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The geometry of good squares of vector bundles (**)

Introduction

In a previous work [8]$_2$ we were interested in defining the notion of a lift of a connection on a vector bundle π: $C \to D$ to a connection on another vector bundle γ: $A \to B$, when these vector bundles are related by a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\gamma"] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] \\ C \arrow[r, "\pi"'] & D \end{tikzcd}$$

which verifies certain properties (that allow us to call it a *good square*). This construction is useful to obtain a general theory of lifts of connections that include as examples several known lifts of connections.

The purpose of the present paper is to offer a complete version of our theory, developing in 1 the main properties of the good squares (and those of vector bundles that we need) and finding sufficient examples of them in order to built the theory (Proposition 1).

We use six equivalent definitions of an infinitesimal connection on a vector bundle. In 2 we define the notion of a lifting of an infinitesimal connection, obtaining some results about covariant derivatives, transitivity of lifts and curvature. Basically, the elements defined by the lifted connection project over the corre-

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sponding ones of the given connection. Finally, we give from $[8]_2$ the local conditions, which are the key to compute the theory in particular cases.

In 3 we offer the first applications, showing that known lifts (as complete and horizontal to TM [19], to T^*M [19], to T_2M [7], [12], [19], and other constructions in [2], $[4]_{1,2}$ [15], [18], each of them defined in a particular way) can be viewed as lifts respect to some good squares in our way.

The second part of the work is devoted to study a generalization of the concepts of an infinitesimal connection and of lift of a connection respect to a good square. Generalized connections were intoduce by Spesivykh [17]: a generalized connection on a vector bundle π: $E \to M$ is a (1, 1)-tensor field on E. We shall use them because (1) an infinitesimal connection on a vector bundle and a linear pseudo-connection on a manifold [6], $[8]_1$, [9] are generalized connections and (2) because this construction can be extended to arbitrary fibre bundles. In fact, in $[8]_4$ we have used generalized connections to solve other problems.

1 - Vector bundles and good squares

Let π: $E \to M$ be a vector bundle of rank r over a smooth real n-dimensional manifold M. If f: $X \to M$ is a smooth mapping, we denote by $\pi^{-1}(X) \to E$ the pull-back of E.

We often shall use the exact sequence of vector bundles over E,

$$0 \longrightarrow VE \xrightarrow{i} TE \xrightarrow{\pi'} \pi^{-1}(TM) \longrightarrow 0 \tag{1}$$

where $VE = \ker \pi_*$ and π' is the map induced by the universal property of the pull-back. VE is called the *vertical vector bundle* and there exists a canonical map ρ: $VE \to E$ given by

$$VE \cong \pi^{-1}(E) \subseteq E \times E \to E$$

where the last map is the second projection.

Now, we shall define one of the main concepts of the work. Consider a diagram as

$$\begin{array}{ccc} A & \xrightarrow{\gamma} & B \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ C & \xrightarrow[\pi]{} & D \end{array} \tag{2}$$

Def. 1. A *good square of vector bundles* is a diagram as above verifying

(1) α and β are fibre bundles, but not necessarily vector bundles;

(2) γ and π are vector bundles;

(3) the square is commutative, i.e., $\pi \circ \alpha = \beta \circ \gamma$;

(4) for each $d \in D$ there exists an open neighbourhood U of d such that U is a local chart for π and β, $\pi^{-1}(U)$ is a local chart for α and $\beta^{-1}(U)$ is a local chart for γ;

(5) the following diagram is commutative

$$\begin{array}{ccc} VA & \xrightarrow{\bar{\rho}} & A \\ \downarrow{\scriptstyle \alpha_*} & & \downarrow{\scriptstyle \alpha} \\ VC & \xrightarrow[\rho]{} & C \end{array}$$

where ρ (resp. $\bar{\rho}$) is the canonical map.

Remark. $\alpha_*(VA) \subseteq VC$, because $VA = \ker \gamma_*$ and $VC = \ker \pi_*$.

Properties (4) and (5) can be changed by

(4)′ The local expression of the good square is

$$\begin{array}{ccc} A & \xrightarrow{\gamma} & B \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ C & \xrightarrow[\pi]{} & D \end{array} \qquad \begin{array}{ccc} U^n \times R^r \times G^s \times R^t & \longrightarrow & U^n \times G^s \\ \downarrow & & \downarrow \\ U^n \times R^r & \longrightarrow & U^n \end{array} \qquad \begin{array}{ccc} (x^i, a^\alpha, g^\lambda, b^\sigma) & \longrightarrow & (x^i, g^\lambda) \\ \downarrow & & \downarrow \\ (x^i, a^\alpha) & \longrightarrow & (x^i) \end{array}$$

where G is a manifold and superindices denote the dimension of the manifolds.

The proof of the equivalence of conditions (4) and (5) and the condition (4)′ is not difficult. The geometrical meaning of the last condition is the following: fibres of γ and α are transversal. And by (5) we know that fibres of α keep the distance among them.

Proposition 1. (i) *Let π: $E \to M$ be a fibre bundle. Then, the following is a good square*

$$\begin{tikzcd} TE \arrow[r, "\pi_E"] \arrow[d, "\pi_*"'] & E \arrow[d, "\pi"] \\ TM \arrow[r, "\pi_M"'] & M \end{tikzcd}$$

(ii) *Let π: $E \to M$ be a vector bundle. Then the following is a good square*

$$\begin{tikzcd} TE \arrow[r, "\pi_*"] \arrow[d, "\pi_E"'] & TM \arrow[d, "\pi_M"] \\ E \arrow[r, "\pi"'] & M \end{tikzcd}$$

(iii) *Let π: $C \to D$ be a vector bundle and β: $B \to D$ be a fibre bundle. Then, the following is a good square*

$$\begin{tikzcd} \pi^{-1}(B) \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & B \arrow[d, "\beta"] \\ C \arrow[r, "\pi"'] & D \end{tikzcd}$$

(iv) *The tangent square of a good square is also a good square.*

Examples. *TTM* has two vector bundle structures over *TM*. Then, we can consider four diagrams as the following

$$\begin{tikzcd} TTM \arrow[r, "\gamma"] \arrow[d, "\alpha"'] & TM \arrow[d, "\pi_M"] \\ TM \arrow[r, "\pi_M"'] & M \end{tikzcd}$$

where α and γ are π_{TM} or $(\pi_M)_*$. If $\alpha = \gamma$ the diagram is not a good square (use property (5) of a good square); in other case it is a good square.

Def. 2. A (global) section $\overline{X}$: $B \to A$ is said *projectable* respect to a good square (2) if there exists a (global) section X: $D \to C$ such that $\alpha \circ \overline{X} = X \circ \beta$.

Examples. Projectable vector fields on TM [19] are projectable sections respect to the good square of the above example, when $\alpha = (\pi_M)_*$ and $\gamma = \pi_{TM}$. The same result for projectable vector fields on the cotangent bundle.

If π: $E \to M$ is a vector bundle and X is a section, then X_*: $TM \to TE$ is a projectable section respect to (ii) of Proposition 1.

2 - Connections and good squares

Let π: $E \to M$ be a vector bundle with local expression

$$\pi\colon E \to M \qquad \pi\colon U^n \times R^r \to U^n \qquad \pi\colon (x^i,\ a^\alpha) \to (x^i).$$

Remember the exact sequence (1) defined by the vector bundle

$$0 \longrightarrow VE \xrightarrow{i} TE \xrightarrow{\pi'} \pi^{-1}(TM) \longrightarrow 0. \tag{1}$$

There are several equivalent definitions of a connection (see, for example [1], [3], [5], [8]$_4$, [13], [14], [18].

Def. 3. An *infinitesimal connection* on the above vector bundle is given by one of the following elements:

(1) A left smooth splitting V of the exact sequence (1) defined by the vector bundle. In local coordinates

$$V\colon TE \to VE \qquad V(x,\ a_1,\ c,\ a_2) = (x,\ a_1,\ 0,\ a_2 + \omega(x,\ a_1)c)$$

V is the vertical projection defined by the connection.

(2) A right smooth splitting H of the exact sequence (1). In local coordinates

$$H\colon \pi^{-1}(TM) \to TE \qquad H(x,\ a_1,\ c) = (x,\ a_1, c,\ -\omega(x,\ a_1)c).$$

(3) A *connection map* K: $TE \to E$, such that: (a) $(K,\ \pi_M)$ is a bundle morphism over $(\pi_*,\ \pi)$; (b) $(K,\ \pi)$ is a bundle morphism over $(\pi_E,\ \pi)$; (c) $K|_{VE} = \rho$.

In local coordinates

$$K\colon TE \to E \qquad K(x, a_1, c, a_2) = (x, a_2 + \omega(x, a_1)c).$$

(4) A horizontal vector bundle HE such that $TE = VE \oplus HE$.

(5) A projectable 1-form h which projects over the identity. This is the horizontal projection defined by the connection. In local coordinates

$$h\colon TE \to HE \qquad h(x, a_1, c, a_2) = (x, a_1, c, -\omega(x, a_1)c).$$

(6) A map $F\colon TE \to TE$ such that F^2 is the identity and each $e \in E$, the vertical subspace V_eE is the eigenspace corresponding to the eigenvalue -1. In local coordinates

$$F\colon TE \to TE \qquad F(x, a_1, c, a_2) = (x, a_1, c, -a_2 - \omega(x, a_1)c).$$

Then, $F = h - V$.

In the above local expressions we have used the *local component*

$$\omega\colon U^n \times R^r \to L(R^n, R^r)$$

which verifies

$$(\omega(x, a_1)c)^\alpha = \omega_j^\alpha(x, a_1)c^j.$$

Observe that F admits the following local expression as a (1, 1)-tensor field on E

$$F = \frac{\partial}{\partial x^i} \otimes dx^i + \frac{\partial}{\partial a_1^\alpha} \cdot (-2\omega_i^\alpha(x, a_1)) \otimes dx^i - \frac{\partial}{\partial a_1^\alpha} \otimes da_1^\alpha$$

which allows us to write

$$F = \begin{pmatrix} \delta_j^i & 0 \\ -2\omega_j^\alpha(x, a_1) & \delta_\beta^\alpha \end{pmatrix}$$

and so we obtain a generalization of the expression of [11].

An infinitesimal connection on the tangent bundle $\pi_M\colon TM \to M$ is said a *connection* on M. Non-homogeneous connections are connections on the slit tangent bundle, and can be defined using the canonical almost tangent structure on TM. (See [11]).

A connection is said to be *linear* if HE is invariant by translations. In local coordinates

$$\omega(x,\ a_1)\, c = \Gamma^{\beta}_{i\alpha}(x)\, a^{\alpha}_{i}\, c^{i}\,.$$

The *covariant derivative* [18] of an infinitesimal connection Γ is

$$\nabla_X \Sigma = K \circ \Sigma_* \circ X\,.$$

The *curvature* of a connection is

$$R = -\frac{1}{2}[h,\ h] = -\frac{1}{2} N_h$$

where N_h is the Nijenhuis tensor field of the horizontal projection h. It is easy to find the relationship between this curvature and the classical curvature tensor field of a linear connection [16], [18]. Moreover, this definition of curvature is a particular use of the Frolicher-Nijenhuis bracket [10] as one can see in [3], [5], [13], [14].

When we have connection on M there are many different definitions of torsion [5], [11], [16], [18], but we shall not use them.

Assume that (2) is a good square and that $\bar{\Gamma}$ (resp. Γ) is an infinitesimal connection on γ: $A \to B$ (resp. on π: $C \to D$). We are looking for the definition of $\bar{\Gamma}$ as a lift of Γ with respect to (2). First of all, we show the following diagram, considered before giving both connections $\bar{\Gamma}$ and Γ

$$\begin{tikzcd}
0 \arrow[r] & VA \arrow[rr, "\bar{i}"] \arrow[dr, "\bar{p}"] \arrow[ddd, "\alpha_*"'] & & TA \arrow[r, "\gamma'"] \arrow[ddd, "\alpha_*"] & \gamma^{-1}(TB) \arrow[r] \arrow[ddd, dashed, "\beta'"] & 0 \\
 & & A \arrow[d, "\alpha"'] & & & \\
 & & C & & & \\
0 \arrow[r] & VC \arrow[rr, "i"'] \arrow[ur, "p"] & & TC \arrow[r, "\pi'"'] & \pi^{-1}(TD) \arrow[r] & 0
\end{tikzcd}$$

Observe that it is commutative and that β' is the unique morphism defined by the universal property of the pull-back, verifying $\pi' \circ \alpha_* = \beta' \circ \gamma'$.

The first idea (that we have used in $[8]_2$) is the following: $\bar{\Gamma}$ is a lift of Γ if the morphisms obtained from $\overline{\Gamma}$ keep the commutativity of the above diagram. Now, using six definitions of an infinitesimal connection, we say:

Def. 4. $\overline{\Gamma}$ is a *lift* of Γ with respect to (2) if it is verified one of the following six conditions:

(1) $\alpha_* \circ \overline{V} = V \circ \alpha_*$ (2) $\alpha_* \circ \overline{H} = H \circ \beta'$ (3) $\alpha_* \circ \overline{K} = K \circ \alpha_*$ (4) $\alpha_*(VA) = VC$ $\alpha_*(HA) = HC$, (5) $\alpha_* \circ \overline{h} = h \circ \alpha_*$ (6) $\alpha_* \circ \overline{F} = F \circ \alpha_*$ where the bar $^-$ denotes the elements defined by the connection $\overline{\Gamma}$. This notation will be ever used.

As it is hoped, we have

Proposition 2. *The above six conditions are equivalent, i.e., if it is verified one of them, all of them are true.*

Proof. Some of the implications are trivial

$$1 \Rightarrow 5 \Rightarrow 6 \Rightarrow 1 \Rightarrow 4 \Rightarrow 1 \Rightarrow 3.$$

On the other hand, $3 \Rightarrow 2$ and $2 \Rightarrow 1$ can be obtain by chasing on the diagram.

We postpone to the end of this section the local expressions (that are the key, when we have concrete cases) because we don't need in order to proof the next results.

Proposition 3. (Covariant derivatives). *Let* (2) *be a good square and* $\overline{\Gamma}$ *(resp.* Γ*) an infinitesimal connection on* γ: $A \to B$ *(resp. on* π: $C \to D$*). Suppose that* $\overline{\Gamma}$ *is a lift of* Γ *with respect to* (2),

Let $\overline{\Sigma}$ *(resp.* Σ*) be a section of* γ *(resp.* π*) such that* $\overline{\Sigma}$ *projects over* Σ.

Let $\overline{X} \in \mathfrak{C}_0^1(A)$ *and* $X \in \mathfrak{C}_0^1(C)$ *such that* $\overline{X}$ *projects over* X.

Then, $\overline{\nabla}_{\overline{X}}\overline{\Sigma}$ *projects over* $\nabla_X \Sigma$, $\overline{\nabla}$ *(resp.* ∇*) being the covariant derivative defined by* $\overline{\Gamma}$ *(resp.* Γ*).*

Proposition 4. (Transitivity of lifts). *Let*

$$\begin{tikzcd}
& A_2 \arrow[rrr, "\pi_2"] \arrow[dd, "\alpha_1"] & & & B_2 \arrow[dd, "\beta_1"] \\
A_3 \arrow[ur, "\alpha_2"] \arrow[rrr, "\pi_3"] \arrow[dr, "\alpha_3"'] & & & B_3 \arrow[ur, "\beta_2"] \arrow[dr, "\beta_3"] & \\
& A_1 \arrow[rrr, "\pi_1"] & & & B_1
\end{tikzcd}$$

be a commutative diagram such that the three squares are good squares. Let Γ_i be an infinitesimal connection on π_i: $A_i \to B_i$ $(i = 1, 2, 3)$. Then, if Γ_3 is a lift of Γ_2 and Γ_2 is a lift of Γ_1, Γ_3 is a lift of Γ_1.

Caution. It is not true that if Γ_3 and Γ_2 are lifts of Γ_1 then Γ_3 is a lift of Γ_2. In fact, we have a counter example: consider a linear connection ∇ on M and its horizontal lift ∇^H to TM and its prolongation ∇^* to T_2M [19]. Then, these are lifts in our sense and ∇^* is not a lift of ∇^H.

Proposition 5. (Curvature). *Let* (2) *be a good square and $\overline{\Gamma}$ a lift of Γ respect to* (2). *Let $\overline{X}, \overline{Y} \in \mathfrak{X}^1_0(A)$ and $X, Y \in \mathfrak{X}^1_0(C)$ such that $\overline{X}$ (resp. $\overline{Y}$) projects over X (resp. Y). Then $\overline{R}(\overline{X}, \overline{Y})$ projects over $R(X, Y)$.*

Proof. We have to obtain $\alpha_* \circ \overline{R}(\overline{X}, \overline{Y}) = R(X, Y) \circ \alpha$, knowing that $\alpha_* \circ \overline{X} = X \circ \alpha$ and $\alpha_* \circ \overline{Y} = Y \circ \alpha$. The result is easy if we develop the expression of the curvature and prove the following claims

(1) $\alpha_* \circ [\overline{X}, \overline{Y}] = [X, Y] \circ \alpha$ (2) $\alpha_* \circ \overline{h}(\overline{X}) = h(X) \circ \alpha$ (3) $\alpha_* \circ \overline{hh}(\overline{X}) = hh(X) \circ \alpha$.

A similar construction can be made for the curvature tensor field of linear connections, and for the several notions of torsion of connections on manifolds.

Finally, show the local expressions

Proposition 6. $[8]_2$ *Let ω and Ω be the local components of connections Γ on π: $\mathrm{C} \to \mathrm{D}$ and $\overline{\Gamma}$ on γ: $\mathrm{A} \to \mathrm{B}$. Let*

$$\omega\colon U^n \times R^r \to L(R^n, R^r)$$

$\Omega: U^n \times R^r \times G^s \times R^t \to L(R^n \times R^s, R^r \times R^t) \cong L(R^n \times R^s, R^r) \times L(R^n \times R^s, R^t)$

be their local expressions, what allows writing $\Omega = (\Omega_1, \Omega_2)$. *Then* $\overline{\Gamma}$ *is a lift of* Γ *with respect to* (2) *if and only if*

$$\Omega_1(x, a, g, b)(c, d) = \omega(x, a)c.$$

Corollary 6.1. *Let a good square as* (i) *of Proposition* 1 *be, and two linear connections* $\overline{\Gamma}$ *one* E *and* Γ *on* M *with local symbols*

$$\overline{\Gamma}^i_{jk} \quad \overline{\Gamma}^i_{j\tau} \quad \overline{\Gamma}^i_{\sigma k} \quad \overline{\Gamma}^i_{\sigma\tau} \quad \overline{\Gamma}^\nu_{jk} \quad \overline{\Gamma}^\nu_{j\tau} \quad \overline{\Gamma}^\nu_{\sigma k} \quad \overline{\Gamma}^\nu_{\sigma\tau} \quad \Gamma^i_{jk}.$$

Then $\overline{\Gamma}$ *is a lift of* Γ *if and only if*

$$\overline{\Gamma}^i_{jk} = \Gamma^i_{jk} \qquad \overline{\Gamma}^i_{j\tau} = 0 \qquad \overline{\Gamma}^i_{\sigma k} = 0 \qquad \overline{\Gamma}^i_{\sigma\tau} = 0.$$

Corollary 6.2. *Let*

$$\begin{array}{ccc} E' & \xrightarrow{\gamma} & M \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle id} \\ E & \xrightarrow[\pi]{} & M \end{array} \qquad \begin{array}{ccc} U^n \times R^r \times R^s & \longrightarrow & U^n \\ \downarrow & & \downarrow \\ U^n \times R^r & \longrightarrow & U^n \end{array} \qquad \begin{array}{ccc} (x^i, a^\alpha, b^\lambda) & \longrightarrow & (x^i) \\ \downarrow & & \downarrow \\ (x^i, a^\alpha) & \longrightarrow & (x^i) \end{array}$$

be a good square, and let $\overline{\Gamma}$ *and* Γ *be linear connections on* γ *and* π. *Then,* $\overline{\Gamma}$ *is a lift of* Γ *respect to this good square if and only if their local symbols*

$$\overline{\Gamma}^\alpha_{j\beta} \quad \overline{\Gamma}^\alpha_{j\mu} \quad \overline{\Gamma}^\lambda_{j\beta} \quad \overline{\Gamma}^\lambda_{j\mu} \quad \Gamma^\alpha_{j\beta}$$

verify

$$\overline{\Gamma}^\alpha_{j\beta} = \Gamma^\alpha_{j\beta} \qquad \overline{\Gamma}^\alpha_{j\mu} = 0.$$

This kind of results are very useful to study particular cases, as we can show in the next section.

3 - First application

Using Corollaries 6.1 and 6.2 we obtain that the following known lifts are lifts respect to some good squares. Each of them is defined using particular proprieties of the bundles considered, and for each of them there exists a particular version of Propositions 3 and 5.

(a) *Lifts to the tangent bundle TM.*

If ∇ is a linear connection on M, its complete and horizontal lifts to TM, ∇^C and ∇^H [19] are lifts of ∇ with respect to the good square (3)

(3)
$$\begin{tikzcd} TTM \arrow[r, "\pi_{TM}"] \arrow[d, "(\pi_M)_*"'] & TM \arrow[d, "\pi_M"] \\ TM \arrow[r, "\pi_M"'] & M \end{tikzcd}$$

(b) *Lifts to the second order tangent bundle T_2M.*

If ∇ is a linear connection on M, its lift ∇^* [19] is a lift of ∇ with respect to the good square (4)

(4)
$$\begin{tikzcd} TT_2M \arrow[r, "\pi_{T_2M}"] \arrow[d, "(\pi_0^2)_*"'] & T_2M \arrow[d, "\pi_0^2"] \\ TM \arrow[r, "\pi_M"'] & M \end{tikzcd}$$

Moreover, the horizontal lift [7], [12] and the complete lift [7] are also lifts respect to the above good square. We shall denote them as ∇^{H2} and ∇^{C2} to avoid confusion with the other lifts to TM.

(c) *Lifts from TM to T_2M.*

Taking the above lifts and the good square

$$\begin{tikzcd} TT_2M \arrow[r, "\pi_{T_2M}"] \arrow[d, "(\pi_1^2)_*"'] & T_2M \arrow[d, "\pi_1^2"] \\ TTM \arrow[r, "\pi_{TM}"'] & TM \end{tikzcd}$$

we obtain: ∇^{C2} and ∇^* are lifts of ∇^C, ∇^{H2} is a lift of ∇^H. If the curvature tensor field of ∇ doesn't vanish, there are not more lifts. (See the Caution after Proposition 4).

(d) *Lifts to the cotangent bundle T^*M.*

The complete and horizontal lifts of a linear symmetric connection on M to T^*M [19] are lifts respect to

$$\begin{tikzcd} TT^*M \arrow[r, "\pi_{T^*M}"] \arrow[d, "\overline{\pi}_*"'] & T^*M \arrow[d, "\overline{\pi}"] \\ TM \arrow[r, "\pi_M"'] & M \end{tikzcd}$$

(e) *Tangent connection.*

Let Γ be an infinitesimal connection on a vector bundle π: $E \to M$. Then [18] is possible to define an induced connection on the vector bundle π_*: $TE \to TM$ given by its connection map $\overline{K} = K_* \circ S_E$, S_E being the automorphism of TTE which permutes the fibred structures over TE, and K the connection map of Γ. Then, this induced connection is a lift of the given one with respect to

$$\begin{tikzcd} TE \arrow[r, "\pi_*"] \arrow[d, "\pi_E"'] & TM \arrow[d, "\pi_M"] \\ E \arrow[r, "\pi"'] & M \end{tikzcd}$$

(f) *Second order connection.*

Given an infinitesimal connection on π_M: $TM \to M$ there exists [2] an induced infinitesimal connection on π_0^2: $T_2M \to M$. It is a lift of the first one with respect to

$$\begin{tikzcd} T_2M \arrow[r, "\pi_0^2"] \arrow[d, "\pi_1^2"'] & M \arrow[d, "id"] \\ TM \arrow[r, "\pi_M"'] & M \end{tikzcd}$$

(g) *Lifts to the frame bundle FM.*

If ∇ is a linear connection on M its complete [4]$_1$ [15] and horizontal [4]$_2$ lifts to the frame bundle are lifts respect to the good square

$$\begin{tikzcd} TFM \arrow[r, "\pi_{FM}"] \arrow[d, "\Pi_*"'] & FM \arrow[d, "\Pi"] \\ TM \arrow[r, "\pi_M"'] & M \end{tikzcd}$$

The proofs of the above examples are obtained using local coordinates. The construction is a tedious and straighforward calculation.

4 - Pseudo-connections and generalized connections

As we have said in the Introduction, there exist another important notion which generalizes that of a linear connection on a manifold. This the concept of a linear pseudo-connection, which was introduced by Di Comite [6].

A *linear pseudo-connection* on M with fundamental tensor field $G \in \mathfrak{T}_1^1(M)$ is a $\mathfrak{T}_0^0(M)$-linear map ∇: $\mathfrak{T}_0^1(M) \to \mathcal{D}(M)$, $\mathcal{D}(M)$ being the Lie algebra of directional derivatives on M, such that ∇_X is a derivative whose direction vector field is $G(X)$.

Now, we are looking for a global notion of a connection in such a way that infinitesimal connections on vector bundles and linear pseudo-connections on manifolds can be considered included in that concept. This problem was solved by Spesivykh in a series of papers [17], with the following simple idea.

Def. 5. [17]$_3$ A *generalized connection* on a vector bundle π: $E \to M$ is a tensor field $F \in \mathfrak{T}_1^1(E)$.

Then, obviously, infinitesimal connections are generalized connections, using (6) of Def. 3, and a linear pseudo-connection Γ on M can be understood as a generalized connection on π_M: $TM \to M$, in the sense that the covariant derivative of Γ and that of the corresponding generalized connection will be the same. (See $[8]_{3,4}$ for a complete exposition of the local expressions of the covariant derivative).

A generalized connection has the following local expression

$$F = \begin{pmatrix} C_j^i & M_\beta^i \\ \Psi_j^\alpha & B_\beta^\alpha \end{pmatrix}$$

where i, $j \in \{1, \ldots, n\}$ and α, $\beta \in \{1, \ldots, r\}$, n being the dimension of M and r the rank of π.

In this case, the definition of a lift is the unique possible.

Def. 6. Let (2) be a good square and $\overline{F}$ (resp. F) a generalized connection on γ: $A \to B$ (resp. on π: $C \to D$). Then, $\overline{F}$ is a *lift* of F with respect to (2) if $\alpha_* \circ \overline{F} = F \circ \alpha_*$.

Observe that if $\overline{F}$ and F are infinitesimal connections, this definition coincides with Def. 4 (6).

Using local coordinates one can proof Proposition 3 and Proposition 4 for generalized connections. The local condition of a lift is the following:

Remember that $F \in \mathfrak{C}_1^1(C)$ and $\overline{F} \in \mathfrak{C}_1^1(A)$. Suppose that their local expressions are

$$F = \begin{pmatrix} C_j^i & M_\beta^i \\ \Psi_j^\alpha & B_\beta^\alpha \end{pmatrix} \qquad \overline{F} = \begin{pmatrix} \overline{C}_j^i & \overline{C}_\beta^i & \overline{M}_\mu^i & \overline{M}_\tau^i \\ \overline{C}_j^\alpha & \overline{C}_\beta^\alpha & \overline{M}_\mu^\alpha & \overline{M}_\tau^\alpha \\ \overline{\Psi}_j^\lambda & \overline{\Psi}_\beta^\lambda & \overline{B}_\mu^\lambda & \overline{B}_\tau^\lambda \\ \overline{\Psi}_j^\sigma & \overline{\Psi}_\beta^\sigma & \overline{B}_\mu^\sigma & \overline{B}_\tau^\sigma \end{pmatrix}.$$

Proposition 7. *With the above notation, $\overline{F}$ is a lift of F respect to* (2) *if and only if*

$$\begin{array}{llll} \overline{C}_j^i = C_j^i & \overline{C}_\beta^i = M_\beta^i & \overline{M}_\mu^i = 0 & \overline{M}_\tau^i = 0 \\ \overline{C}_j^\alpha = \psi_j^\alpha & \overline{C}_\beta^\alpha = B_\beta^\alpha & \overline{M}_\mu^\alpha = 0 & \overline{M}_\tau^\alpha = 0 \end{array}.$$

If we want to understand an infinitesimal connection Γ on γ: $A \to B$ as a generalized connection, we obtain

$$\begin{pmatrix} \delta^i_j & 0 & 0 & 0 \\ -2\Omega^\alpha_{11j} & -\delta^\alpha_\beta & -2\Omega^\alpha_{12\mu} & 0 \\ 0 & 0 & \delta^\lambda_\mu & 0 \\ -2\Omega^\sigma_{21j} & 0 & -2\Omega^\sigma_{22\mu} & -\delta^\sigma_\tau \end{pmatrix}$$

where the local component of the connection is

$$\Omega = ((\Omega_{11}+\Omega_{12}),\ (\Omega_{21}+\Omega_{22})) \qquad U^n \times R^r \times G^s \times R^r \to L(R^n \times R^s,\ R^r \times R^t)$$

$$\cong (L(R^n, R^r) \times L(R^s,\ R^r) \times (L(R^n, R^t) \times L(R^s,\ R^t)).$$

5 - More applications

We have seen in 3 several examples of our construction. Now, we shall study some others concernig pseudo-connections.

(h) *Lifts of linear pseudo-connections to the tangent bundle TM.*

Let ∇ be a linear pseudo-connection on M and let ∇^C and ∇^V its complete and vertical lifts to TM [**9**]. Then, ∇^C is a lift of ∇ respect to the good square (3), but ∇^V is a lift of the zero pseudo-connection respect to this good square.

(i) *Lifts of linear pseudo-connections to the second order tangent bundle* T_2M.

Let ∇ be linear pseudo-connection on M and $\nabla^{(0)}$, $\nabla^{(1)}$ and $\nabla^{(2)}$ its lifts to T_2M in the sense of $[8]_1$. Only the last one is a lift of ∇ respect to the good square (4).

Another kind of problems, such as the relationship between the lifts of connections where they are considered as such connections and as (1, 1)-tensor fields, are studied in $[8]_4$.

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Abstract

The aim of this work is to find a general theory of lifts of connections on vector bundles, which contains as examples several known lifts. We introduce the notion of «good squares of vector bundles» and firstly develop the theory of lifts of infinitesimal connections with respect to good squares, obtaining some properties and several examples. After that, we extend the construction to generalized connections, which give some results about linear pseudo-connections.

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