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On existence and uniqueness for viscoelastic compressible fluids (**)

1 - Introduction

In some recent papers [9], [7] a careful study has been made of the connections between the laws of Thermodynamics and the constitutive equations for *viscoelastic* materials (solids and fluids).

Thys type of viscous continua is characterized for admitting properties of *fading memory* on the parameters related to viscosity: as a consequence, the actual motions turn out to be influenced by the past ones through the *history* of some basic kinematical field [4], [2], [3], [6].

Of course, application of Thermodynamics introduces certain restrictions in this context, and several papers are devoted to the search of what conditions can be claimed to be equivalent to the statement of the thermodynamic laws and further, possibly, to assure the well posedness of the evolution problems.

In the ambit of viscoelastic fluids, this program has been widely worked out in the incompressible (linear) case: for an account we refer to [5], [8], [10], [12], [14], where various results concerning existence, uniqueness and stability for the typical initial-boundary value problems can be found.

In this paper, we consider the existence and uniqueness issue for a linear, compressible viscoelastic fluid; dealing with a purely mechanical context, in which temperature is supposed constant (1), we assume the following as constitu-

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(1) As in most of the previously cited papers.

tive equation for the stress tensor $\boldsymbol{T}$ [6], [3], [7]

$$(1) \qquad \boldsymbol{T}(\boldsymbol{x},\ t) = -p(\boldsymbol{x},\ t)\boldsymbol{I} + \int_{\mathbb{R}^+} [2\mu(\tau)\boldsymbol{D}(\boldsymbol{x},\ t-\tau) + \lambda(\tau)\,\mathrm{tr}\boldsymbol{D}(\boldsymbol{x},\ t-\tau)\boldsymbol{I}]\,\mathrm{d}\tau$$

$$(\boldsymbol{x},\ t) \in \Omega \times T.$$

Above, and throughout the paper, Ω denotes the (bounded, regular) domain of the physical space ($\equiv \mathbb{R}^3$) occupied by the fluid, and T a time interval; $\boldsymbol{D}$ is the symmetric velocity gradient, $\mathrm{tr}\boldsymbol{D}$ its trace, and p the pressure field, which is given as a function of the mass density ρ. Finally, the scalar (continuous) functions $\mu = \mu(\tau)$ and $\lambda = \lambda(\tau)$, $\tau \in [0,\ +\infty)$ are the *relaxation moduli* of the viscosity: their dependence from a time-like parameter accounts for the hereditary properties of the viscoelastic fluids here concerned (2).

We shall admit the *fading memory* hypothesis on these moduli, that implies $\mu,\ \lambda \in L^1(\mathbb{R}^+)$, and both of them tending to zero as $\tau \to +\infty$ [3]. We shall also assume that the *barotropic* relation $p = p(\rho)$ has a strictly positive derivative [11], and, according to the linear context in which we confine, that this derivative be a constant.

The basic equations and definitions will be stated in 2, together with a Laplace-transformed formulation of the field equations. In 3, starting from the thermodynamical restrictions on the relaxation moduli established in [7], we derive two strict inequalities involving these moduli. Then, in 4, we shall prove that such conditions can be crucially related with a theorem of existence and uniqueness for (weak) solutions to the initial-boundary value problem of a fluid as above.

2 - Basic equations and definitions

Consider the constitutive equation (1), and recall that, as previously assumed, $p'(\rho) = k$, a (positive) constant. Let also $T = (0,\ +\infty)$. The classical balance laws of continuum mechanics can be linearly approximated in the present context to

(2) Note that the classical Newtonian form is recovered from (1) for $\mu(\tau) = \mu\delta(\tau)$ and $\lambda(\tau) = \lambda\delta(\tau)$, with $\mu,\ \lambda$ the usual viscosity coefficients and δ the Dirac delta.

give the following evolution equations

$$\begin{aligned} \partial_t \boldsymbol{v} &= -k\nabla\rho + \mu * \Delta \boldsymbol{v} + (\lambda+\mu) * \nabla \operatorname{div} \boldsymbol{v} + \boldsymbol{b} + \boldsymbol{\pi} \\ \partial_t \rho &= -\operatorname{div} \boldsymbol{v} \end{aligned} \qquad \text{in } \Omega \times (0, +\infty)\,. \tag{2}$$

In these equations, where for convenience we put equal to 1 the referential (constant) density, $\boldsymbol{b} = \boldsymbol{b}(\boldsymbol{x}, t)$ denotes the external body force and $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{x}, t)$ the velocity field of the fluid; $\rho = \rho(\boldsymbol{x}, t)$ is the density field. Moreover, in (2) we mean $\Delta = \operatorname{div} \nabla$,

$$\boldsymbol{\pi} = \boldsymbol{\pi}(\boldsymbol{x}, t) = \int_{-\infty}^{0} [\mu(t-\tau)\Delta \boldsymbol{v}(\boldsymbol{x}, \tau) + (\lambda+\mu)(t-\tau)\nabla \operatorname{div} \boldsymbol{v}(\boldsymbol{x}, \tau)]\, d\tau$$

and by $a * b$ the usual time-convolution, so that, for example,

$$(\mu * \Delta \boldsymbol{v})(\boldsymbol{x}, t) = \int_0^t \mu(\tau)\Delta \boldsymbol{v}(\boldsymbol{x}, t-\tau)\, d\tau\,.$$

The vector field $\boldsymbol{\pi}$ is of course determined by the *past hystory* of the fluid — that is, the motion for $t<0$ — through some assignment $\boldsymbol{v} = \tilde{\boldsymbol{v}}$ in $\Omega \times (-\infty, 0)$.

Consider now the following set of initial-boundary conditions for (2):

$$\boldsymbol{v} = \boldsymbol{v}_0 \qquad \rho = \rho_0 \qquad \text{in } \Omega \times \{0\} \tag{3}$$

$$\boldsymbol{v} = 0 \qquad \text{in } \partial\Omega \times (0, +\infty)\,. \tag{4}$$

Along with $\boldsymbol{b}$ and $\boldsymbol{\pi}$, the fields $\boldsymbol{v}_0$ and ρ_0 represent the *data* of the initial-boundary value problem (2), (3), (4). For *solution* to this problem we shall mean a pair of velocity and density fields $\boldsymbol{v}$, ρ on $\Omega \times (0, +\infty)$ satisfying all equations (2) to (4), for given data, in some (weak) sense.

The main result of the paper needs a Laplace-transformed version of the above problem. To this end, recall that given a (smooth) function $f\colon (0, +\infty) \to \mathbb{R}$, the Laplace-transform $\hat{f}$ is defined by

$$\hat{f}(z) = \int_{\mathbb{R}^+} \exp(-zt) f(t)\, dt$$

for all $z \in \boldsymbol{C}$ making sense. With a view towards the next developments, we note that $f \in L^2(0, +\infty)$ admits well-defined Laplace transform $\forall z \in \boldsymbol{C}^+ \equiv \{z \in \boldsymbol{C}\colon \operatorname{Re}\{z\} \geqslant 0\}$.

Proceeding formally on the equations in concern, we easily get

$$(5)\qquad \begin{aligned} & z\hat{\boldsymbol{v}} = -k\nabla\hat{\rho} + \hat{\mu}(z)\Delta\hat{\boldsymbol{v}} + [\hat{\lambda}(z) + \hat{\mu}(z)]\nabla \operatorname{div}\hat{\boldsymbol{v}} + (\hat{\boldsymbol{b}} + \hat{\boldsymbol{\pi}} + \boldsymbol{v}_0) \\ & z\hat{\rho} = -\operatorname{div}\hat{\boldsymbol{v}} + \rho_0 \qquad \hat{\boldsymbol{v}} = 0 \end{aligned}$$

Insertion of $(5)_2$ in $(5)_1$ for $z \neq 0$ finally gives

$$(6)\qquad \begin{aligned} & z\hat{\boldsymbol{v}} = \hat{\mu}(z)\Delta\hat{\boldsymbol{v}} + [(\hat{\lambda} + \hat{\mu})(z) + k/z]\nabla \operatorname{div}\hat{\boldsymbol{v}} + \tilde{\boldsymbol{F}} && \text{in } \Omega \\ & \hat{\boldsymbol{v}} = 0 && \text{in } \partial\Omega \end{aligned}$$

where

$$(7)\qquad \tilde{\boldsymbol{F}} = \tilde{\boldsymbol{F}}(\boldsymbol{x},\ z) = \hat{\boldsymbol{b}}(\boldsymbol{x},\ z) + \hat{\boldsymbol{\pi}}(\boldsymbol{x},\ z) + \boldsymbol{v}_0(\boldsymbol{x}) - (k/z)\nabla\rho_0 .$$

Given $z \in \boldsymbol{C}^+ - \{0\}$ as a parameter, this system is of course a linear *elliptic* boundary value problem in Ω in the (only) unknown $\hat{\boldsymbol{v}}$. When the dependence on z must be taken into account, we shall write $\hat{\boldsymbol{v}} = \hat{\boldsymbol{v}}(\boldsymbol{x};\ z)$ (and likewise for any other field in concern).

3 - Thermodynamic restrictions

The classical conditions on the viscosity coefficients

$$\mu > 0 \qquad 3\lambda + 2\mu > 0$$

usually presented as originated from thermodynamic principles [**11**], need an obvious generalization when hereditary effects are admitted in the viscosity. This topic has been recently investigated in [7], where an elegant characterization of the second law of Thermodynamics in terms of $\mu(\tau)$ and $\lambda(\tau)$ is established; as a matter of fact, the author proves that the constitutive equation (1) is compatible with this law (reduced to isothermal-mechanical context) if and only if

$$(8)\qquad \int_{\mathrm{R}^+} \mu(\tau)\cos\xi\tau \mathrm{d}\tau > 0 \qquad \int_{\mathrm{R}^+} [3\lambda(\tau) + 2\mu(\tau)]\cos\xi\tau \mathrm{d}\tau > 0 \qquad \forall \xi \in \mathrm{R}\ (^3).$$

We now deduce some simple consequences from (8) (cf. [8], [1]). Let us begin

(3) Note these inequalities reduce to the preceding ones for $\mu(\tau) = \mu\delta(\tau)$, $\lambda(\tau) = \lambda\delta(\tau)$ (see footnote on p. 160).

with $(8)_1$; note firstly that left hand side defines the Fourier (cosine) transform $\bar{\mu}(\xi)$ of $\mu(\tau)$, so that, assuming $\bar{\mu} \in L^1(\mathrm{R}^+)$, it is also

$$\mu(\tau) = \frac{2}{\pi} \int_{\mathrm{R}^+} \bar{\mu}(\xi) \cos \xi\tau \mathrm{d}\xi \tag{9}$$

by Fourier integral theorem.

Take then Laplace-transform of (9); we get

$$\hat{\mu}(z) = \int_{\mathrm{R}^+} \exp(-z\tau)\mu(\tau)\,\mathrm{d}\tau = \frac{2}{\pi} \int_{\mathrm{R}^+} \exp(-z\tau)[\int_{\mathrm{R}^+} \bar{\mu}(\xi) \cos \xi\tau \mathrm{d}\xi]\,\mathrm{d}\tau$$

whence, by reversing the order of integration and letting $\mathrm{Re}\{z\} > 0$,

$$\hat{\mu}(z) = \frac{2}{\pi} \int_{\mathrm{R}^+} \bar{\mu}(\xi)[\int_{\mathrm{R}^+} \exp(-z\tau) \cos \xi\tau \mathrm{d}\tau]\,\mathrm{d}\xi = \frac{2}{\pi} \int_{\mathrm{R}^+} \bar{\mu}(\xi) \frac{z}{(z^2 + \xi^2)}\,\mathrm{d}\xi .$$

Put now $z = s + \mathrm{i}\omega$ $(s > 0)$, and consider the real part of equality above; it gives

$$\mathrm{Re}\{\hat{\mu}(z)\} = \int_{\mathrm{R}^+} \exp(-s\tau) \cos \omega\tau \qquad \mu(\tau)\,\mathrm{d}\tau = \frac{2}{\pi} \int_{\mathrm{R}^+} \bar{\mu}(\xi) \frac{s}{(s^2 + \omega^2 + \xi^2)}\,\mathrm{d}\xi$$

which is strictly positive $\forall \omega \in \mathrm{R}$ in view of $(8)_1$.

Of course, $\mathrm{Re}\{\hat{\mu}(z)\}$ is also strictly positive $\forall\omega \in \mathrm{R}$ when $\mathrm{Re}\{z\} = 0$, since, in this case, $\mathrm{Re}\{\hat{\mu}(\mathrm{i}\omega)\} \equiv \bar{\mu}(\omega)$.

From $(8)_2$, and analogous result can be derived for $\mathrm{Re}\{(3\hat{\lambda} + 2\hat{\mu})(z)\}$ as well: we only need to assume $\bar{\lambda} \in L^1(\mathrm{R}^+)$ and consider linearity of Fourier transform.

We summarize these results in form of the following

Lemma. *Let the second law of Thermodynamics hold in the present context of (isothermal) compressible viscoelastic fluids described by equation* (1). *Then, the real part of Laplace transforms of* $\mu(\tau)$ *and* $(3\lambda + 2\mu)(\tau)$ *is positive definite in the complex half-plane* $\boldsymbol{C}^+$.

We conclude by noting that linearity of Laplace transform trivially implies the same result for $(\lambda + \mu)(\tau)$.

4 - Existence and uniqueness

The main result of the paper is the following

Theorem. *Consider the initial-boundary value problem* (2), (3), (4), *and let the Lemma of previous section hold. Then, assigned*

$$\boldsymbol{b} \qquad \pi \in L^2(0, +\infty; H^{-1}(\Omega)) \qquad \boldsymbol{v}_0 \in H_0^1(\Omega) \qquad \rho_0 \in L^2(\Omega)$$

there exists one and only one solution

$$(10) \qquad \boldsymbol{v} \in L^2(0, +\infty; H_0^1(\Omega)) \qquad \rho \in C(0, +\infty; L^2(\Omega))$$

to this problem.

The proof of the Theorem will be performed in three steps: 1st: Existence and uniqueness for the (Laplace-transformed) problem (6). 2nd: Behaviour of the solution $\hat{\boldsymbol{v}}$ with respect to the parameter z. 3rd: Inverse transformation of $\hat{\boldsymbol{v}}(\boldsymbol{x}; z)$ to give a (unique) solution $\boldsymbol{v}(\boldsymbol{x}, t)$, $\rho(\boldsymbol{x}, t)$ as in (10) to the original problem. Throughout, local relations are to be interpreted in the distribution sense.

Proof of first step. We consider the variational formulation of the linear elliptic problem (6), and call *weak solution* to this problem for given $\tilde{\boldsymbol{F}} \in H^{-1}(\Omega)$, a vector field $\hat{\boldsymbol{v}} \in H_0^1(\Omega)$ such that

$$(11) \qquad \int_\Omega \{z\hat{\boldsymbol{v}} \cdot \boldsymbol{u}^* + \hat{\mu}(z)\nabla\hat{\boldsymbol{v}} : \nabla\boldsymbol{u}^* + [\hat{\lambda}(z) + \hat{\mu}(z) + k/z] \operatorname{div}\hat{\boldsymbol{v}} \operatorname{div}\boldsymbol{u}^*\} \, d\Omega$$

$$= \int_\Omega \tilde{\boldsymbol{F}}(z) \cdot \boldsymbol{u}^* \, d\Omega \qquad \forall \boldsymbol{u} \in H_0^1(\Omega) \ \ (\boldsymbol{u}^* \equiv \text{conjugated of } \boldsymbol{u}).$$

Existence and uniqueness of weak solutions to (6) for $z \in \boldsymbol{C}^+ - \{0\}$ are straightly assured by the thermodynamic restrictions. Indeed, consider the bilinear form $a(.\,, \,.\,; z)$ defined on $H_0^1 \times H_0^1$ by the left side of (11): aiming to show the coerciveness of this form in H_0^1, we easily get

$$|a(v,\ v;\ z)| \geqslant \operatorname{Re} a(v,\ v;\ z)$$

$$= \int_\Omega \{s|\hat{v}|^2 + (\operatorname{Re}\hat{\mu})|\nabla\hat{v}|^2 + [\operatorname{Re}(\hat{\lambda} + \hat{\mu}) + sk/(s^2 + \omega^2)](\operatorname{div}\hat{v})^2\}\, d\Omega$$

$$\geqslant c(z) \int_\Omega |\nabla\hat{v}|^2\, d\Omega$$

where $c(z) \equiv \operatorname{Re}\{\hat{\mu}(z)\} > 0$ and $z = s + i\omega \in C^+ - \{0\}$.

An application of Poincaré inequality finally gives

$$|a(v,\ v;\ z) \geqslant K\|\hat{v}\|^2_{H_0^1}$$

in which $K = K(\Omega,\ z) > 0$.

The form is trivially continuous on $H_0^1 \times H_0^1$. Lax-Milgram theorem then applies and tells us that problem (6) has one and only one weak solution $\hat{v} = \hat{v}(x;\ z)$.

Proof of second step. To study the behaviour of $\hat{v}(x;\ z)$ with respect to z, consider the Green (tensor) function $H = H(x,\ y;\ z)$ of problem (6). It formally solves the equation

$$\int_\Omega \{zH(x,\ y;z)\, u^*(y) + \hat{\mu}(z)\, \nabla_y H(x,\ y;\ z)\, \nabla_y u^*(y)$$
$$+[\hat{\lambda}(z) + \hat{\mu}(z) + \frac{k}{z}]\operatorname{div}_y H(x,\ y;\ z)\operatorname{div}_y u^*(y)\}\, d\Omega_y = \int_\Omega \delta(x-y)\, u^*(y)\, d\Omega_y \tag{12}$$

$\forall u \in H_0^1(\Omega)$, where δ is the Dirac delta on $H_0^1(\Omega)$ and subscript y denotes the spatial variable to be concerned.

It is a simple matter to prove that:

(i) $H(x,\ .;\ z)$ as a solution of (12) exists and is unique in $H_0^1(\Omega)$ $\forall x \in \Omega$, $\forall z \in C^+ - \{0\}$.

(ii) $H(x,\ y;\ .)$ is continuous on $C^+ - \{0\}$.

(iii) $\lim_{z\to\infty} \int_\Omega zH(x,\ y;\ z)\, u(y)\, d\Omega_y = u(x)$ $\quad \forall(\text{real})\, u \in H_0^1(\Omega)$.

(iv) $H(x,\ y;\ z) = o(z^{-1+\varepsilon})$ as $z \to \infty$; $\quad H(x,\ y;\ z) = o(z)$ as $z \to 0$. $\quad \forall \varepsilon > 0,\ z \in C^+ - \{0\}$.

Indeed, property (i) is assured — via Lax Milgram theorem — by the coerciveness of the form a and since $\delta \in H^{-1}(\Omega)$. Property (ii) follows from the continuity of $a(v, u; .)$ with respect to $z \in C^+ - \{0\}$ [13].

As regards properties (iii), (iv), consider equation (12) for real $u \in C_0^\infty(\Omega)$ and apply the divergence theorem, to give

$$\int_\Omega z^{\alpha+1} H(x, y; z)[z^\beta u(y) - z^{\beta-1}\hat{\mu}(z)\Delta_y u(y) - z^{\beta-1}(\hat{\lambda}+\hat{\mu})(z)\nabla_y \operatorname{div}_y u(y)$$

$$-z^{\beta-2}k\nabla_y \operatorname{div}_y u(y)]\, d\Omega_y = z^{\alpha+\beta} u(x) \qquad \forall \alpha, \beta \in \mathbb{R}.$$

$\hat{\mu}$ and $\hat{\lambda}$ are of course bounded functions of z as $z \to 0$ or $z \to \infty$. Then, (iii) follows from above by letting $\alpha+\beta = 0$, $z \to \infty$, and since C_0^∞ is dense in H_0^1; (iv)′ follow from above by letting $\alpha + \beta = -\varepsilon$ and $z \to \infty$ for the first, $\alpha = -2$, $\beta = 2$ and $z \to 0$ for the second, and since u is arbitrary in $C_0^\infty(\Omega)$.

In terms of H, the weak solution to (6) is given by

$$\hat{v}(x; z) = \int_\Omega H(x, y; z)\tilde{F}(y, z)\, d\Omega_y. \tag{13}$$

A similar representation can be set up for $\nabla\hat{v}(x; z)$, namely,

$$\nabla\hat{v}(x; z) = \int_\Omega \nabla_x H(x, y; z)\tilde{F}(y, z)\, d\Omega_y \tag{14}$$

by introducing the (third-order) tensor function $(\nabla_x H)(x, y; z)$ such that

$$\int_\Omega \{z\nabla_x H(x, y; z)\, u^*(y) + \hat{\mu}(z)\nabla_y(\nabla_x H)(x, y; z)\nabla_y u^*(y) \tag{15}$$

$$+[\hat{\lambda}(z) + \hat{\mu}(z) + k/z]\operatorname{div}_y(\nabla_x H)(x, y; z)\operatorname{div}_y u^*(y)\}\, d\Omega_y = \int_\Omega \delta_x(x-y)\, u^*(y)\, d\Omega_y$$

$\forall u \in H_0^1(\Omega)$. By the same arguments used above in connection with H, the following properties can be proved for $\nabla_x H$:

(i)′ $\nabla_x H(x, y; z)$ as a solution of (15) exists and is unique in $L^2(\Omega)$ $\forall x \in \Omega$, $\forall z \in C^+ - \{0\}$ [13].

(ii)′ $\nabla_x H(x, y; .)$ is continuous on $C^+ - \{0\}$.

(iii)′ $\begin{aligned} &\nabla_x H(x, y; z) = o(z^{-1+\varepsilon}) &&\text{as } z \to \infty \\ &\nabla_x H(x, y; z) = o(z) &&\text{as } z \to 0. \end{aligned} \qquad \forall \varepsilon > 0, \quad z \in C^+ - \{0\}.$

Recall now equation (7) defining $\tilde{F}(x, z)$ ($\hat{b}$ and $\hat{\pi}$ are Laplace-transforms);

since $\lim_{z\to\infty} \tilde{F}(x, z) = v_0(x)$, by (13), (14) and properties of H, $\nabla_x H$, we get

$$\text{(16)} \qquad \begin{aligned} &\lim_{z\to\infty} z^{1-\varepsilon}\hat{v}(x;\ z) = 0 \\ &\lim_{z\to\infty} z^{1-\varepsilon}\nabla\hat{v}(x;\ z) = 0 \end{aligned} \qquad \forall\varepsilon > 0,\ z \in C^+ - \{0\}$$

as well as

$$\text{(17)} \qquad \lim_{z\to\infty} z\hat{v}(x;\ z) = v_0(x)\,.$$

Further, by (13), (14), $(\mathrm{iv})_2$ and $(\mathrm{iii})_2'$, we also deduce that $\hat{v}(x;\ z)$ and $\nabla\hat{v}(x;\ z)$ remain bounded as $z \to 0$.

Proof of third step. Choose $\varepsilon \in (0,\ 1/2)$ and put $\mathrm{Re}\{z\} = 0$ in equations (16): what is proved above assure that $\hat{v}(x;\ \mathrm{i}\omega)$ and $\nabla\hat{v}(x;\ \mathrm{i}\omega)$ belong to $L^2(-\infty,\ +\infty)$ with respect to $\omega = \mathrm{Im}\{z\}$, so that both of them can be regarded as Fourier transformation of some functions $v^0(x,\ t)$, $\nabla v^0(x,\ t)$ on $\Omega \times (-\infty,\ +\infty)$. Of course, it must be $(\nabla v^0)_{ij} = \partial v_i^0/\partial x_j$.

Let $v(x,\ t) \equiv v^0(x,\ t)$ on $\Omega \times [0,\ +\infty)$; by Parseval theorem

$$\int_\Omega \int_0^{+\infty} |v(x,\ t)|^2 \mathrm{d}t\,\mathrm{d}\Omega \leqslant \int_\Omega \int_{-\infty}^{+\infty} |v^0(x,\ t)|^2 \mathrm{d}t\,\mathrm{d}\Omega = \frac{1}{2\pi} \int_\Omega \int_{-\infty}^{+\infty} |\hat{v}(x,\ \mathrm{i}\omega)|^2 \mathrm{d}\omega\,\mathrm{d}\Omega$$

$$\int_\Omega \int_0^{+\infty} |\nabla v(x,\ t)|^2 \mathrm{d}t\,\mathrm{d}\Omega \leqslant \int_\Omega \int_{-\infty}^{+\infty} |\nabla v^0(x,\ t)|^2 \mathrm{d}t\,\mathrm{d}\Omega = \frac{1}{2\pi} \int_\Omega \int_{-\infty}^{+\infty} |\nabla\hat{v}(x,\ \mathrm{i}\omega)|^2 \mathrm{d}\omega\,\mathrm{d}\Omega$$

Then, the vector field $v(x,\ t)$ belongs to $L^2(0,\ +\infty;\ H_0^1(\Omega))$, and, in view of (17), it holds $v(x,\ 0) = v_0(x)$ in Ω.

Moreover, by inverse transformation of (6), we deduce

$$\text{(18)} \qquad \partial_t v = \mu * \Delta v + (\lambda + \mu) * \nabla\,\mathrm{div}\, v + k * \nabla\,\mathrm{div}\, v + b + \pi - k\nabla\rho_0$$

in $\Omega \times (0,\ +\infty)$ as well as $v = 0$ in $\partial\Omega \times (0,\ +\infty)$.

Define now

$$\text{(19)} \qquad \rho(x,\ t) = \rho_0(x) - 1 * \mathrm{div}\, v(x,\ t) \qquad \forall (x,\ t) \in \Omega \times (0,\ +\infty)\,.$$

Of course, such a scalar field belongs to $C(0,\ +\infty;\ L^2(\Omega))$, since $\mathrm{div}\, v \in L^1_{\mathrm{loc}}(0,\ +\infty;\ L^2(\Omega))$, and uniquely solves the equations

$$\text{(20)} \qquad \partial_t\rho = -\mathrm{div}\, v \quad \text{in } \Omega \times (0,\ +\infty) \qquad \rho = \rho_0 \quad \text{in } \Omega \times \{0\}\,.$$

Taking the spatial gradient of (19) and inserting what results in (18), we finally get equation $(2)_1$, namely:

$$\partial_t \boldsymbol{v} = -k\nabla\rho + \mu * \Delta \boldsymbol{v} + (\lambda+\mu) * \nabla \operatorname{div} \boldsymbol{v} + \boldsymbol{b} + \boldsymbol{\pi} \qquad \text{in } \Omega \times (0, \ +\infty).$$

The existence item is fully achieved. The uniqueness' one follows since null data imply $\hat{\boldsymbol{v}} \equiv 0$ (see (13) for $\dot{\tilde{\boldsymbol{F}}} = 0$), and by uniqueness of the inverse transformation.

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Summary

We prove a theorem of existence and uniqueness for the linear initial-boundary value problem of a compressible viscous fluid with hereditary properties. Thermodynamical restrictions on the relevant moduli are seen to be crucial.

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