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Existence, continuous dependence and periodic solutions of nonlinear integrodifferential equation with infinite delay (**)

1 - Introduction

Theory of differential and integrodifferential equations with unbounded delays are studied by several authors (see the survey article by Corduneau and Lakshmikantham [3]). Burton [2] has established existence theorems for nonlinear integrodifferential equations with infinite delay. He also proved one type of continuous dependence result and using this he established the periodicity. Balachandran [1] has proved an existence theorem for nonlinear integrodifferential equations having implicit derivative and infinite delay. In [5] Kaminago discussed the continuous dependence of solutions for nonlinear integrodifferential equations with infinite delay. The general question of continual dependence is treated by Haddock [4] and Kappel and Schappacher [6].

In this paper we shall prove existence theorem for more general class of nonlinear integrodifferential equations by including an operator in the nonlinear term and infinite delay. We shall also prove the continuous dependence of solutions and existence of periodic solutions.

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2 - Basic assumptions

Consider the nonlinear integrodifferential equation with infinite delay

$$\text{(1)} \qquad \begin{aligned} &\dot{x}(t) = h(t,\ x(t),\ Ax(t)) + \int_{-\infty}^{t} q(t,\ s,\ x(s))\,\mathrm{d}s \qquad t \geqslant t_0 \\ &x(t) = \phi(t) \qquad -\infty < t \leqslant t_0 \end{aligned}$$

or an equivalent system

$$\text{(2)} \qquad \begin{aligned} &\dot{x}(t) = h(t,\ x(t),\ Ax(t)) + \int_{-\infty}^{t_0} q(t,\ s,\ \phi(s))\,\mathrm{d}s + \int_{t_0}^{t} q(t,\ s,\ x(s))\,\mathrm{d}s \qquad t \geqslant t_0 \\ &x(t_0) = \phi(t_0). \end{aligned}$$

Here h: $(-\infty,\ \infty) \times R^n \times R^n \to R^n$, q: $(-\infty,\ \infty) \times (-\infty,\ \infty) \times R^n \to R^n$ are continuous and A: $R^n \to R^n$ is a continuous operator such that for $x_1,\ x_2 \in R^n$, there exists a constant N and a continuous function a: $[t_0,\ \infty) \to [t_0,\ \infty)$

$$\text{(3)} \qquad |Ax_1(t) - Ax_2(t)| < N \int_{t_0}^{t} a(s)|x_1(s) - x_2(s)|\,\mathrm{d}s.$$

Further, the functions q and h satisfy the local Lipschitz conditions

$$\text{(4)} \qquad |q(t,\ s,\ x_1) - q(t,\ s,\ x_2)| \leqslant L|x_1 - x_2|$$

$$\text{(5)} \qquad |h(t,\ x_1, Ax_1) - h(t,\ x_2,\ Ax_2)| \leqslant L|x_1 - x_2| + K|Ax_1 - Ax_2|$$

for $x_1,\ x_2 \in R^n$ where L and K are positive constants.

Assume that for each $t_0 \in R$ there exists a nonempty convex subset $B(t_0)$ of the space of continuous functions ϕ: $(-\infty,\ t_0] \to R^n$ such that $\phi \in B(t_0)$ implies

$$\text{(6)} \qquad \int_{-\infty}^{t_0} q(t,\ s,\ \phi(s))\,\mathrm{d}s = Q(t,\ t_0,\ \phi)$$

is continuous on $[t_0,\ \infty)$.

For a given t_0 let $\phi \in B(t_0)$, and let β_1 be a positive number. Now for $|x - \phi(t_0)| \leqslant 1$ and $t_0 < s \leqslant t \leqslant t_0 + \beta_1$ there is an $M > 0$ with

$$\text{(7)} \qquad \begin{gathered} \beta_1 |q(t,\ s,\ x)| + |h(t,\ x,\ Ax)| + |Q(t,\ t_0,\ \phi)| \leqslant M \\ 0 < \beta < \beta_1 \qquad \beta < \frac{1}{M}. \end{gathered}$$

Consider the complete metric space $(X,\ \rho)$ of continuous functions

x: $(-\infty,\ t_0+\beta] \to R^n$ with $x(t) = \phi(t)$ on $(-\infty,\ t_0]$, $|x(t_1) - x(t_2)| \leqslant M|t_1 - t_2|$ for $t_0 \leqslant t_1 \leqslant t_0 + \beta$, and with $\rho(x_1,\ x_2) = \max\limits_{t_0 \leqslant t \leqslant t_0+\beta} |x_1(t) - x_2(t)|$. Assume that

$$L\beta(\beta + 1 + KN\alpha) < 1 \qquad \text{where} \qquad \alpha = \sup_{t_0 \leqslant t \leqslant t_0+\beta} \int_{t_0}^{t} a(s)\,\mathrm{d}s. \tag{8}$$

3 - Existence

Theorem 1. *Under the assumptions* (3)-(8), *the equation* (1) *has a unique solution* $x(t,\ t_0,\ \phi)$ *defined on an interval* $[t_0,\ t_0+\beta)$ *for* $\beta > 0$, *for each* t_0 *and each* $\phi \in B(t_0)$.

Proof. Define a mapping P: $X \to X$ by

$$(Px)(t) = \phi(t) \qquad \text{for } -\infty < t \leqslant t_0\,;$$

$$(Px)(t) = \phi(t_0) + \int_{t_0}^{t} h(s,\ x(s),\ Ax(s))\,\mathrm{d}s$$

$$+ \int_{t_0}^{t}\int_{t_0}^{u} q(u,\ s,\ x(s))\,\mathrm{d}s\,\mathrm{d}u + \int_{t_0}^{t} Q(u,\ t_0,\ \phi)\,\mathrm{d}u \qquad \text{for } t_0 < t \leqslant t_0 + \beta.$$

Now:

$$|(Px)(t) - \phi(t_0)|$$

$$\leqslant \Big| \int_{t_0}^{t} h(s, x(s),\ Ax(s))\,\mathrm{d}s\Big| + \Big| \int_{t_0}^{t}\int_{t_0}^{u} q(u,\ s,\ x(s))\,\mathrm{d}s\,\mathrm{d}u\Big| + \int_{t_0}^{t} Q(u,\ t_0,\ \phi)\,\mathrm{d}u\Big|$$

$$\leqslant \beta|h(t,\ x,\ Ax)| + \beta\beta_1|q(t,\ s,\ x)| + \beta|Q(t,\ t_0,\ \phi)| \leqslant \beta M < 1;$$

$$|(Px)(t_1) - (Px)(t_2)|$$

$$\leqslant \Big| \int_{t_1}^{t_2} h(s,\ x(s),\ Ax(s))\,\mathrm{d}s\Big| + \Big| \int_{t_1}^{t_2}\int_{t_0}^{u} q(u,\ s,\ x(s))\,\mathrm{d}s\,\mathrm{d}u\Big| + \Big|\int_{t_1}^{t_2} Q(u,\ t_0,\ \phi)\,\mathrm{d}u\Big|$$

$$\leqslant M|t_1 - t_2|.$$

So P maps X into itself.

To see that P is a contraction for small β,, we have for $x_1, x_2 \in X$

$$\rho(Px_1, Px_2)$$

$$\leqslant \sup_{t_0 < t \leqslant t_0+\beta} [\int_{t_0}^{t} \int_{t_0}^{u} |q(u, s, x_1(s)) - q(u, s, x_2(s))| \, ds \, du$$

$$+ \int_{t_0}^{t} |h(u, x_1(u), Ax_1(u)) - h(u, x_2(u), Ax_2(u))| \, du]$$

$$\leqslant L\beta^2 \|x_1 - x_2\| + \beta L\|x_1 - x_2\| + \beta K\|Ax_1 - Ax_2\|$$

$$\leqslant L\beta^2 \|x_1 - x_2\| + \beta L\|x_1 - x_2\| + \beta KN\|x_1 - x_2\| \int_{t_0}^{t} a(s) \, ds$$

$$\leqslant L\beta^2 \|x_1 - x_2\| + \beta L\|x_1 - x_2\| + \beta KN\alpha\|x_1 - x_2\| = L\beta(\beta + 1 + KN\alpha) \|x_1 - x_2\|$$

$$< \|x_1 - x_2\|.$$

Hence P is a contractions mapping. Consequently P has a unique fixed point $x(t)$ which is the solutions of the equation (1).

Theorem 2. *Under the assumptions* (3), (6) *and* (7), *the equation* (1) *has at least one solution* $x(t, t_0, \phi)$ *for each* t_0 *and each* $\phi \in B(t_0)$.

Proof. Let X be the space of continuous functions x: $(-\infty, t_0 + \beta] \to R^n$ with $x(t) = \phi(t)$ on $(-\infty, t_0]$ and with supremum norm. Define

$$G = \{x \in X: |x - \phi(t_0)| \leqslant \beta M; \ |x(t_1) - x(t_2)| \leqslant M|t_1 - t_2|\}.$$

By Ascoli's theorem G is compact. Also G is convex.

Define an operator P: $G \to G$ by

$$(Px)(t) = \phi(t) \qquad \text{for } -\infty < t \leqslant t_0;$$

$$(Px)(t) = \phi(t_0) + \int_{t_0}^{t} h(s, x(s), Ax(s)) \, ds$$

$$+ \int_{t_0}^{t} \int_{t_0}^{u} q(u, s, x(s)) \, ds \, du + \int_{t_0}^{t} Q(u, t_0, \phi) \, du \qquad \text{for } t_0 \leqslant t \leqslant t_0 + \beta.$$

Now:
$$|(Px)(t)-\phi(t_0)|$$
$$\leqslant \Big|\int_{t_0}^{t} h(s,\ x(s),\ Ax(s))\,ds\Big| + \Big|\int_{t_0}^{t}\int_{t_0}^{u} q(u,\ s,\ x(s))\,ds\,du\Big| + \Big|\int_{t_0}^{t} Q(u, t_0,\ \phi)\,du\Big|$$
$$\leqslant \beta|h(t,\ x, Ax)| + \beta\beta_1|q(t,\ s,\ x)| + \beta|Q(t,\ t_0, \phi)| \leqslant \beta M;$$
$$|(Px)(t_1)-(Px)(t_2)| \leqslant M|t_1-t_2|\,.$$

Hence P maps G into itself.

To prove that P is continuous let $x \in G$ and let $\mu > 0$ be given. We must find $\eta > 0$ such that $y \in G$ and $\|x-y\| < \eta$ implies $\|Px-Py\| \leqslant \mu$.

Now
$$|(Px)(t)-(Py)(t)|$$
$$\leqslant \int_{t_0}^{t} |h(s,\ x(s),\ Ax(s)) - h(s,\ y(s),\ Ay(s))|\,ds + \int_{t_0}^{t}\int_{t_0}^{u} |q(u,\ s,\ x(s)) - (u,\ s,\ y(s))|\,ds\,du.$$

But since q is uniformly continuous so for the $\mu > 0$ there is an $\eta > 0$ such that $|x(s)-y(s)| < \eta$ implies
$$|q(u,\ s,\ x(s)) - q(u,\ s,\ y(s))| < \frac{\mu}{2\beta^2}\,.$$

Also since h is uniformly continuous and A is continuous, given $\mu > 0$ there is an $\eta > 0$ such that $|x(s)-y(s)| < \eta$ implies
$$|h(s,\ x(s)),\ Ax(s)) - h(s,\ y(s),\ Ay(s))| < \frac{\mu}{2\beta}\,.$$

Thus for $\|x-y\| < \eta$ we have
$$|(Px)(t)-(Py)(t)| \leqslant \frac{\mu\beta}{2\beta} + \frac{\mu\beta^2}{2\beta^2} = \mu\,.$$

Hence P is continuous.

Therefore by Schauder's fixed point theorem, there is a fixed point which is a solution of the equation (1)

Remark 1. In the above proof take the set G as it is and define
$$H = \{x \in X\colon\ |x-\phi(t_0)| \leqslant \beta M\} \qquad P\colon\ H \to G\,.$$

Clearly

$$|(Px)(t) - \phi(t_0)| \leqslant \beta M \qquad |(Px)(t_1) - (Px)(t_2)| \leqslant M|t_1 - t_2|.$$

Hence P: $H \to G$, and $P(H) \subset G$. Here G is compact and convex. Since P is continuous and $P(H)$ contained in a compact set G, P is a compact mapping. Therefore by another version of the Schauder's fixed point theorem P has a fixed point.

4 - Continuous dependence

Let X denote the set of continuous functions ϕ: $(-\infty, 0] \to R^n$: (i) $|\phi(t)|$ denote the Euclidean length of $\phi(t)$; (ii) $\|\phi\| = \sup_{-\infty < t \leqslant 0} |\phi(t)|$, if it exists; (iii) if g: $(-\infty, 0] \to (0, \infty)$ is continuous, then

$$|\phi|_g = \sup_{-\infty < t \leqslant 0} |\phi(t)/g(t)| \quad \text{if it exists.}$$

Let $(X, |\cdot|_g)$ be the Banach space of continuous functions ϕ. Let ρ be a metric as defined in [2] and let (X, ρ) be a locally convex topological vector space.

Now we shall assume the following

Assumption 1. There is a continuous function g: $(-\infty, 0] \to [0, \infty)$, $g(0) = 1$, $g(r) \to \infty$ as $r \to -\infty$ and g is decreasing such that $[\phi \in X,\ |\phi| \leqslant \gamma g(s)$ for some $\gamma > 0$ and $-\infty < s \leqslant 0$ and $t \geqslant 0]$ imply that $\int_{-\infty}^{0} q(t, s, \phi(s))\, ds$ is continuous.

Def. 1. Equation (1) is said to have a *fading memory* if for each $\varepsilon > 0$ and for each $B > 0$, there exists $K > 0$ such that $[\phi \in X,\ \|\phi\| \leqslant B,\ t \geqslant 0]$ imply that $\int_{-\infty}^{-K} |q(t, s, \phi(s))|\, ds < \varepsilon$.

Def. 2. Solutions of (1) are *g-uniform bounded at* $t = 0$ if for each $B_1 > 0$ there exists $B_2 > 0$ such that $[\phi \in X,\ |\phi|_g \leqslant B_1,\ t \geqslant 0]$ imply that $|x(t, 0, \phi)| < B_2$.

Def. 3. Solutions of (1) are *g-uniform ultimate bounded for bound B at* $t = 0$ if for each $B_3 > 0$ there is a $K > 0$ such that $[\phi \in X,\ |\phi|_g \leqslant B_3,\ t \geqslant K]$ imply that $|x(t, 0, \phi)| < B$.

Theorem 3. *Let the equation* (1) *have a fading memory; let its solutions be g-uniform bounded; let h satisfy a local Lipschitz condition in its arguments; let q satisfy a Lipschitz condition of the following type: for each* $H>0$, *each* $J>0$ *there exists* $M>0$ *such that* $|x_i| \leqslant H$ *and* $-\infty \leqslant s \leqslant t \leqslant J$ *imply that*

$$|q(t,\ s,\ x_1)-q(t,\ s,\ x_2)| \leqslant M|x_1-x_2|.$$

If S is any bounded (*sup norm*) *subset of X, then solutions of* (1) *are continuous in* ϕ *relative to S and* ρ.

Proof. Let $B_1>0$ be given and find $B_2>0$ such that $[\phi \in X,\ |\phi|_g < B_1,\ t \geqslant 0]$ imply that $|x(t,\ 0,\ \phi)| < B_2$. Define $S = \{\phi \in X/\|\phi\| < B_1\}$ and $H = B_2$. Let $J>0$, $\varepsilon>0$ and $\phi \in S$ be given. It will suffice to find $\delta>0$ such that $[\psi \in S,\ \rho(\phi,\ \psi) < \delta]$ imply $|x(t,\ 0,\ \phi)-x(t,\ 0,\ \psi)| < \varepsilon$ for $0 \leqslant t \leqslant J$. For $\varepsilon_1>0$ satisfying $2\varepsilon_1 J\, e^{[M(1+J)+LN\alpha]J} < \varepsilon/2$ find $D>0$ such that $[\phi \in S,\ t \geqslant 0]$ imply that $\int_{-\infty}^{-D} |q(t,\ s, \phi(s))|\, ds < \varepsilon_1$. Here M and L are Lipschitz constants for h and q when $|x_i| \leqslant H$ and $-\infty \leqslant s \leqslant t < J$. Let $\psi \in S$ and $\rho(\phi,\ \psi) < \delta$ so that $|\phi(t)-\psi(t)| < K\delta$ for some $K>0$ when $-D \leqslant t \leqslant 0$ and $K\delta(1+MJD)\, e^{[M(1+J)+LN\alpha]J} < \varepsilon/2$.

Let $x_1(t) = x(t,\ 0,\ \phi)$ and $x_2(t) = x(t,\ 0,\ \psi)$. Then

$$\dot{x}_1(t)-\dot{x}_2(t) = h(t,\ x_1,\ Ax_1)-h(t,\ x_2,\ Ax_2)+\int_{-\infty}^{t}[q(t,\ s,\ x_1(s))-q(t,\ s,\ x_2(s))]ds$$

$$x_1(t)-x_2(t) = \phi(0)-\psi(0)+\int_0^t[h(s,\ x_1(s),\ Ax_1(s))-h(s,\ x_2(s),\ Ax_2(s)]\, ds$$

$$+\int_0^t\int_{-\infty}^{u}[q(u,\ s,\ x_1(s)-q(u,\ s, x_2(s))]\, ds\, du$$

$$|x_1(t)-x_2(t)| \leqslant |\phi(0)-\psi(0)|+\int_0^t |h(s,\ x_1,\ Ax_1)-h(s,\ x_2, Ax_2)|\, ds$$

$$+\int_0^t\int_{-\infty}^{-D} |q(u,\ s,\ \phi(s))-q(u,\ s,\ \psi(s))|\, ds\, du+\int_0^t\int_{-D}^{u} M|x_1(s)-x_2(s)|\, ds\, du$$

$$\leqslant K\delta+M\int_0^t |x_1(s)-x_2(s)|\, ds+L\int_0^t |Ax_1(s)-Ax_2(s)|\, ds+2\xi_1 J$$

$$+\int_0^t\int_{-D}^{0} M|\phi(s)-\psi(s)|\, ds\, du+\int_0^t\int_0^{u} M|x_1(s)-x_2(s)|\, ds\, du$$

$$\leqslant K\delta + M\int_0^t |x_1(s) - x_2(s)|\, ds + L\int_0^t |Ax_1(s) - Ax_2(s)|\, ds$$

$$+2\varepsilon_1 J + K\delta MDJ + M\int_0^t (t-s)|x_1(s) - x_2(s)|\, ds$$

$$\leqslant K\delta + M(1+J)\int_0^t |x_1(s) - x_2(s)|\, ds + LN\alpha\int_0^t |x_1(s) - x_2(s)|\, ds + 2\varepsilon_1 J + K\delta MDJ$$

$$\left(\text{where} \sup_{t_0 \leqslant t \leqslant J} \int_{t_0}^{t} a(s)\, ds = \alpha\right)$$

$$\leqslant K\delta(1+MDJ) + 2\varepsilon_1 J + [M(1+J) + LN\alpha]\int_0^t |x_1(s) - x_2(s)|\, ds$$

(by Gronwall's inequality)

$$\leqslant [K\delta(1+MDJ) + 2\varepsilon_1 J]\; e^{[M(1+J)+LN\alpha]J} < \varepsilon .$$

Therefore $|x_1(t) - x_2(t)| < \varepsilon$, hence the proof.

Remark 2. The fading memory definition can also be altered as $[\phi \in X,\ |\phi|_g \leqslant B,\ t \geqslant 0]$ imply that $\int_{-\infty}^{-K} |q(t, s, \phi(s)|\, ds < \varepsilon$. With this change the above theorem can be proved when (X, ρ) is replaced by $(X, |\cdot|_g)$.

5 - Periodicity

Most investigations into the existence of periodic solutions of differential equations require that one can verify that $x(t+T)$ is a solution whenever $x(t)$ is a solution. It is easy to verify that a sufficient condition for that property to hold for (1) is that

$$h(t+T,\ x,\ Ax) = h(t,\ x,\ Ax) \qquad q(t+T,\ s+T,\ x) = q(t,\ s,\ x).$$

The basic idea is to find a set S of initial functions and define a mapping $P\colon S \to S$ by $\phi \in S$ implies

$$P\phi = x(t+T,\ 0,\ \phi) \qquad \text{for } -\infty < t \leqslant 0 .$$

Thus if P has a fixed point ϕ then $x(t+T,\ 0,\ \phi)$ is a solution which has initial fuction ϕ and by uniqueness $x(t,\ 0,\ \phi) = x(t+T,\ 0,\ \phi)$.

Now assume the following conditions: (a) If $\phi \in X$ and $|\phi|_g \leqslant \gamma$ for any $\gamma > 0$, then there is a unique solution $x(t, 0, \phi)$ on $[0, \infty)$. (b) Solutions of (1) are g-uniform bounded and g-uniform ultimate bounded for bound B at $t = 0$. (c) For each $\gamma > 0$ there is an $L > 0$ such that $|\phi(t)| \leqslant \gamma$ on $(-\infty, 0]$ implies that $|\dot{x}(t, 0, \phi)| \leqslant L$ on $[0, \infty)$. (d) For each $\gamma > 0$, if $U = \{\phi \in X / |\phi(t)| \leqslant \gamma$ on $(-\infty, 0]\}$ then solutions of (1) depend continuously on ϕ in U relative to (X, ρ). (e) If $x(t)$ is a solution of (1) on $[0, \infty)$ so is $x(t+T)$.

Theorem 4. *Under the assumptions* (a)-(e) *the equation* (1) *has an mT-periodic solution for some positive integer m.*

Theorem 5. *Let the conditions of Theorem* (4) *hold with (d) modified to require continuity in* $(X, |\cdot|_g)$. *Then* (g) *has a T-periodic solution.*

(f) *If $\phi \in X$, then $x(t, 0, \phi)$ exists on* $[0, \infty)$ *and is continuous in ϕ relative to X and $|\cdot|_g$.*

(g) *For each $H > 0$ there exists $L > 0$ such that $\phi \in X$ and* $|\phi(t)| \leqslant B\sqrt{g(t-H)}$ *on* $(-\infty, 0]$ *imply that* $|\dot{x}(t, 0, \phi)| \leqslant L$ *on* $[0, \infty)$.

Theorem 6. *Under assumptions* (b), (e), (f) *and* (g) *the equation* (1) *has a T-periodic solution.*

The proofs of the Theorems 4.5 and 6 are similar to that of Theorem 4.33, 4.34 and 4.35 in [2] and hence they are omitted.

References

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Abstract

We consider the nonlinear integrodifferential equation with infinite delay

$$\dot{x}(t) = h(t,\ x(t),\ Ax(t)) + \int_{-\infty}^{t} q(t,\ s,\ x(s)\,ds \qquad t \geqslant t_0$$

$$x(t) = \phi(t) \qquad -\infty < t \leqslant t_0 .$$

Existence theorems are proved by using the contraction mapping principle and Schauder's fixed point theorem. Continuous dependence of solutions is studied and theorems related with periodicity of solutions are stated.

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