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MICHAEL G. VOSKOGLOU (*)

Prime and semiprime ideals of skew Laurent polynomial rings (**)

1 - Preliminaries

All the rings considered in this paper are with identities.

Let R be a ring and let $H = \{f_1, \ldots, f_n\}$ be a finite set of automorphisms of R. Then an ideal I of R is called a *H-ideal* if $f_i(I) = I$, for all f_i in H. Furthermore a H-ideal I of R is called a *H-prime ideal* if, given any two H-ideals A and B of R such that $AB \subseteq I$, is either $A \subseteq I$ or $B \subseteq I$ and R is called a *H-prime ring* if (0) is a H-prime ideal of R.

In the same way, a H-ideal I of R is called a *H-semiprime ideal* if, given any H-ideal A of R such that $A^k \subseteq I$ for some positive integer k, is $A \subseteq I$ and R is called a *H-semiprime ring* if (0) is a H-semiprime ideal of R. It turns easily out that in the definition above one can take always $k = 2$ (cf. $[4]_4$ Lemma 1.1).

Assume now that $f_i \circ f_j = f_j \circ f_i$, for all $i, j = 1, \ldots, n$ and consider the set S_n of all polynomials in n variables, say $x_1, \ldots, x_n$, over R.

Define addition in S_n in the usual way and define multiplication by the relations $x_i r = f_i(r) x$ and $x_i x_j = x_j x_i$ for all r in R and all $i, j = 1, \ldots, n$. Then, as a consequence of Theorem 2.4 of $[4]_1$ S_i becomes a skew polynomial ring over S_{i-1} (cf. [2], p. 35) for each $i = 1, \ldots, n$, where $S_0 = R$.

We call the ring constructed above a *skew polynomial ring in n variables over R* (by automorphisms) and we denote it by $S_n = R[x_1, f_1] \ldots$

(*) Indirizzo: Technological and Educational Institute (T.E.I), New Buildings, GR-30200 Mesolongi.

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$\dots [x_n, f_n] = R[x, H]$. Notice that, under these conditions, f_i extends to an automorphism of S_n by $f_i(x_j) = x_j$, $j = 1, \dots, n$, for al f_i in H (cf. $[4]_1$ Theorem 2.2).

In $[4]_2$ $[4]_3$ and $[4]_4$ we study relations among the prime (semiprime) ideals of R and those of S_n in a more general context.

It is easy now to check that the set $C\{x_1^{a_1} \dots x_n^{a_n},\ a_1, \dots, a_n \in Z_0^+\}$ (Z_0^+ the set of non negative integers) is an ore subset (cf. [3], p. 170) of S_n. We call the *quotient ring T_n of S_n with respect to C* a skew Laurent polynomial ring in n variables over R and we denote it by $T_n = R[x, x^{-1}, H]$.

For reasons of brevity we write $x^{(a)}$ instead of $x_1^{a_1} \dots x_n^{a_n}$. Then the elements of T_n can written in the form $hx^{-(a)}$, with h in S_n and $x^{(a)}$ in C.

2 - Main results

We need first the following two lemmas.

Lemma 2.1. (i) *If I is a H-ideal of S_n, then $I \cap R$ is a H-ideal of R.*

(ii) *If A is a H-ideal of R, then AS_n is a H-ideal of S_n.*

Proof. $x_i A \subseteq f_i(A) x_i \subseteq Ax_i \subseteq AS_n$, for each $i = 1, \dots, n$, therefore AS_n is an ideal of S_n. The rest of the proof is a straightforward consequence if the way in which f_i extends to an automorphism of S_n, for all f_i in H (see **1**).

Lemma 2.2. (i) *If I is an ideal of T_n, then $I \cap S_n$ is a H-ideal of S_n.*

(ii) *If A is a H-ideal of S_n, then AT_n is an ideal of T_n.*

Proof. (i) Let h be $I \cap S_n$, then $x_i h = f_i(h) x_i$ for each $i = 1, \dots, n$ and therefore $f_i(h) = x_i h x_i^{-1}$ is in I.

Conversely, put $f_i^{-1}(h) = h'$, then $x_i h' x_i^{-1} = f_i(h') = h$ and therefore $f_i^{-1}(h) = x_i^{-1} h x_i$ is in I.

(ii) It is obvious that AT_n is a right ideals of T_n.To show that AT_n is a left ideal of T_n it is enough to show that $x^{-(a)} h$ belongs to AT_n, for all h in A. But it si clear hat this happens if $x_1^{-1} \dots x_n^{-1} h$, $x_2^{-1} \dots x_n^{-1} h$, $\dots$, $x_n^{-1} h$ are all in AT_n. For this, since $f_n(A) = A$, $f_n^{-1}(h) = x_n^{-1} h x_n$ is in A, therefore $f_n^{-1}(h) x_n^{-1} = x_n^{-1} h = h'$ is in A. Repeat the same argument for h', x_{n-1} and f_{n-1} to show that $x_{n-1}^{-1} x_n^{-1} h$ is in A and keep going in the same way until you show that $x_1^{-1} \dots x_n^{-1} h$ is in A.

We are ready now to show

Theorem 2.3. *Let P be a prime (semiprime) ideal of T_n, then $P \cap R$ is a H-prime (semiprime) ideal of R.*

Proof. Since P is an ideal of T_n, by Lemma (2.2)(i), $P \cap S_n$ is H-ideal and therefore, by Lemma 2.1 (i), $(P \cap S_n) \cap R = P \cap R$ is a H-ideal of R.

Assume first that P is a prime ideal of T_n and let A and B be any H-ideals of R, such that $AB \subseteq P \cap R$. Then, by Lemma 2.1 (ii), AS_n and BS_n are H-ideals of S_n and therefore, by Lemma 2.2 (ii), $(AS_n)T_n = AT_n$ and BT_n are ideals of T_n.

But, since B is a H-ideals of R, is $T_nB \subseteq BT_n$. For this, observe that, for all b in B and each $i = 1, \ldots, n$, is $x_i b = f_i(b) x_i$ and $x_i^{-1} b = f_i^{-1}(b) x_i$ (for the second relation work as in the proof of Lemma 2.2 (i) for h).

Thus $(AT_n)(BT_n) = A(T_nB)T_n \subseteq A(BT_n)T_n = (AB)T_n \subseteq P$ and therefore is either $AT_n \subseteq P$, or $BT_n \subseteq P$, fact which shows that $A = AT_n \cap R \subseteq P \cap R$, or $B \subseteq P \cap R$.

Next, assuming that P is a semiprime ideal of T_n, set $A = B$ and repeat the previous argument, to show that $P \cap R$ is a H-semiprime ideals of R.

Theorem 2.4. *Let I a H-prime ideal of R, then IT_n is an ideal of T_n having the folloving property: Given any ideals A and B of T_n such that $AB \subseteq IT_n$, is either $A \cap R \subseteq I$, or $B \cap R \subseteq I$; furthermore, if $A \cap R \neq I$ and $B \cap R \neq I$, is either $A \subset IT_n$, or $B \subset IT_n$.*

Proof. Since I is a H-ideal of R, by Lemma 2.1(ii), IS_n is a H-ideal of S_n and therefore, by Lemma 2.2(ii), $(IS_n)T_n = IT_n$ is an ideal of T_n.

Also, by Lemma 2.2(i), $A \cap S_n$ and $B \cap S_n$ are H-ideals of S_n and therefore, by Lemma 2.1(i), $A \cap R$ and $B \cap R$ are H-ideals of R. Then $(A \cap R)(B \cap R) \subseteq (AB) \cap R \subseteq IT_n \cap R = I$ and therefore is either $A \cap R \subseteq I$, or $B \cap R \subseteq I$.

Without loss of the generality assume that $A \cap R \subseteq I$, then, if $A \supseteq IT_n$, $A \cap R \supseteq IT_n \cap R = I$ and therefore $A \cap R = I$, fact which contradicts our hypothesis.

Theorem 2.5. *Let I be a H-semiprime ideal of R, then IT_n is an ideal of T_n having the following property: Given any ideal A of T_n such that $A^2 \subseteq IT_n$, is $A \cap R \subseteq I$; furthermore, if $A \cap R \neq I$, is $A \subset IT_n$.*

Proof. Set $A = B$ and repeat the proof of Theorem 2.4.

Corollary 2.6. *Let T_n be a prime (semiprime) ring, then R is a H-prime (semiprime) ring. Conversely if R is a H-prime (semiprime) ring and $AB = (0)$ ($A^2 = (0)$) with A, B ideals of T_n, is either $A \cap R = (0)$, or $B \cap R = (0)$ ($A \cap R = (0)$).*

Proof. Apply Theorems 2.3, 2.4 and 2.5 for $I = (0)$.

Moreover we prove

Theorem 2.7. *Let R be a $\{f_1\}$-prime ring, then T_n is a prime ring.*

Proof. Since P is a $\{f_1\}$-prime ring, $T_1 = R[x_1, x_1^{-1}; f_1]$ is a prime ring (cf. [1] Lemma 1.4). But (0) is obviously a $\{f_2\}$-ideal of T_1, therefore T_1 is a $\{f_2\}$-prime ring. Also $S_2 = S_1[x_2, f_2]$ and therefore it is easy check that $T_2 \cong T_1[x_2, x_2^{-1}, f_2]$. Thus, repeating the previous argument, we find that T_2 is a prime ring.

We keep going in the same way until we find, after n steps, that T_n is a prime ring.

At this point we recall that a ring R is said to be a *(left) Goldie ring* if R satisfies the ascending chain condition on (left) annihilators and R contains no infinite direct sums or left ideals.

We close this section by proving the following

Theorem 2.8. *If one of the ring R, S_n and T_n is a semiprime left Goldie ring, then so are orther two.*

Proof. Assume first that R is a semiprime left Goldie ring, then so are S_1 and T_1 (cf. [1], Proposition 2.2) and therefore so are $S_2 = S_1[x_2, f_2]$ and $T_2 \cong T_1[x_2, x_2^{-1}, f_2]$. We keep going in the same way until we find, after n steps, that S_n and T_n are semiprime left Goldie rings as well.

Assume now that S_n is a semiprime left Goldie ring. Then, since $S_n = S_{n-1}[x_n, f_n]$, by the previous, so is S_{n-1}. We keep going in the same way until we find that $S_0 = R$ is as semiprime left Goldie ring. Then, so is T_1 and therefore so are $T_2, T_3, \ldots, T_n$.

Finally if $T_n \cong T_{n-1}[x_n, x_n^{-1}, f_n]$ is a semiprime left Goldie ring, so are T_{n-1}, $T_{n-2}, \ldots, T_1$, therefore so are R and S_1 and therefore so are $S_2, S_3, \ldots, S_n$.

Notice that in [1], Proposition 2.2 speaks about right Goldie rings, but this has to do with the way in which multiplication is defined in T_1 (cf. [1] section 1).

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Abstract

In the present paper we study relations among the prime and semiprime ideals of a given ring R and those of a skew Laurent polynomial ring in finitely many variables over R.

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