma}(p))|}{|x-p|^{n+2}}\,\mathrm{d}x.$$

This proves (a) and (b). Statement (c) is verified directly. Statement (d) is the direct consequence of our definition of C_B.

When B is not a manifold, but any closed set in $\mathbb{R}^n$, we *define* the metric C_B in $\mathbb{R}^n \sim B$ by formula $(*)$.

Proposition 3. *Statement* (d) *of Theorem* 1 *still holds for any closed* B.

Proof. We will show in Appendix that integrand in $(*)$ is conformally invariant. More rigorously, consider the double bundle

$$\begin{array}{ccc} & \mathfrak{A} & \\ \swarrow & & \searrow \\ R^n & & \Sigma \end{array}$$

consisting of all pairs (p, σ) such that $p \in \sigma$. (See [9] for general theory, and also a survey [7]). Let (p, x, R) be natural coordinates in $\mathfrak{A}$. Then we'll show that $(n+1)$-form

$$\Lambda = \alpha_{(x-p)} \wedge \frac{\bigwedge dx_i}{|x-p|^{n+2}}$$

is invariant modulo ideal generated by $dp_i \wedge dp_j$ under the natural action of $G_n = O(n+1, 1)$ on $\mathfrak{A}$. Here $\alpha_{(x-p)}$ is the 1-form in $\mathbb{R}^n$ dual to the vector $(x-p)$. This implies the conformal invariancy of the Finsler metric defined by $(*)$. Another application will appear in some other place.

Remark. The same trick as before together with conformal invariance shows that full analogue of the theorem holds in S^n. This will enable us to make some applications to the actions of subgroups of G_n.

Theorem 2. *Let* G *be a subgroup of* G_n, M *is an invariant set in* S^n, $\natural M > 1$. *Then* $S^n \sim \overline{M}$ *carries a* G*-invariant Finsler metric. In particular, if* V *is the* (*open*) *union of the main type orbits and* G *is a connected Lie group then* V *carries a* G*-invariant Finsler metric. The same is true if* G *is a lattice and* M *is its limit set.*

Proof. The statement of the theorem is a direct corollary of Theorem 1 and Proposition 3.

Example 1. Let M be a sphere in S^n so $M \in \tilde{\Sigma}$ and $H \subset G_n$ its *stabilizer* in $\tilde{\Sigma}$, so M is an orbit of H-action. Then the two components of $S^n \sim M$ carry an H_0-invariant metrices. It's nothing else but the hyperbolic metric. This can be computed directly by the explicit formula $(*)$. It also follows from the fact that for H acts transitively on the projectivization of the tangent bundle of $S^n \sim M$.

Our metric yields the following hyperbolicity property.

Theorem 3. *Let M be a closed n-dimensional submanifold of S^n (resp $\mathbb{R}^n$) with a border B. Let $\rho(\cdot, \cdot)$ be the distance function on* $\operatorname{Int} M \times \operatorname{Int} M$ *determined by the Finsler metric C_B. Then for every $g \in G_n$ such that $g(M) \subseteq M$, $g|_{\operatorname{Int} M}$ is the contraction of the metric space* $(\operatorname{Int} M, \rho)$.

Proof. Let $p \in \operatorname{Int} M$, γ: $[-b, b] \to \operatorname{Int} M$ be a C^2-curve, $\gamma(0) = p$. Let $0 < b_1 < b$ and $\sigma \in \tilde{\Sigma}(g \circ \gamma([-b_1, b_1]), B)$ so σ divides both $g \circ \gamma([-b_1, b_1])$ and B. We claim that σ divides $g(B)$. Indeed, suppose that one of the components of $S^n \sim \sigma$, say P, doesn't consist any point of $g(B)$. Consider $q_1 \in g \circ \gamma([-b_1, b_1] \cap P$ and $q_2 \in B \cap P$. Let μ be an arc in P connecting q_1 and q_2. For $g(M)$ is the n-manifold with boundary, $q_1 \in g(M)$ and $q_2 \overline{\in} g(M)$, this arc must intersect $g(B)$ which contradicts $g(B) \cap P = \emptyset$. So $\tilde{\Sigma}(g \circ \gamma[-b_1, b_1]), B) \subseteq \tilde{\Sigma}(g \circ \gamma([-b_1, b_1]), g(B))$, hence $C_B(g_* \dot{\gamma}(p)) \leq C_{g(B)}(g_* \dot{\gamma}(p)) = C_B(\dot{\gamma}(p))$ which proves the theorem.

2 – In the last part of the paper our attention will be focused on the metrics g_Σ, $g_{\tilde{\Sigma}}$ on $\Sigma, \tilde{\Sigma}$. The main idea of all following applications is to construct a lift for all immersed submanifolds $N \hookrightarrow S^n$ of codimension one to the sphere space $\tilde{\Sigma}$ and to use the conformal invariance of this construction for studying the geometry of N.

We being with the case $n = 2$ so N will be curve γ: $[0, b] \to S^2$ such that $\dot{\gamma} \neq 0$. For every $t \in [0, b]$ let $\hat{\gamma}(t)$ be the oscullating circle in the point $\gamma(t)$ of γ (see [3]). We want to introduce a conformally invariant natural parameter on γ—a «conformal length». The first idea is to use the metric $g_{\tilde{\Sigma}}$ and to declare $L_c(\gamma)$ to be $\int_0^b \sqrt{|g_{\tilde{\Sigma}}(\dot{\hat{\gamma}}, \dot{\hat{\gamma}})|}\, dt$. The result will be: $L_c(\gamma)$ is always equal to zero because of the next

Proposition. 4. *For all γ, $\hat{\gamma}(t)$ is isotropic in $\hat{\Sigma}$.*

Proof. We obviously can come down to $\mathbb{R}^2$ and use coordinates (x, R) on Σ. The centers $x(t)$ of $\hat{\gamma}(t)$ form the evolute of γ [3], and the identity $|\dot{x}(t)|^2 = |\dot{R}|^2$ means nothing else than the fact the length of a segment of the evolute is equal to the variation of the curvature radius (see [3]).

Therefore, to introduce the conformal length, we are supposed to invent a length of an isotropic curve in Lorentz manifold.

Theorem 4. (On isotropic length). *Let W a smooth manifold of dimension $n \geq 2$ and let g be a non-degenerate pseudometric on W. If ∇ denotes a canonical Levi-Civita connection on (W, g), then for every immersed isotropic curve*

$\hat{\gamma}$: $[0, b] \to W$ *an integral*

$$(**) \qquad I = \int_0^b \sqrt[4]{|g(\nabla_{\dot{\hat{\gamma}}} \dot{\hat{\gamma}}, \nabla_{\dot{\hat{\gamma}}} \dot{\hat{\gamma}})|}\, dt$$

does not depend on parametrization of $\hat{\gamma}$.

Proof. Let's write (X, Y) instead of $g(X, Y)$. For $(\dot{\hat{\gamma}}, \dot{\hat{\gamma}}) = 0$ we have $(\nabla_{\dot{\hat{\gamma}}} \dot{\hat{\gamma}}, \dot{\hat{\gamma}}) = 0$. Consider a reparametrization $t = t(\tau)$ and the reparametrizied curve $\hat{\gamma}(t(\tau))$. Denote $f(\tau) = (t'_\tau)^{-1}$, then we have $\dot{\hat{\gamma}}_t = f(\tau)\dot{\hat{\gamma}}_\tau$ so

$$\nabla_{\dot{\hat{\gamma}}_t} \dot{\hat{\gamma}}_t = f(\tau)\nabla_{\dot{\hat{\gamma}}_\tau} \dot{\hat{\gamma}}_t = f(\tau)\nabla_{\dot{\hat{\gamma}}_\tau}(f(\tau)\dot{\hat{\gamma}}_\tau) = f(\tau) f'(\tau)\dot{\hat{\gamma}}_\tau + f^2(\tau)\nabla_{\dot{\hat{\gamma}}_\tau} \dot{\hat{\gamma}}_\tau .$$

Recalling that $\hat{\gamma}(\tau)$ is isotropic together with $\hat{\gamma}(t)$ we have

$$(\nabla_{\dot{\hat{\gamma}}_t} \dot{\hat{\gamma}}_t, \nabla_{\dot{\hat{\gamma}}_t} \dot{\hat{\gamma}}_t) = f^4(\tau)(\nabla_{\dot{\hat{\gamma}}_\tau} \dot{\hat{\gamma}}_\tau, \nabla_{\dot{\hat{\gamma}}_\tau} \dot{\hat{\gamma}}_\tau)$$

which implies invariancy of I.

Def. If γ is a smooth curve γ: $[0, b] \to S^2$ (resp. $\mathbb{R}^2$) then the conformal length L_c of γ is the isotropic length of its lift $\hat{\gamma}$: $[0, b] \to \tilde{\Sigma}$ (resp. Σ).

Along with the relation of oscillation, $L_c(\gamma)$ is conformally invariant. We wish to express $L_c(\gamma)$ through standard differential invariants of γ. The metric g_Σ is actualy a hyperbolic Lorentz metric and one determine explicit formulas for the Christoffel coefficients in coordinates (x, R). Let (x_1, x_2, R) be the coordinates in Σ, then we can represent g_Σ by $\dfrac{dx_1^2 + dx_2^2 - dR^2}{R^2}$. Direct computation which is the same that in case of positively-definite hyperbolic metric [10] shows that the Chistoffel coefficients are

$$(***) \qquad \begin{aligned} &\Gamma^1_{11} = \Gamma^2_{11} = \Gamma^1_{22} = \Gamma^2_{22} = \Gamma^1_{12} = \Gamma^2_{12} = 0 \\ &\Gamma^3_{11} = \Gamma^3_{22} = -\frac{1}{R} \qquad \Gamma^3_{33} = -\frac{1}{R} \qquad \Gamma^1_{13} = \Gamma^2_{23} = -\frac{1}{R} \\ &\Gamma^3_{12} = 0 \qquad \Gamma^1_{33} = \Gamma^2_{33} = 0 \qquad \Gamma^3_{13} = \Gamma^3_{23} = 0 \qquad \Gamma^2_{13} = \Gamma^1_{23} = 0. \end{aligned}$$

Next, if $n(t)$ denotes normal to $\gamma(t)$, then we have $\hat{\gamma}(t) = (\gamma(t) + \dfrac{1}{K(t)} n(t), \dfrac{1}{K(t)})$, where $K(t)$ is the curvature. Substituting this to $(***)$, choosing the length parameter t (so $|\dot{\gamma}(t)| = 1$) and using $(***)$ we obtain that

$$L_c(\gamma) = \int_\gamma \sqrt{\frac{dK}{dt}}\, dt.$$

Therefore, we've obtained the following classical result of Liebmann-Pick.

Theorem 5. *The integral* $\int_\gamma \sqrt{\left|\frac{dK}{dt}\right|}\, dt$ *is invariant under the action of Möbius group* G_2 *on* $\mathbb{R}^2$, *where* t *is the length parameter along* γ.

We'll now show the connection between this theorem and $SL_2(\mathbb{R})$-invariance of Schwartzian and Virasoro cocycle. Let $\rho(\theta)$ be smooth function on $S^1 \subset \mathbb{R}^2$ and $\varepsilon > 0$ and let γ_ε be the convex curve with the supporting function $1 + \varepsilon\rho(\theta)$. It is well-known [4] that the curvature of a curve with supporting function $\lambda(\theta)$ is $\frac{1}{\lambda(\theta) + \lambda''(\theta)}$ at the point $x(\theta) = \lambda(\theta)\, e^{i\theta} + i\lambda'(\theta)\, e^{i\theta}$. Here $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ is the angle parameter. Next, if $x \in \gamma_\varepsilon$, and $e^{i\theta}$ is the normal at x, then one knows that

$$\varepsilon\rho(\theta) = \rho(x,\ S^1) + O(\varepsilon^2)$$

where $\rho(x,\ S^1)$ denotes the usual distance and $O(\varepsilon^2)$ in the sense of C^∞-metric. Let g: $\mathbb{R}^2 \to \mathbb{R}^2 \cup \{\infty\}$ be a Möbius map leaving S^1 invariant. We wish to represent the supporting function $1 + \varepsilon\rho_1(\theta)$ of the curve $g \circ \gamma_\varepsilon$. For $z \in \mathbb{R}^2$ let $\mu(g,\ z)$ be the dilatation coefficient of g at z (i.e. $\|g_* X\| = \mu(g,\ z)\|X\|$ for $X \in T_z\mathbb{R}^2$). The normal to $g \circ \gamma_\varepsilon$ at $g(x)$ will be

$$e^{i\eta} = \frac{1}{\mu(g,\ x)}\, g_*(x)(e^{i\theta})$$

and the value of $\varepsilon\rho_1$ at η will be

$$\varepsilon\rho_1(\eta) = \rho(g(x),\ S^1) + O(\varepsilon^2).$$

As $x = e^{i\theta} + O(\varepsilon)$ we see that $e^{i\eta} = g(e^{i\theta}) + O(\varepsilon)$ (recall that $g_*(e^{i\theta})\, e^{i\theta}$ is colinear to $g(e^{i\theta})$ by conformity of g). Again by conformity of g, $\rho(g(x), S^1) = \mu(g,\ e^{i\theta})\rho(x,\ S^1) + O(\varepsilon^2)$ hence we have $\varepsilon\rho_1(\eta) = \mu(g,\ e^{i\theta})\, \varepsilon\rho(\theta) + O(\varepsilon^2)$. Let $z = e^{i\theta}$, $w = e^{i\eta}$, then we obtain $1 + \varepsilon\rho_1(w) = 1 + \mu(g,\ g^{-1}w)\varepsilon \cdot \rho(g^{-1}w) + O(\varepsilon^2)$. Next, let $\lambda(\theta)$ be a supporting function of some $\gamma(t)$, then we can express the integral in Theorem 6 as follows. The curvature $K(t) = \frac{1}{\lambda(\theta) + \lambda''(\theta)}$ and $\left|\frac{dt}{d\theta}\right| = |\lambda(\theta) + \lambda''(\theta)|$ (see [4]), so $I = \int_\gamma \sqrt{\left|\frac{dK}{dt}\right|}\, dt = \int_0^{2\pi} \sqrt{|\lambda'(\theta) + \lambda'''(\theta)|}\, d\theta$. If

$\lambda(\theta) = 1 + \varepsilon\rho(\theta) + O(\varepsilon^2)$, then $I = \sqrt{\varepsilon}\int_0^{2\pi}\sqrt{|\rho' + \rho'''|}\,\mathrm{d}\theta + O(\varepsilon)$. For ρ, ρ_1 above we obtain immediately from Theorem 6 that $\int_0^{2\pi}\sqrt{|\rho' + \rho'''|}\,\mathrm{d}\theta = \int_0^{2\pi}\sqrt{|\rho_1' + \rho_1'''|}\,\mathrm{d}\theta$. Moreover, we could consider part of a curve γ_ε defined in an interval $\theta_0 \leqslant \theta \leqslant \theta_0 + \Delta\theta$ to obtain the point-wise identity

$$\sqrt{|\rho_\theta'(\mathrm{e}^{i\theta}) + \rho_\theta'''(\mathrm{e}^{i\theta})|} = \sqrt{|\rho_1'(g(\mathrm{e}^{i\theta})) + \rho_1'''(g(\mathrm{e}^{i\theta}))|}\,\mu(g,\ \mathrm{e}^{i\theta}).$$

We summarize this in the following well-known

Proposition 5. *Let $\Gamma(TS^1)$ be a space of smooth vector fields on S^1 (each of the type $\rho(\varphi)\frac{\mathrm{d}}{\mathrm{d}\varphi}$) and let $\Gamma(T^*S^1 \otimes T^*S^1)$ be the space of metric on S^1 (each of the type $p(\varphi)\,\mathrm{d}\varphi^2$). Consider the natural action of $SL_2(\mathbb{R})$ on $\Gamma(TS^1)$ and $\Gamma(T^*S^1 \otimes T^*S^1)$. Then the linear map*

$$C\colon \rho\frac{\mathrm{d}}{\mathrm{d}\varphi} \mapsto (\rho' + \rho''')\,\mathrm{d}\varphi^2$$

is a homomorphism of $SL_2(\mathbb{R})$-modules. Hence its kernel is 3-dimensional $SL_2(\mathbb{R})$ submodule of $\Gamma(TS^1)$.

The last statement is clear: Ker C has a basis that consists of $\frac{\mathrm{d}}{\mathrm{d}\varphi}$, $\cos\varphi\frac{\mathrm{d}}{\mathrm{d}\varphi}$, $\sin\varphi\frac{\mathrm{d}}{\mathrm{d}\varphi}$ which simply are the generating vector fields of $SL_2(\mathbb{R})$-action, or in another words, Ker C is the image of the Lie algebra homomorphism $\tau\colon sl_2(\mathbb{R}) \to \Gamma(TS^1)$, $\Gamma(TS^1)$ being a Lie algebra of Diff(S^1). Of course, weight decomposition of both modules can be performed, $\mathbb{R}\frac{\mathrm{d}}{\mathrm{d}\varphi}$ being Cartan subalgebra and the proposition above directly verified.

Our map C is in fact a Virasoro cocycle (see [**15**], [**12**]). If $h\colon S^1 \to \mathbb{R}$ is a smooth function we can produce canonically a vector field $(\mathrm{d}h)^0$, dual to $\mathrm{d}h$ (in local coordinate θ, $(\mathrm{d}h)^0 = \frac{1}{h'(\theta)}\frac{\partial}{\partial\theta}$). Consider a 1-form $\phi(h) = C((\mathrm{d}h)^0)((\mathrm{d}h)^0,\ \cdot)$. Its primitive function is a Schwartzian of h, which explains the $SL_2(\mathbb{R})$-invariance of Schwartzian. So in a sense, our invariant I in Theorem 6 is an *integrated Virasoro cocycle* for curves. See [**12**] for another conformal construction leading to the «group-theoretical» integrated Virasoro cocycle on Diff(S^1).

Let N now be a hypersurface in $\mathbb{R}^n$, $z \in N$, let Π_z be the second fundamental form on T_zN (suppose that local orientation is chosen and $n(y)$ is a positive unit normal vector at $y \in N$). Let $\lambda_1(z), \ldots, \lambda_{n-1}(z)$ be eigenvalues of Π_z identified with some symmetric linear operator on T_zN, i.e. the main curvature numbers of N, let $\sigma_i(z)$ be the spheres tangent to T_zN in point z of radii $R_i(z) = 1/\lambda_i(z)$ i.e. curvature spheres (if $\lambda_i = 0$ then $\sigma_i \equiv T_zN$). Let $g \in G_n$ with no pole in z, let $N' = gN$, λ_i', σ_i' means the same for $z' = gz$. Is it true that for all i there exists j such that $g\sigma_i = \sigma_j'$? The answer is: yes, it is and much more can be said. True, from the singularity theory viewpoint the fact is of no wonder, because the curvature spheres have the order of touching with T_zN which is grater, than the one that a «common» sphere has. But we'll show it in another way to get some formulas we're interested in.

Let ρ be any real function in the neighbourhood V of z, $\rho(z) \neq 0$, and $\hat{\gamma}$: $V \to \Sigma$ is defined by coordinates (x, R) on Σ: $x(y) = y + \rho n(y)$, $R(y) = |\rho(y)|$. We want to compute the metric which is induced on V by the map $\hat{\gamma}$. So let $X \in T_zN$ then

$$\hat{\gamma}_*(X) = (X + \rho_X' n(z) + \rho n_X'(z),\ \operatorname{sgn}\rho \cdot \rho_X').$$

Remember that $n_X'(z) = \Pi_z(X)$ where we look at Π_z as linear symmetric operator, so $(n,\ n_X'(z)) = 0$ and

$$g_\Sigma(\hat{\gamma}_*(X),\ \hat{\gamma}_*(X)) = \frac{1}{\rho^2}[(X + \rho_X' n(z) + \rho n_X'(z),\ X + \rho_X' n(z) + \rho n_X'(z)) - (\rho_X')^2]$$

$$= \frac{1}{\rho^2}[(X,\ X) + (\rho_X')^2 + \rho^2(n_X'(z),\ n_X'(z)) + 2\rho(n_X'(z),\ X) - (\rho_X')^2]$$

$$= ((\frac{E}{\rho} + \Pi_z)^2 X,\ X))$$

where E is the identity operator in T_zN (all scalar products is euclidean metric or in its restriction on T_zN). From this we immediately conclude that:

(1) the induced metric on T_zN depends only on $\rho(z)$ but not on its values at other points;

(2) the induced metric is degenerate iff $1/\rho = -\lambda_i(z)$ for some i;

(3) the trace of the induced metric in respect to the first fundamental form is minimal when $1/\rho = -\dfrac{\Sigma\lambda_i}{n}$.

Now we'll say the same in other wolds. Consider a map $\hat{\gamma}$: $V \to \Sigma$ such that the sphere $\hat{\gamma}(y)$ is tangent to T_yN for all y. Let $\hat{\gamma}^*(g_\Sigma)$ be the induced metric in V. Then:

(1) $\hat{\gamma}^*(g_\Sigma)|T_zN$ depends only on $\hat{\gamma}(z)$ but not on neighbour behaviour of $\hat{\gamma}$;

(2) this metric is degenerate iff $\hat{\gamma}(z)$ coincides with some of the curvature spheres in z;

(3) the trace of this metric is minimal when $\hat{\gamma}(z)$ coincides with the mean curvature sphere in z and in this case the induced metric is expressed by formula

$$\hat{\gamma}^*(g_\Sigma)(X, X) = ((\Pi_z - H_z E)^2 X, X)$$

where H_z is the mean curvature. This immediately implies the following

Theorem 6. *Let N be a hypersurface in $\mathbb{R}^n$, let $z \in N$, let $g \in G_n$ be a conformal map with no pole in z, let $N' = gN$, $z' = gz$, let $\sigma_i(z)$ be the curvature spheres in z, let $\sigma_m(z)$ be the mean curvature sphere, let $\sigma_i'(z')$, $\sigma_m'(z')$ denote the same for N'. Then*

(1) $g\sigma_i(z) = \sigma_j'(z')$ for some $j = j(i)$.

(2) $g\sigma_m(z) = \sigma_m'(z')$.

(3) $g_*\colon T_zN \to T_{z'}N'$ acts as an isometry when both spaces are supplied with metrics

$$((\Pi_z - H_z E)^2 \cdot, \cdot) \quad \text{and} \quad (\Pi_{z'} - H_{z'} E)^2 \cdot, \cdot).$$

In particular,

$$|\lambda_i - H_z| = \mu(z, g)\,|\lambda_i' - H_{z'}|$$

where $\mu(z, g)$ is the dilatation coefficient of the conformal map g in point z. If all λ_i are different, the same holds for λ_j' and g_* maps the main curvature directions in T_zN to the main curvature directions in $T_z'N'$.

(4) An integral

$$\int_N \prod_{i=1}^{n-1} |\lambda_i(z) - H_z|\,\mathrm{ds}$$

(ds *is the volume element on N) is the conformal invariant of the hypersurface $N \hookrightarrow \mathbb{R}^n$ (this is the result of* [11]).

Proof. The theorem is actually proved by the above construction since the metric g_Σ is conformally invariant. We only will make two remarks. First, one can easily see that the curvature spheres $\sigma_i(z)$ having the center in $z - 1/\lambda_i(z)\,n(z)$ do not depend on the choice of orientation. Secondly, all the statements above remain true if some $\lambda_i = 0$ so σ_i is tangent hyperplane. Of course, the integral in (4) is nothing else but the volume of the induced metric. In the case $n = 3$ it has the form

$$\frac{1}{4}\int_N (\lambda_1 - \lambda_2)^2 \, ds$$

so it vanishes iff N is a part of a sphere. In this special case $n = 3$, assuming that N is closed, we can add $\int_N \lambda_1 \lambda_2 \, ds = 2\pi\chi(N)$ (by the Gauss-Bonnet theorem) to obtain the theorem of J. White [15] $\int_N H^2 \, ds$ is the conformal invariant.

Remark. In [11] Hsiung and Lewko introduce other conformal invariants of N of any codimension. They verify the conformal invariance by direct computations. For codimension one their invariants are in fact algebraic invariants of the operator $(\Pi - HE)^2$ and the conformal invariance follows from Theorem 7.

We will point out briefly two applications. Let N be the unbounded surface in $\mathbb{R}^3$ closed as a subset of $\mathbb{R}^3$. We will say that N has *a negative curvature at infinity* if form some $R > 0$ the part of N lying out of the ball $|x| \leqslant R$ has negative curvature.

Theorem 7. *Let N be a unbounded surface in the halfspace $x_3 \geqslant 0$ closed as a subset of $\mathbb{R}^3$, having negative curvature at infinity. Then one of the two following statements holds:*

(1) *N lies in the proper cone $(x_1^2 + x_x^2) < \mu x_3^2$ for some $\eta > 0$.*

(2) *Out of any ball $|x| \leqslant R$ there are such points $z \in N$ that*

$$\min(|\lambda_1(z)|,\ |\lambda_2(z)|) < \frac{2|(z,\ n_2)|}{|z|^2}.$$

Proof. Let a be the point $(0,\ 0,\ -1)$, let g be the inversion in $\mathbb{R}^3$ with the center a leaving the sphere $|x - a| = 1$ fixed. It is clear that gN lies in the halfplane $x_3 \geqslant -1$. We will show that if (1) doesn't hold then in any neighbourhood of a there are points z' of $N' = gN$ in which N' has a nonnegative curvature. Suppose this is shown. It means that the curvature spheres $\sigma_1'(z')$, $\sigma_2'(z')$ lie in the same halfspace that the tangent plane $T_z' N'$ divides $\mathbb{R}^3$ onto, so their interiors have a nonempty intersection. However, since the curvature of N is negative, the interiors of the spheres $\sigma_1(z)$, $\sigma_2(z)$ haven't got common points. From this and Theorem 7, we conclude that a lies inside one of the spheres $\sigma_1(z)$ and $\sigma_2(z)$ which is equivalent to the inequality in (2). Since $|z| < |z-a|$ we have $\min(|\lambda_1(z)|, |\lambda_2|(z)|) < \frac{2}{|z|}$.

To find a positive curvature (we'll say «convex») point, let's take a plane close to $0x_1$, x_2 lying below a and move it up. The point that will touch N' is obviously convex or it is exactly a. If it is always a than N lies in a cone.

It can be added that without any alteration the similar statement concerning non-convex at infinity hypersurfaces lying in $\{x|x_n \geqslant 0\} \subset \mathbb{R}^n$ can be proved.

Another application deals with $O(3,\ 1)$ modules. Again consider the standard sphere $S^2 \subset \mathbb{R}^3$ and let $r(\nu) = 1 + \varepsilon\rho(\nu)$, $\nu \in S^2$, be a supporting function of a convex surface N, C^∞-close to S^2. We can express our invariant $\int\limits_N (\lambda_1 - \lambda_2)^2 \, ds$ in the form (see [4])

$$I = \int\limits_{S^2} \frac{2\,\mathrm{Tr}\,(rE + \mathrm{Hess}\, r)^2 - [\mathrm{Tr}\,(rE + \mathrm{Hess}\, r)]^2}{\det\,(rE + \mathrm{Hess}\, r)} \, d\nu$$

where $\mathrm{Hess}\, r(\nu)$ is the Hessian operator in $T_\nu S^2$ of the function $r(\nu)$. Replacing $r(\nu)$ by $1 + \varepsilon\rho(\nu)$ and expanding I by the powers of ε we'll see that the first term (containing ε^2) will be of the form

$$Q(\rho) = \int\limits_{S^2} (2\,\mathrm{Tr}\,(\mathrm{Hess}\,\rho)^2 - (\mathrm{Tr}\,\mathrm{Hess}\,\rho)^2)\, d\nu .$$

Let H be the isotropy group of S^2 in Σ, i.e. $g \in H$ leaves S^2 invariant. Let $1 + \varepsilon \cdot g\rho(\nu)$ be the supporting function of gN, then just as before we see that

$$g\rho(\nu) = \mu(g,\ g^{-1}\nu)\,\rho(g^{-1}\nu) + O(\varepsilon)$$

where $\mu(g,\ \alpha)$ is again the dilatation coefficient of g in point α. In other words, let $\mathcal{L}$ be the linear bundle $\sqrt{TS^2 \wedge TS^2}$ of half-forms (see [9]). Each section of $\mathcal{L}$ can written in the form $\rho(\nu)\sqrt{w}$ where w is the canonical coarea 2-form on S^2 and the natural action of H as a subgroup of $\mathrm{Diff}(S^2)$ in $\Gamma(\mathcal{L})$ is exactly such as we've just written. We've obtained that the quadratic form $Q(\rho)$ is invariant under this action of $H \approx O(3,\ 1)$. In particular, it is invatiant under $O(3)$ action. Let $B(\rho_1, \rho_2)$ be the associate symmetric bilinear form, then standard arguments show that it admits an expression of the type $B(\rho_1,\ \rho_2) = \int\limits_{S^2} D_4(\rho_1)\cdot\rho_2\, d\nu$ where D_4 is a self-adjoint differential operator of order 4. Since $B(\rho_1,\ \rho_2)$ is $O(3)$-invariant and such is $d\nu$, then D_4 is also $O(3)$-invariant, hence it is a polynomial of Laplace-Beltrami operator Δ: $D_4 = \alpha[(\Delta + \beta E)^2 + \gamma E]$. Further, $Q(\rho_0) = 0$ when $\rho_0 = \text{const} + (\nu,\ p)$ for some $p \in \mathbb{R}^3$ because this is the support function of some sphere (recall that Q vanishes on spheres). Since Q is semidefinite, ρ_0 lies in the kernel of D_4. But, ρ_0's form the direct sum of the first and second eigenspaces of Δ: $\Delta(\text{const}) = 0$ and $\Delta(\nu,\ p) = -2(\nu,\ p)$ thus $D_4 = \alpha\Delta(\Delta + 2E)$. We've obtained the following

Theorem 8. *The natural action of* $O(3,\ 1)$ *in the space of sections* $\Gamma(\sqrt{TS^2 \wedge TS^2})$ *of the square root of the determinant bundle of the tangent bundle leaves invariant the nonnegative quadratic form*

$$Q(\rho\sqrt{w}) = \int_{S^2} ((\Delta\rho)^2 - 2\,|\text{grad}\,\rho|^2)\,d\nu.$$

The 4-dimensional kernel of this form is isomorphic to the space of the natural representation of $O(3,\ 1)$.

Remark. The existence of such a form can be shown by usual method of spherical harmonic decomposition [**17**].

We can say that the natural representation is «glued» to some orthogonal (therefore unitar) representation which arises after factorization and completion. This seems to be a sort of quantization of the fact that the conformal action of $O(3,\ 1)$ on S^2 is glued to the isometrical action on the hyperbolic space H^3.

We will conclude with generalization of Theorem 9 to the upper dimensions. The proof of the next result is similar to the speculations abowe if we consider the trace of metric $\hat{\gamma}^*(g_\Sigma)$ in Theorem 7 in relation to the induced metric from the euclidian space.

Theorem 9. *Consider the sphere* S^n *with the standard immersion in* $\mathbb{R}^{n+1}$. *Let* w, Ω *be the coarea and area* n*-forms on* S^n, $(\wedge TS^n)^{k/n}$ *and* $(\wedge T^*S^n)^{k/n}$ – k*-th powers of the* n*-th root of the determinant bundles of* TS^n *and* T^*S. *Then*:

(1) *The quadratic map* $\rho\sqrt[n]{w} \mapsto (n\,\text{Tr}\,(\text{Hess}\,\rho)^2 - (\Delta\rho)^2)(\sqrt[n]{\Omega})^2$ *from* $\Gamma(\wedge TS^n)^{1/n}$ *to* $\Gamma(\wedge T^*S^n)^{2/n}$ *is* $O(n+1,\ 1)$*-invariant.*

(2) *There exists the nonnegative* $O(n+1,\ 1)$*-invariant norm in* $\Gamma(\wedge TS^n)^{1/n}$ *of the form*

$$\|\rho\sqrt[n]{w}\| = \Big(\int_{S^n} (n\,\text{Tr}\,(\text{Hess}\,\rho)^2 - (\Delta\rho)^2)^{n/2}\,\Omega\Big)^{1/n}.$$

(3) *The $(n+2)$-dimensional $O(n+1,\ 1)$-invariant space of the null-norm vectors is equivariantly isomorphic to the space $\mathbb{R}^{n+2}$ of the natural representation of $O(n+1,\ 1)$.*

Appendix. A conformally invariant $(n+1)$-form

Consider a double bundle

$$\begin{array}{ccc} & \mathfrak{A} & \\ \swarrow & & \searrow \\ R^n & & \Sigma \end{array}$$

consisting of such pairs $(p,\ \sigma)$ that $p \in \sigma$. We use coordinates $(x,\ R)$ on Σ. Let α_{x-p} be the 1-form $\Sigma_i(x_i - p_i)\,\mathrm{d}p_i$. Then we state that $(n+1)$-form

$$\Lambda = \alpha_{x-p} \wedge \frac{\mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n}{|x-p|^{n+2}}$$

is up to sign G_n-invariant modulo ideal, generated by $\mathrm{d}p_i \wedge \mathrm{d}p_j$. It means, that given $g \in G_n$, a point $(p, x, |x-p|)$ in $\mathfrak{A}$ and vector $v \in T_p\mathbb{R}^n$ one has

$$(*) \qquad g^*((x-P,\ V)\,\frac{\mathrm{d}X_1 \wedge \ldots \wedge \mathrm{d}X_n}{|X-p|^{n+2}}) = \pm\,(x-p,\ v)\,\frac{\mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n}{|x-p|^{n+2}}.$$

(The equality of differential forms of variable x only). Here

$$(P,\ X,\ |X-P|) = g(p,\ x,\ |x-p|) \qquad V = g_*(p)\,v.$$

Proof. The invariance under the group of euclidean motions is trivial, so we need only show the invariance under inversions. Consider the inversion $g\colon y \to \dfrac{y}{|y|^2}$. Then one knows $X = \dfrac{x}{|x|^2 - r^2}$, $R = |X-P| = \dfrac{r}{|x|^2 - r^2}$, $P = \dfrac{p}{|p|^2}$, where $r = |x-p|$. For $q \in \mathbb{R}^n$ and $Q = \dfrac{1}{|q|^2} = g(q)$ one has

$$|X-Q| = \left|\frac{x}{|x|^2 - r^2} - \frac{q}{|q|^2}\right| = \sqrt{\frac{|q|^4\,|x|^2 + (|x|^2 - r^2)^2\,|q|^2 - 2|q|^2(|x|^2 - r^2)(x,\ q)}{|q|^2\,\big||x|^2 - r^2\big|}}$$

Let $q = p + tv$. Then $\dfrac{\mathrm{d}}{\mathrm{d}t}|x-q|_{t=0} = -\dfrac{(x-p,\ v)}{|x-p|}$ and $\dfrac{\mathrm{d}}{\mathrm{d}t}\,|X-Q|_{t=0} = -\dfrac{(X-P,\ V)}{|X-P|}$. Substituting $q = p + tv$ to the last formula and omitting terms, containing t^2, we'll see using $r = |x-p|$, that

$$|X-Q| = \frac{|p|^2 r + \frac{t}{r}(v,\ (|x|^2+r^2)p - (|x|^2-r^2)x)}{(|p|^2+2t(p,\ v))\,\big||x|^2-r^2\big|} + O(t^2).$$

It gives

$$\frac{(V,\ X-P)}{|X-P|} = \operatorname{sgn}(|x|^2-r^2)\frac{1}{|p|^2 r}(v, x-p).$$

But, $|X-P| = R = \dfrac{r}{\big||x|^2-r^2\big|}$, so $(V,\ X-P) = \dfrac{(v,\ x-p)}{(|x|^2-r^2)|p|^2}$. Next, for p fixed we have

$$X = \frac{x}{|x|^2-|x-p|^2} = \frac{x}{2(x,\ p)-|p|^2}.$$

Let $w \in T_x\mathbb{R}^n$, then

$$\frac{\mathrm{d}X}{\mathrm{d}x}(x) = \frac{1}{2(x,\ p)-|p|^2}\Big(w - \Big(\frac{2p}{2(x,\ p)-|p|^2},\ w\Big)x\Big).$$

For any operator A of rank one in $\mathbb{R}^n$ we know $\det(E+A) = 1 + \operatorname{tr} A$, so

$$\det\frac{\mathrm{d}X}{\mathrm{d}x} = \frac{1}{(|x|^2-r^2)^n}\Big(1 - \frac{2(p,\ x)}{2(x,\ p)-|p|^2}\Big) = -|p|^2\Big(\frac{1}{|x|^2-r^2}\Big)^{n+1}.$$

In other words,

$$g^*(\mathrm{d}X_1\wedge\ldots\wedge\mathrm{d}X_n) = -|p|^2\frac{1}{(|x|^2-r^2)^{n+1}}\mathrm{d}x_1\wedge\ldots\wedge\mathrm{d}x_n.$$

Next, as $|X-P| = R = \dfrac{r}{\big||x|^2-r^3\big|}$, we have $|X-P|^{n+2} = \dfrac{r^{n+2}}{\big||x|^2-r^2\big|^{n+2}}$, so

$$g^*\Big((X-P,\ V)\frac{\mathrm{d}X_1\wedge\ldots\wedge\mathrm{d}X_n}{|X-P|^{n+2}}\Big) = -(\operatorname{sgn}(|x|^2-r^2))^{n+2}\Big((x-p,\ v)\frac{\mathrm{d}x_1\wedge\ldots\wedge\mathrm{d}x_n}{|x-p|^{n+2}}\Big).$$

If we pass from forms to densities, we obtain

$$|(X-P,\ V)|\ \frac{|\mathrm{d}X_1\wedge\ldots\wedge\mathrm{d}X_n|}{|X-P|^{n+2}}=|(x-p,\ v)|\ \frac{\mathrm{d}x_1\wedge\ldots\wedge\mathrm{d}x_n}{|x-p|^{n+2}}$$

which completes the proof of Proposition 3.

Remark. If n is even, we obtain invariance of forms (not densities) under the action of $G_n^+=SO(n+1,\ 1)$.

A different approach to conformal invariants is presented in preprints [5] and the paper [**16**].

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Summary

We show that: (1) *the complement of the limit set of a Kleinian group acting in* S^n *carries an invariant Finsler metric;* (2) *every conformal map of a compact domain in* S^n *in itself is a contraction of some Finsler metric given by an explicit formula;* (3) *for a length-parametrized smooth curve* $\gamma(t)$ *in* $\mathbb{R}^2$ *the integral* $\int_\gamma \sqrt{\frac{dK}{dt}}$ *is a conformal invariant* (*K is the curvature*), *which is an «integrated» version of a conformal invariance of the Virasoro cocycle and Schwartzian;* (4) *every non-convex near infinity hypersurface in half-space of* $\mathbb{R}^n$ *should yield strong curvature decaying conditions;* (5) *natural representation of* $O(n+1,\ 1)$ *in* $\mathbb{R}^{n+1,\,1}$ *is «glued» to some isometrical Banach representation* (*unitar when* $n=2$), *which is a sort of quantization of «gluing» the conformal action on* S^n *to isometrical action in the hyperbolic space* H^{n+1}.

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