Riv. Mat. Univ. Parma (4) **17** (1991), 207-210

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Ideals in antiflexible rings (**)

1 - Introduction

A nonassociative ring A is called *antiflexible* in case the following identities hold

(1) $(x, y, z) = (z, y, x)$ (2) $(x, x, x) = 0$

where $(x, y, z) = (xy)z - x(yz)$. Antiflexible rings have been studied by Anderson and Outcalt [1], Celik [2], Rodabough [4] and others.

A straightforward verification shows that any ring satisfies

(T) $(wx, y, z) - (w, xy, z) + (w, x, yz) = w(x, y, z) + (w, x, y)z$

which is known as the Teichmüller identity. Also, it is known [1] that an antiflexible ring with characteristic $\neq 2$ satisfies the following identities:

(2)′ $(x, y, z) + (y, z, x) + (z, x, y) = 0$

(3) $(w, (x, y), z) = 0$

where $(x, y) = xy - yz$.

2 - In what follows, an expression of the form (A, a, b) means the set of all finite sums (x, a, b) for $x \in A$, analogous arguments are meant for other form of similar expressions. Let A be a ring. Then $M = \{m \in A\colon (A, m, A) = (0)\}$ is called

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(**) MR classification: 17A30. – Ricevuto: 21-XII-1990.

the *middle nucleus* of A. (3) implies that

$$(A,\ A) \subseteq M\,. \tag{4}$$

Theorem 1. *Let A be an antiflexible ring with characteristic $\neq 2$, and let R be a right ideal of A.*

(a) *If R is maximal and nil, then R is a two-sided ideal of A.*

(b) *If R is minimal, then either R is a two-sided ideal of A of the ideal generated in A by R is contained in M.*

Proof. Suppose first the right ideal R is maximal and nil. If $aR \nsubseteq R$ for some $a \in A$, we consider $R + aR$. This is a right ideal, since using (2)′ and (1) we have

$$(aR)A \subseteq (a,\ R,\ A) + a(RA) \subseteq (R,\ A,\ a) + (A,\ a,\ R) + aR$$

$$\subseteq (R,\ A,\ a) + (R,\ a,\ A) + aR \subseteq R + aR\,.$$

Thus $R \subsetneqq R + aR$ and R maximal imply

$$A = R + aR\,. \tag{5}$$

Let $a = x_1 + ax_2$ where $x_1, x_2 \in R$. Then n iterations for a in the right side of this equation give $a = x_3 + (((ax_2)\,x_2)\ldots x_2)\,x_2$, where $x_3 \in R$ and x_2 is a factor n times. Now $(A,\ R,\ R) \subseteq (R,\ R,\ A) \subseteq R$ by (1), and so by finite induction we see that $a = x_4 + a(x_2)^n$ where $x_4 \in R$. But since R is nil, $(x_2)^n = 0$ for some n. Thus $a \in R$ which means $aR \subseteq R$, a contradiction. We therefore have $aR \subseteq R$ for all $a \in A$, i.e. R is a two-sided ideal of A.

Let us next assume that the right ideal R is minimal, but not a two-sided ideal. Then there exists an $a \in A$ such that $aR \nsubseteq R$. Let $R' = \{x \in R\colon\ ax \in R\}$. Now by (1) and (2)′ $x \in R'$ imples $xr \in R$ and

$$a(xr) = (ax + xa)\,r - x(ra) + (xr)\,a - x(ar) \in R$$

for all $r \in A$. Thus it follows $R' \subsetneqq R$ is a right ideal, and so by the minimality of R we have $R' = (0)$. Clearly using (2)′ and (1)

$$(A,\ R,\ A) \subseteq (R,\ A,\ A) + (A,\ A,\ R) \subseteq (R,\ A,\ A) \subseteq R\,. \tag{6}$$

By (1), (T) and (2)′ $a(x,\ r,\ y) = (ax,\ r,\ y) - (a,\ xr,\ y) + (a,\ x,\ ry)$, $(a,\ x,\ r)\,y = -(r,\ y,\ ax) - (r,\ ax,\ y) - (a,\ xr,\ y) + (ry,\ x,\ a) - (r,\ x,\ a)\,y$ and by (3)

$(a, xr, y) = (a, rx, y) = -(rx, y, a) - (y, a, rx) = -(rx, y, a) - (rx, a, y)$. So $a(x, r, y) \in R$. This implies that $(A, R, A) \subseteq R' = (0)$, i.e. $R \subseteq M$.

We next set $W_0 = R$ and $W_{i+1} = W_i + AW_i$ for $i \geqslant 0$. Suppose W_i is a right ideal of A contained in M. Then $W_{i+1}A \subseteq (W_i + AW_i)A \subseteq W_i + (AW_i)A \subseteq W_i + A(W_iA) \subseteq W_i + AW_i = W_{i+i}$, i.e. W_{i+1} is a right ideal. Also, using (3) and $W_i \subseteq M$, $(A, W_{i+1}, A) = (A, W_i, A) + (A, AW_i, A) \subseteq (A, W_iA, A) \subseteq (A, W_i, A) = (0)$, i.e. $W_{i+1} \subseteq M$. Thus it follows by induction that each W_i is a right ideal contained in M. Since the ideal generated in A by R is simply $\bigcup_{i=0}^{\infty} W_i$, this completes the proof of the theorem.

A right ideal R of A is called *regular* if there exists an element $g \in A$, such that $x - gx \in R$ for all $x \in A$. A is called *primitive* if it contains a regular maximal right ideal, which contains no two-sided ideal of A other than the zero ideal (0). Define an ideal P of A to be a *primitive ideal* if the ring A/P is a primitive ring. The intersection of all regular maximal right ideals in A is called the *radical of A* and is denoted by $\mathrm{rad}\, A$.

THEOREM 2. *Let A be an antiflexible ring with characteristic $\neq 2$. Then* $\mathrm{rad}\, A$ *is contained in P for any primitive ideal P of A.*

PROOF. Suppose that P is a primitive ideal of A. A/P is a primitive ring. Therefore by Theorem 3.5 in [2] A/P is either a simple ring with an identity element or it is an associative ring. In either case $\mathrm{rad}\, A/P = (0)$.

If A/P is simple then by Lemma 3.1 in [1], A/P has no one sided proper ideals. If A/P is associative, by Theorems 6.16 and 6.20(a) in [3], the intersection of all the regular maximal right ideals of the ring A/P is zero. But the regular maximal right ideals of the ring A/P are of the form P_i/P where P_i is a regular maximal right ideal of the ring $A \supseteq P$. Let $\{P_i\colon i \in \Gamma\}$ be the set of all the regular maximal right ideals of the ring $A \supseteq P$. Then we have

$$\bigcap_{i \in \Gamma} \left(\frac{P_i}{P}\right) = (0) = \text{zero ideal of } \frac{A}{P} = P.$$

This implies that $\dfrac{\bigcap_{i \in \Gamma} P_i}{P} = P$. Therefore, $\bigcap_{i \in \Gamma} P_i \subseteq P$. That is, $\mathrm{rad}\, A$ is contained in P for any primitive ideal P of A.

References

[1] C. T. ANDERSON and D. L. OUTCALT, *On simple antiflexible rings*, J. Algebra **10** (1968), 310-320.

[2] H. A. CELIK, *On primitive and prime antiflexible rings*, J. Algebra **21** (1972), 428-440.

[3] N. H. MCCOY, *The theory of rings*, Macmillan, 1964.

[4] D. J. RODABAUGH, *A generalization of the flexible law*, Trans. Amer. Math. Soc. **114** (1965), 468-487.

Abstract

Let R be a right ideal of an antiflexible ring A with characteristic $\neq 2$. If R is maximal and nil, then R is a two-sided ideal. If R is minimal then it is either a two-sided ideal, or the ideal it generates is contained in the middle nucleus of A, $\operatorname{rad} A$ *is contained in P for any primitive ideal P of A.*

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