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A continued fraction and some identities associated with Ramanujan (**)

1 - Introduction

In an earlier paper [4] we have considered a generalization of a *continued fraction of Ramanujan* $[\mathbf{1}]_1$

$$F(a,\ b,\ c,\ \lambda,\ q)=1+\cfrac{(1-\frac{1}{c})(aq+\lambda q)}{(1+\frac{aq}{c})+\cfrac{bq+\lambda q^2}{1+\cfrac{(1-\frac{1}{cq})(aq^2+\lambda q^3)}{(1+\frac{aq}{c})+\cfrac{bq^2+\lambda q^4}{\vdots}}}}$$

where

$$F(a,\ b,\ c,\ \lambda,\ q)=\frac{P(a,\ b,\ c,\ \lambda,\ q)}{P(aq,\ bq,\ c,\ \lambda q,\ q)} \qquad P(a,\ b,\ c,\ \lambda,\ q)=\sum_{n=0}^{\infty}\frac{(-\frac{\lambda}{a})_n(c)_n(-\frac{aq}{c})^n}{(q)_n(-bq)_n}$$

which in the limit gives most of the classical results involving continued fractions. In fact, for $c\to\infty$, it reduces to the *unusual* continued fraction of Ramanujan, which Andrews discovered in the «Lost» manuscript of Ramanujan $[\mathbf{1}]_2$.

In this paper, we consider a special case of this continued fraction and obtain some identities.

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2 – We put $a=0$, $b=0$, $\lambda=1$ and $c=-q$ in our continued fraction to obtain

$$(2.1)\quad C(-q;\,q)=1+\cfrac{(1+\frac{1}{q})q}{1+\cfrac{q^2}{1+\cfrac{(1+\frac{1}{q^2})q^3}{1+\cfrac{q^4}{\ddots}}}}=\sum_{n\geq 0}\frac{q^{\frac{n^2-n}{2}}(-q)_n}{(q)_n}\Big/\sum_{n\geq 0}\frac{q^{\frac{n^2+n}{2}}(-q)_n}{(q)_n}.$$

Using the summation formula Slater [3] (eqn. (8) and (13)) viz.

$$\sum_{n\geq 0}\frac{q^{\frac{n^2+n}{2}}(-q)_n}{(q)_n}=\frac{(-q)_\infty}{(q)_\infty}(q^3;\,q^4)_\infty(q;\,q^4)_\infty(q^4;\,q^4)_\infty$$

and

$$\sum_{n\geq 0}\frac{q^{\frac{n^2-n}{2}}(-q)_n}{(q)_n}=\frac{(-q)_\infty}{(q)_\infty}[(q^3;\,q^4)_\infty(q;\,q^4)_\infty(q^4;\,q^4)_\infty+(q^2;\,q^4)_\infty(q^4;\,q^4)_\infty]$$

we get

$$(2.2)\qquad C(-q;\,q)-1=\frac{(q^2;q^4)^2_\infty}{(q^3;\,q^4)_\infty(q;\,q^4)_\infty}.$$

3 – We now prove the following

Lemma 3.1.

$$(3.1)\qquad C(-q;\,q)=1+\frac{1}{(q^4;\,q^4)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n q^{6n^2+n}(1+q^{4n+1}).$$

Proof. From (2.2)

$$C(-q;\,q)-1=\frac{(q^2;\,q^4)^2_\infty}{(q^3;\,q^4)_\infty(q;\,q^4)_\infty}$$

$$=\frac{1}{(q^4;\,q^4)_\infty}\cdot\frac{(q^2;\,q^4)_\infty(q^4;\,q^4)_\infty}{(q^3;\,q^4)_\infty(q;\,q^4)_\infty}=\frac{1}{(q^4;\,q^4)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n q^{6n^2+n}(1+q^{4n+1})$$

by using the quintuple product identity Andrews [1]$_4$ (p. 456, Theorem 3.9), with q replaced by q^4 and then z replaced by q.

4 – We now prove the following four identities with the help of the above Lemma.

$$(4.1)\qquad \sum_{m=0}^{\infty} c_{4m} q^{2m} = \frac{1}{(q^2;q^2)_\infty}\Big[\sum_{n=-\infty}^{\infty}(-1)^n q^{12n^2-n} + q\sum_{n=-\infty}^{\infty}(-1)^n q^{12n^2+7n}\Big]$$

$$(4.2)\qquad \sum_{m=0}^{\infty} c_{4m+1} q^{2m} = \frac{1}{(q^2;q^2)_\infty}\Big[q^3\sum_{n=-\infty}^{\infty}(-1)^n q^{12n^2-13n} - \sum_{n=-\infty}^{\infty}(-1)^n q^{12n^2-5n}\Big]$$

$$(4.3)\qquad \sum_{m=0}^{\infty} c_{4m+2} q^{2m} = \frac{1}{(q^2;q^2)_\infty}\Big[q^{12}\sum_{n=-\infty}^{\infty}(-1)^n q^{12n^2-25n} + q^5\sum_{n=-\infty}^{\infty}(-1)^n q^{12n^2-17n}\Big]$$

$$(4.4)\qquad \sum_{m=0}^{\infty} c_{4m+3} q^{2m} = \frac{1}{(q^2;q^2)_\infty}\Big[q^{27}\sum_{n=-\infty}^{\infty}(-1)^n q^{12n^2-37n} - q^{16}\sum_{n=-\infty}^{\infty}(-1)^n q^{12n^2-29n}\Big].$$

In proving the above identities we shall use the Hecke operator u_4 ([1]$_3$, [3]) operating on $f(q) = \sum_{n\geqslant 0} a_n q^n$, defined as

$$U_4 f(q) = \sum_{n=0}^{\infty} a_{4n} q^n = \frac{1}{4}\sum_{j=0}^{3} f(\rho^j q^{\frac{1}{4}}) \qquad \text{where} \quad \rho = e^{2\pi i/4}.$$

Now, for $0 \leqslant \alpha \leqslant 3$, by Lemma 3.1

$$\sum_{m=0}^{\infty} c_{4m+\alpha} q^m = U_4 q^{-\alpha} C(-q;\ q) = \frac{1}{4(q)_\infty}\sum_{j=0}^{3} q^{-\alpha/4}\Big[\sum_{-\infty}^{\infty}(-1)^n q^{\frac{6n^2-n}{4}} \rho^{\frac{j(12n^2-2n-2\alpha)}{2}}$$

$$+ q^{\frac{1}{4}}\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{6n^2-5n}{4}} \rho^{\frac{j(12n^2-10n+2-2\alpha)}{2}}\Big].$$

Now $12n^2-2n-2\alpha \equiv 0 \pmod 4$, for $n \equiv -\alpha \pmod 2$ and $12n^2-10n-2\alpha \equiv 0 \pmod 4$, for $n \equiv 1-\alpha \pmod 2$. Hence

$$\sum_{m=0}^{\infty} c_{4m+\alpha} q^m = \frac{1}{(q)_\infty}\Big[\sum_{n=-\infty}^{\infty}(-1)^\alpha q^{\frac{6}{4}(2n-\alpha)^2 - \frac{2n-\alpha}{4} - \frac{\alpha}{4}}$$

$$+q^{\frac{1}{4}} \sum_{n=-\infty}^{\infty} (-1)^{\alpha} q^{\frac{6}{4}(2n+1-\alpha)^2 - \frac{5}{4}(2n+1-\alpha) - \frac{\alpha}{4}}]$$

$$= \frac{1}{(q)_\infty} [q^{\frac{3\alpha^2}{2}} \sum_{n=-\infty}^{\infty} (-1)^{\alpha} q^{\frac{12n^2-(12\alpha+1)n}{2}} + q^{\frac{3(1-\alpha)^2}{2}} - (1-\alpha) \sum_{n=-\infty}^{\infty} (-1)^{\alpha} q^{\frac{12n^2+(7-12\alpha)n}{2}}].$$

Replacing q by q^2, we have

$$\sum_{m=0}^{\infty} c_{4m+\alpha} q^{2m} = \frac{1}{(q^2;\ q^2)_\infty} [q^{3\alpha^2} \sum_{n=-\infty}^{\infty} (-1)^{\alpha} q^{12n^2-(12\alpha+1)n}$$

$$+ q^{3(1-\alpha)^2-2(1-\alpha)} \sum_{n=-\infty}^{\infty} (-1)^{\alpha} q^{12n^2+(7-12\alpha)n}].$$

Putting $\alpha = 0,\ 1,\ 2,\ 3$ we get the identities (4.1)-(4.4), respectively, after writing $-q$ for q.

5 - Two other results

We can obtain two more identities of a similar nature, by applying Hecke operator U_2 to $C(-q;\ q)$. The identies are

$$(5.1) \qquad \sum_{m=0}^{\infty} (-1)^m c_{2m} q^m = \frac{1}{(q^2;\ q^2)} [\sum_{n=-\infty}^{\infty} (-1)^n q^{12n^2-n} + q \sum_{n=-\infty}^{\infty} (-1)^n q^{12n^2+7n}]$$

$$(5.2) \qquad \sum_{m=0}^{\infty} (-1)^m c_{2m+1} q^m$$

$$= \frac{1}{(q^2;\ q^2)_\infty} [q^3 \sum_{n=-\infty}^{\infty} (-1)^n q^{12n^2-13n} + \sum_{n=-\infty}^{\infty} (-1)^n q^{12n^2-5n}].$$

6 - Two further identities

We shall now prove the following two identities:

$$(6.1) \qquad \sum_{m=0}^{\infty} (-1)^m c_{2m} q^m = \frac{(-q;\ q)_\infty (q^3;\ q^8)_\infty (q^5;\ q^8)_\infty (q^4;\ q^4)_\infty}{\sum_{n=-\infty}^{\infty} (1)^n q^{4n^2}}$$

$$(6.2) \qquad \sum_{m=0}^{\infty} (-1)^m c_{2m+1} q^m = \frac{(-q;\, q)_\infty (q;\, q^8)_\infty (q^7;\, q^8)_\infty (q^4;\, q^4)_\infty}{q^3 \sum_{n=-\infty}^{\infty} (-1)^n q^{4n^2}}.$$

Proof of (6.1). By (5.1)

$$\sum_{m=0}^{\infty} (-1)^m c_{2m} q^m = \frac{1}{(q^2;\, q^2)_\infty} [\sum_{n=-\infty}^{\infty} (-1)^n q^{12n^2-n} + q \sum_{n=-\infty}^{\infty} (-1)^n q^{12n^2+7n}]$$

$$= \frac{1}{(q^2;\, q^2)} [\sum_{n=-\infty}^{\infty} (-1)^n q^{12n^2-n} (1 + q^{8n+1})].$$

Using the quintuple identity with q replaced by q^8 and putting $z = q$, we have

$$\sum_{m=0}^{\infty} (-1)^m c_{2m} q^m = \frac{1}{(q^2; q^2)_\infty} (-q^7; q^8)_\infty (-q; q^8)_\infty (q^6; q^{16})_\infty (q^{10}; q^{16})_\infty (q^8; q^8)_\infty$$

$$= \frac{(-q^7; q^8)_\infty (-q; q^8)_\infty (q^3; q^8)_\infty (-q^3; q^8)_\infty (q^5; q^8)_\infty (-q^5; q^8)_\infty (q^4; q^4)_\infty (-q^4; q^4)_\infty}{(q^2; q^2)_\infty}.$$

Using Jacobi's triple product identity [**1**]$_3$ (p. 21)

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{4n^2} = \frac{(q^4;\, q^4)_\infty}{(-q^4;\, q^4)_\infty}$$

we get

$$\sum_{m=0}^{\infty} (-1)^m c_{2m} q^m$$

$$= \frac{(-q^7;\, q^8)_\infty (-q;\, q^8)_\infty (q^3;\, q^8)_\infty (-q^3;\, q^8)_\infty (q^5;\, q^8)_\infty (-q^5;\, q^8)_\infty (q^4;\, q^4)_\infty^2}{(q^2;\, q^2)_\infty \sum_{n=-\infty}^{\infty} (-1)^n q^{4n^2}}$$

$$= \frac{(-q;\, q)_\infty (q^3;\, q^8)_\infty (q^5;\, q^8)_\infty (q^4;\, q^4)_\infty}{\sum_{n=-\infty}^{\infty} (-1)^n q^{4n^2}},$$

which proves (6.1). The proof of (6.2) follows in a similar way.

7 - Three identities for $C(-q;\ q)$

In this section we shall prove three identities for continued fraction $C(-q;\ q)$. These are

$$(7.1)\qquad (q^4;\ q^4)_\infty[C(-q;\ q)-1]=\sum_{n=0}^{\infty}\frac{(-1)^n q^{2n^2}}{(q;\ q^4)_{n+1}}-\sum_{n=0}^{\infty}\frac{(-1)^n q^{2n^2+4n+2}}{(q^3;\ q^4)_{n+1}}.$$

$$(7.2)\qquad \sum_{n=0}^{\infty}\frac{(-1)^n q^{2n^2+4n+2}}{(q^3;q^4)_{n+1}}$$

$$=-\frac{1}{2}(q^4;\ q^4)_\infty[c(-q;\ q)-1]+\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n q^{6n^2+n}(1+q^{10n+5})$$

$$+\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n q^{6n^2+5n+1}(1+q^{2n+1}).$$

$$(7.3)\qquad \sum_{n=0}^{\infty}\frac{(-1)^n q^{2n^2}}{(q;q^4)_{n+1}}$$

$$=\frac{1}{2}(q^4;\ q^4)_\infty[c(-q;\ q)-1]+\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n q^{6n^2+n}(1+q^{10n+5})$$

$$+\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n q^{6n^2+5n+1}(1+q^{2n+1}).$$

Proof of (7.1). By the Lemma, we have

$$(q^4;\ q^4)[C(-q;q)-1]=\sum_{n=-\infty}^{\infty}(-1)^n q^{6n^2+n}(1+q^{4n+1})$$

$$=\sum_{n=-\infty}^{\infty}(-1)^n q^{6n^2+n}+\sum_{n=-\infty}^{\infty}(-1)^n q^{6n^2-5n+1}$$

$$=\sum_{n=-\infty}^{\infty}(-1)^n q^{6n^2+n}-\sum_{n=-\infty}^{\infty}(-1)^n q^{6n^2-7n+2}$$

$$=\sum_{n=0}^{\infty}(-1)^n q^{6n^2+n}(1+q^{4n+1})-\sum_{n=0}^{\infty}(-1)^n q^{6n^2+7n+2}(1+q^{4n+3}).$$

Applying Rogers-Fine identity [2] (p. 334, eq. (1))

$$\sum_{n=0}^{\infty} \frac{(a)_n t^n}{(b)_{n+1}} = \sum_{n=0}^{\infty} \frac{(a)_n (\frac{at}{b})_n b^n t^n q^{n^2} (1 - atq^{2n})}{(b)_{n+1} (t)_{n+1}}$$

to the first sum with q replaced by q^4 and putting $a = q^2/t$, $b = q$ and then letting $t \to 0$, and to the second sum with q replaced by q^4 and putting $a = q^6/t$, $b = q^3$ and then letting $t \to 0$, we get

$$(q^4;\, q^4)_\infty [C(-q;\, q) - 1] = \lim_{t \to 0} [\sum_{n=0}^{\infty} \frac{(\frac{q^2}{t};\, q^4) t^n}{(q;\, q^4)_{n+1}} - q^2 \sum_{n=0}^{\infty} \frac{(\frac{q^6}{t};\, q^4)_n t^n}{(q^3;\, q^4)_{n+1}}]$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n q^{2n^2}}{(q;\, q^4)_{n+1}} - \sum_{n=0}^{\infty} \frac{(-1)^n q^{2n^2+4n+2}}{(q^3;\, q^4)_{n+1}}$$

which prove (7.1).

Proof of (7.2). By (7.1)

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{2n^2+4n+2}}{(q^3;\, q^4)_{n+1}} = -(q^4;\, q^4)_\infty [C(-q;\, q) - 1] + \sum_{n=0}^{\infty} \frac{(-1)^n q^{2n^2}}{(q;\, q^4)_{n+1}}$$

$$= -\frac{1}{2}(q^4;\, q^4)_\infty [C(-q;\, q) - 1] + \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n q^{6n^2+n} (1 + q^{4n+1})$$

$$+ \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n q^{6n^2+7n+2} (1 + q^{4n+3})$$

$$= -\frac{1}{2}(q^4;q^4)_\infty [C(-q;\, q) - 1] + \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n q^{6n^2+n} (1 + q^{10n+5})$$

$$+ \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n q^{6n^2+5n+1} (1 + q^{2n+1})$$

which proves (7.2).

Proof of (7.3). Adding (7.1) and (7.2)

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{2n^2}}{(q;\ q^4)_{n+1}} = \frac{1}{2}(q^4;\ q^4)_\infty [C(-q;q)-1] + \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n q^{6n^2+n}(1+q^{10n+5})$$

$$+\frac{1}{2} \sum_{n=0}^{\infty} (-1)^n q^{6n^2+5n+1}(1+q^{2n+1})$$

which proves (7.3).

Conclusion. In the identities proved in 4, by considering the coefficients in the power series expansions of $C(-q;\ q)$, a combinatorial interpretation can be given by using a generalization of Gollnitz-Gordon identities $[\mathbf{1}]_3$ (p. 114, Theorem 7.1).

The present paper was motivated by Andrews' treatment of the Rogers-Ramanujan Continued Fraction $[\mathbf{1}]_2$ and the technique employed in the proof is a straight forward modification of his technique.

The continued fraction $C(-q;\ q)$ is a special case of Ramanujan' work (Entry 9 and 13 in chapter 16 of Ramanujan's Second Notebook Memoirs of the AMS 53 (1985) No. 315).

References

[1] G. E. ANDREWS: $[\bullet]_1$ *An introduction to Ramanujan's «Lost» Notebook*, Amer. Math. Monthly **86** (1979), 89-108; $[\bullet]_2$ *Ramanujan's «Lost» Notebook* III. *The Rogers Ramanujan continued fraction*, Adv. in Math., **41** (1981), 186-208; $[\bullet]_3$ *The theory of partitions*, Encyclopedia of Mathematics and Its Applications, Vol. **2**, Addison-Wesley, Reading, Mass., 1976; $[\bullet]_4$ *Applications of basic hypergeometric functions*, SIAM Rev. **16** (1974), 441-484.

[2] L. J. ROGERS, *On two theorems of Combinatory Analysis and some allied identities*, Proc. London Math. Soc. (2), **16** (1916), 315-336.

[3] L. J. SLATER, *Further identities of the Rogers-Ramanujan type*, Proc. London Math. Soc. (2) **54** (1952), 147-167.

[4] BHASKAR SRIVASTAVA, *On a generalization of a continued fraction of Ramanujan* (communicated for publication).

Summary

See Introduction.

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