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Potential estimate for the obstacle problem relative to the sum of squares of vector fields (**)

1 - Introduction

Recently M. Biroli [1] has studied the properties of local solution of equations of the type

$$Lu = f \qquad f \in L^{\infty}(\Omega) \tag{1}$$

where L is the sum of squares of vector fields, i.e.

$$L = -\sum_{i=1}^{m} X_i^* X_i \tag{2}$$

X_i satisfying a uniform Hörmander condition (that is, X_i and commutators up to a fixed order K span the tangent space) and X_i^* being the formal adjoint of X_i.

He has proved the local Hölder continuity by a potential estimate which very much recalls analogous results given for elliptic and degenerate elliptic equations (see, for example [8], [2], [5]).

In the following we will consider an obstacle problem relative to the sum of squares of vector fields as defined above and we will study the local behaviour of local solutions.

The method we will follow is essentially the same already used in [1]. Its main feature is that a Poincaré inequality has been proved on balls but not on anuli and

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therefore the same technical tools have to be used as in [3], where an obstacle problem for a degenerate elliptic operator was considered.

2 - Notations and preliminaries

Let us consider the following $C^\infty(\Re^N)$ functions $a_{il}(x)$, $b_i(x)$, $c_{ij}(x)$, $c_0(x)$ with $i,j = 1\dots m$; $l = 1\dots N$.

We define the vector fields

$$X_i = \sum_{l=1}^{N} a_{il}(x)\frac{\partial}{\partial x_l} \tag{3}$$

and we suppose that they satisfy a uniform Hörmander condition, as already explained in **1**.

We can introduce a natural intrinsic distance associated to the vector fields by

$$\mathrm{d}(x,\ y) = \inf\{b;\ \gamma\colon\ [0,\ b] \to \Re^N \text{ admissible path with } \gamma(0) = x,\ \ \gamma(b) = y\}$$

where an admissible path is a Lipschitz curve such that

$$\gamma'(t) = \sum_{i=1}^{m} \mathrm{d}_i(t) X_i(\gamma(t)) \qquad \sum_{i=1}^{m} |\mathrm{d}_i(t)|^2 \leqslant 1$$

(see, also, [**10**], [7], [9]).

We then denote

$$B(r,\ x) = \{y | \mathrm{d}(x,\ y) < r\}. \tag{4}$$

The main feature of these balls is that they have different dimensions along the directions defined by the commutators of the vector fields.

The following Poincaré inequality has been proved for intrinsic balls.

$$\int_{B(r,\ x)} |f - f_r|^2 \,\mathrm{d}x \leqslant C r^2 \sum_{i=1}^{m} \int_{B(2r,\ x)} |X_i(f)|^2 \,\mathrm{d}x \tag{5}$$

for every function $f \in C^\infty(B(2r,\ x))$ where C is a constant which depends only on N and f_r is the average of f on $B(r,\ x)$.

Given an open set Ω, we introduce

$$\xi(u,\ v,\ \Omega) = \sum_{i=1}^{m} \int_{\Omega} X_i(u) X_i(v)\, dx + \int_{\Omega} uv\, dx$$

which gives rise to the norm $[\xi(u,\ v,\ \Omega)]^{1/2}$, as it is easily shown.

By $H^1(\Omega,\ \xi)$ $(H_0^1(\Omega,\ \xi))$ we understand the closure of $C^\infty(\Omega)$, $(C_0^\infty(\Omega))$ for that norm.

An easy extension of inequality (5) has then been proved in [**1**] for $H^1(\Omega,\ \xi)$ $(H_0^1(\Omega,\ \xi))$ function.

Let us now consider the operator

$$L = -\sum_{i=1}^{m} X_i^* X_i + \sum_{i=1}^{m} b_i X_i + \sum_{i,\ j=1}^{m} c_{ij}[X_i,\ X_j] + c_0 \tag{6}$$

where $b_i(x)$, $c_{ij}(x)$, $c_0(x)$ are the functions introduced at the beginning of this paragraph, and the associated bilinear form on $H^1(\Omega,\ \xi) \times H_0^1(\Omega,\ \xi)$

$$a_\Omega(u,\ v) = \sum_{i=1}^{m} \int_{\Omega} X_i(u) X_i(v)\, dx + \sum_{i=1}^{m} \int_{\Omega} b_i X_i(u) v\, dx$$

$$+ \sum_{i,\ j=1}^{m} \int_{\Omega} c_{ij}[X_i,\ X_j](u) v\, dx + \int_{\Omega} c_0 uv\, dx\,.$$

The existence of the Green function for the Dirichlet problem relative to L in $\mathfrak{R}^N$ has been proved in [**10**] and the following estimates are given

$$\Lambda_2 \frac{r^2}{|B(r,\ x)|} \leq G^x(y) \leq \Lambda_1 \frac{r^2}{|B(r,\ x)|} \tag{7}$$

$$|X_{J_1} \ldots X_{J_S} G^x(y)| \leq \Lambda_S \frac{r^{2-S}}{|B(r,\ x)|} \tag{7$'$}$$

where $G^x(y)$ is the Green function relative to L in $\mathfrak{R}^N$ with singularity at x and $y \in \partial B(r,\ x)$, $r \leq \overline{R}$, $\overline{R} > 0$ suitable, $|B(r,\ x)|$ being the volume of the ball according to Lebesgue measure.

$G^x(y)$ is such as

$$a_{\mathfrak{R}^N}(u,\ G^x) = u(x) \qquad \forall u \in C_0^\infty(\mathfrak{R}^N)\,.$$

As $a(u,\ v;\ B(r,\ x))$ is coercive on $H_0^1(B(r,\ x),\ \xi)$, it has been possible on [**1**] to define the Green function relative to L in $B(r,\ x)$ with singularity at x.

The previous estimates are easily replied and in particular we have

$$\Lambda_4 \frac{r^2}{|B(r,\ x)|} \leqslant G^x_{B(r,\ x)}(y) \leqslant \Lambda_3 \frac{r^2}{|B(r,\ x)|} \tag{8}$$

for $y \in \partial B(qr,\ x)$, $q \in (0,\ q_0]$, $q_0 < 1$ suitable and Λ_3, Λ_4 are constants which do not depend on x. Working as in [5] we also have

$$G^x_{B(\overline{R}_0,\ x)} \cong \int_r^{\overline{R}_0} \frac{s^2}{|B(s,\ x)|} \frac{ds}{s} \tag{9}$$

where by $\cong$ we mean the usual equivalence relation and $r = |x - y|$. Finally we have

$$G^x_{B(\overline{R}_0,\ x)} \in L^{1+\varepsilon}(B(\overline{R}_0,\ x)) \tag{10}$$

where $\varepsilon < \dfrac{1}{N+K-1}$, with obvious meanings for K and N.

We also consider the so-called *regularized Green-function relative to L in* $\Re^N$ $G^x_\varrho(y)$, which converges to $G^x(y)$ in suitable spaces (see, also, [1] and [2]).

The previously mentioned coercitivity of $a(u,\ v;\ B(r,\ x))$ on $H^1_0(B(r,\ x),\ \xi)$ allows to define the potential φ of $B(sr,\ x)$ with respect to $B(r,\ x)$, $r \leqslant \overline{R}_0$, as the solution of the problem

$$a(\varphi,\ \varphi - v;\ B(r,\ x)) \leqslant 0 \tag{11}$$

$$\forall v \in H^1_0(B(r,\ x);\ \xi) \qquad v \geqslant 1 \text{ a.e. on } B(sr,\ x)$$

$$\varphi \in H^1_0(B(r,\ x);\ \xi) \qquad \varphi \geqslant 1 \text{ a.e. on } B(sr,\ x).$$

We observe that $L\varphi$ is a measure on $B(r,\ x)$, which belongs to space $H^{-1}(B(r,\ x),\ \xi) = (H^1_0(B(r,\ x),\ \xi))'$ and whose support is contained in $\partial B(sr,\ x)$.

If we consider the duality pairing between $H^{-1}(B(r,\ x),\ \xi)$ and $H^1_0(B(r,\ x),\ \xi)$ by the previous notations we have

$$\langle L\varphi,\ \varphi \rangle = a(\varphi,\ \varphi;\ B(r,\ x)) \tag{12}$$

and this number is defined to be the capacity of $B(sr,\ x)$ with respect to L and $B(r,\ x)$, which we denote by $\mathrm{cap}_L(B(sr),\ B(r))$ or simply by $\mathrm{cap}\,(B(sr))$ if there is no possibility of mistake.

Similarly, for any set $E \subset B(r, x)$, we can define its capacity with respect to L and $B(r, x)$ in just the same way.

The usual definitions about capacity are still valid: in particular if a property depending on $\bar{x} \in S \subset B(r, x)$ holds for every $\bar{x} \in S$ except a subset N of capacity zero, we say that this property holds quasi everywhere (q.e.) in S.

Moreover a function u: $B(r, x) \to [-\infty, +\infty]$ is said to be *quasi continuous* if for every $\varepsilon > 0$ there exists an open set $A \subset B(r, x)$ with $\mathrm{cap}(A, B(r, x)) < \varepsilon$ such that the restriction of u to $B(r, x) - A$ is continuous on $B(r, x) - A$.

Along the same lines of Proposition 1.27 of [5] we can prove that given $u \in H_0^1(B(r, x), \xi)$ there exists $\tilde{u}$ in $H_0^1(B(r, x), \xi)$ with $u = \tilde{u}$ a.e. and $\tilde{u}'$ quasi continuous.

This is called *a quasi continuous representative*. The substantial unicity of the quasi continuous representative is given by the following

Lemma 1. *If $\tilde{u}_1$ and $\tilde{u}_2$ belong to $H_0^1(B(r, x), \xi)$, are quasi continuous and agree almost everywhere, they agree quasi everywhere.*

The proof is quite simple. As $\tilde{u}_1$, $\tilde{u}_2 \in H_0^1(B(r, x), \xi)$ and are quasi continuous, it is obvious that $\tilde{u}_1 - \tilde{u}_2$ quasi continuous, $\tilde{u}_1 - \tilde{u}_2 \in H_0^1(B(r, x), \xi)$. Therefore, we can reason as in [5] or in [4] and we immediately get to the point.

For arbitrary functions v: $B(r, x) \to [-\infty, +\infty]$ and arbitrary sets $F \subset B(r, x)$, by inf (and sup) we denote the essential infimum (and supremum) with respect to the capacity previously defined.

If we consider functions $v \in H^1(B(r, x), \xi)$ and $B(sr, \bar{x}) \subset B(r, x)$ the condition $v \geq 0$ a.e. in $B(sr, \bar{x})$ and $\tilde{v} \geq 0$ q.e. in $B(sr, \bar{x})$ are equivalent (where by $\tilde{v}$ we understand the quasi continuous representive of v). Then the essential sup and inf on a given $B(sr, \bar{x})$ are defined unambiguously and this will be important in the following.

Let us now get to our obstacle problem. Consider $g \in H^1(B(r, x), \xi)$ and Ψ a function in $B(r, x)$ defined up to set of zero capacity, such that the convex $K = \{v \in H^1(B(r, x), \xi), v - g \in H_0^1(B(r, x), \xi), v \geq \Psi \text{ q.e.}\}$ is not empty.

By standard projection arguments, there exists a unique solution of the following Dirichlet problem

(13) $$u \in H^1(\Omega, \xi) \quad u \geq \Psi \quad \text{q.e. } u - g \in H_0^1(\Omega, \xi)$$

$$a_\Omega(u, u - v; \Omega) \leq 0$$

$$\forall v \in H^1(\Omega, \xi) \quad v \geq \Psi \quad \text{q.e. } v - g \in H_0(\Omega, \xi).$$

In the following, however, we will be interested only in local weak solutions of the obstacle variational inequality, that is in functions $u \in H^1(\Omega, \xi)$ such that

(14) $$u \in H^1(\Omega, \xi) \quad u \geq \Psi \quad \text{q.e. in } \Omega$$

$$a_\Omega(u, u - v; \Omega) \leq 0$$

$$\forall v \in H^1(\Omega, \xi) \quad v \geq \Psi \quad \text{q.e. in } \Omega \quad u - v \in H_0'(\Omega, \xi).$$

Following [9] and [2], given an arbitrary $\varepsilon > 0$ and an arbitrary $\rho > 0$, we consider the one sided level sets of our obstacle Ψ

(15) $$E(\varepsilon, \rho) = E(x_0, \Psi; \varepsilon, \rho) = \{x \in B(\rho/c^*, x_0), \ \Psi \geq \sup_{B(\rho/c^* m, x_0)} \Psi - \varepsilon\}$$

where $m \geq 1$ is a fixed parameter, $c^* > 1$ is a constant, whose meaning will become clear in the proof of Lemma 3, and their relative capacities

(16) $$\delta(\rho) = \delta(\varepsilon, \rho) = \frac{\operatorname{cap}(E(\varepsilon, \rho); B(c^*\rho, x_0))}{\operatorname{cap}(B(\rho, x_0); B(c^*\rho, x_0))}.$$

It is clear that $0 \leq \delta(\varepsilon, \rho) \leq 1 \ \forall \varepsilon, \rho > 0$, and also that $\delta(\varepsilon, \rho)$ is nondecreasing in ε for every fixed ρ.

We define the Wiener modulus for arbitrary $0 < r \leq R$ and $\sigma > 0$ as

(17) $$\omega_\sigma(r, R) = \inf\{\omega > 0: \ \omega \exp \int_r^R \delta(\sigma\omega, \rho) \frac{d\rho}{\rho} \geq 1\}.$$

As $\delta(\sigma\omega, \rho)$ is nondecreasing in ω for fixed $\sigma > 0$, what we have introduced makes sense and we have $\frac{r}{R} \leq \omega_\sigma(r, R) \leq 1$. Furthermore $\omega_\sigma(r, R)$ is nondecreasing in r and nonincreasing in R and σ.

3 - Main theorem and examples

Let us now define the seminorm $V(r) = V(u, x_0, r)$ for every $r > 0$ such that $B(r, x_0) \subset B(2q^{-1}r, x_0) \subset \Omega$ by setting

$$V(r) = (\operatorname*{osc}_{B(r,\, x_0)} u)^2 + \sum_{i=1}^{m} \int_{B(r,\, x_0)} |X_i(u)|^2 G_{B(2q^{-1}r,\, x_0)}^{x_0} \, dx \tag{18}$$

where $q \in (0, 1/5m)$ is a parameter, which is fixed once for all.

Relying on the previously defined $\delta(\varepsilon, \rho)$ and $\omega_\sigma(r, R)$, the main result is then the following estimate of $V(r)$.

Theorem 1. *Let u be a weak local solution of (14) in $H^1(B(R_0, x_0), \xi)$. We have then*

$$V(r) \leqslant C_1 V(R)\, \omega_\sigma(r, R)^\beta + C_2 \sigma \omega_\sigma(r, R) + C_3 R^\alpha \tag{19}$$

where $\alpha, \beta \in (0, 1)$, $C_1 > 0$, $C_2 > 0$ depend on the coefficients in $L(a_{ij}, b_{ij}, c_{ij}, c_0)$ and C_3 on $\sup_{B(R,\, x)} |u|$. If $c_0 = 0$, we have $C_3 = 0$.

We consider now some special cases.

Let us first recall that for $0 < r \leqslant R$, $\varepsilon > 0$ and $\sigma > 0$ verify

$$\sigma = \varepsilon \exp \int_r^R \delta(\varepsilon, \rho) \frac{d\rho}{\rho}$$

if and only if

$$\omega_\sigma(r, R) = \exp\Big(-\int_r^R \delta(\varepsilon, \rho) \frac{d\rho}{\rho}\Big) \qquad \sigma \omega_\sigma(r, R) = \varepsilon .$$

(See, for example, [3] or [9]).

Making direct use of the definition or by a straightforward application of the previous result, it is now possible to give the expression of the Wiener modulus in some simple cases.

(a) *Constant obstacle.* Let $\Psi =$ constant q.e. on $B(R, x_0)$ for some R, the constant possibly being $-\infty$. Then we have $\omega_\sigma(r, R) \equiv \dfrac{r}{R}$ for every $0 < r \leqslant R$, $\sigma > 0$.

(b) *Continuous obstacle.* If Ψ is continuous at x_0 and if it also happens that for some R $\operatorname{osc}_{B(R,\, x_0)} \Psi > 0$, then, choosing $\sigma = \frac{R}{r} \operatorname{osc}_{B(R,\, x_0)} \Psi$, $0 < r \leq R$, we get

$$\omega_\sigma(r,\, R) = \frac{r}{R} \qquad \sigma\omega_\sigma(r,\, R) = \operatorname{osc}_{B(R,\, x_0)} \Psi .$$

Furthermore we find

$$\operatorname{osc}_{B(r,\, x_0)} u \leq C[(\frac{r}{R})^\beta + \operatorname{osc}_{B(R,\, x_0)} \psi + R^\alpha]^{1/2}$$

and Hölder continuity of Ψ at x_0 implies Hölder continuity for every local solution u at the same point x_0.

(c) *Cylindrical obstacle.* By cylindrical obstacle we mean a function Ψ, whose level sets $E(\varepsilon,\, \rho)$ for every $\rho > 0$ have constant relative capacities, that is $\delta(\varepsilon,\, \rho) = \delta(\rho)$. Then we easily find

$$\omega_\sigma(r,\, R) = \exp(-\int_r^R \delta(\rho) \frac{d\rho}{\rho}) . \tag{20}$$

Moreover if $\delta(\rho) \geq \eta > 0$, we have $\omega_\sigma(r,\, R) \equiv (\frac{r}{R})^\eta$ and (20) implies Hölder continuity of our solution u.

(d) *Thin obstacle.* We now define $F = F_\Psi = \{x \in \Re^N \colon \Psi(x) > -\infty \text{ q.e.}\}$ and for every $\rho > 0$

$$B_\rho^F = B(\rho,\, x_0) \cap F \quad \text{if } \operatorname{cap}(B(\rho,\, x_0) \cap F,\, B(c^*\rho,\, x_0)) > 0$$
$$B_\rho^F = B(\rho,\, x_0) \qquad \text{if } \operatorname{cap}(B(\rho,\, x_0) \cap F,\, B(c^*\rho,\, x_0)) = 0 .$$

We also introduce the quantity

$$W(r,\, R) = W_F(r,\, R) = \exp(-\int_r^R \frac{\operatorname{cap}(B_\rho^F,\, B(c^*\rho,\, x_0))}{\operatorname{cap}(B(\rho,\, x_0),\, B(c^*\rho,\, x_0))} \frac{d\rho}{\rho})$$

for arbitrary $0 < r \leq R$.

It is interesting that $W(r,\, R)$ depends on Ψ only via F_Ψ.

As in [9] we can prove the following estimate of the Wiener modulus

$$(21)\qquad \frac{r}{R} \leqslant \omega_\sigma(r,\ R) \leqslant \min\{1,\ \max[W_F(r,\ R),\ \frac{1}{\sigma} \operatorname*{osc}_{F \cap B(R,\ x_0)} \Psi]\}$$

for arbitrary $0 < r \leqslant R$.

In this case the same relationships between (Hölder) continuity of Ψ at x_0 and (Hölder) continuity of u at the same point as discussed on [3] are still valid.

4 - Introductory lemmas

In the proof of our Wiener estimate, we need an inequality of Caccioppoli type. In our setting it is not difficult to extend the analogous proved in [1].

Lemma 2. *For arbitrary q.e. $z \in \Omega$ and $\overline{R}_0 > 0$ such that $B(\overline{R}_0,\ z) \subset \Omega$ and for arbitrary constant $d \geqslant \sup_{B(r,\ z)} \Psi$, we have for u, local bounded solution of* (14) *in $B(\overline{R}_0,\ x_0)$,*

$$(22)\qquad \sum_{i=1}^{m} \int_{B(qr,\ x_0)} |X_i((u-d)^{\pm})|^2 G_{sr}^{x_0}\,dx + \sup_{B(qr,\ x_0)} |(u-d)^{\pm}|^2$$

$$\leqslant \frac{C_1}{|B(r,\ x_0)|} \int_{B(r,\ x_0)-B(qr,\ x_0)} |(u-d)^{\pm}|^2\,dx + C_2 r^{\alpha}$$

where $0 < q \leqslant q_0$ with $q_0 \in (0,\ 1)$, $0 < r < R_0$, R_0 suitable indipendent on x_0, $s \geqslant 1$, C_1 and C_2 constants dependent on $q(\max_{B(R_0,\ x_0)} |u|^2,\ d)$ and α depends on N and K, order of the Hörmander condition.

It is interesting to observe that $C_2 = 0$ only if $c_0 = 0$ in the operator L.

We can also prove the following refinement of the Poincaré inequality.

Lemma 3. *Let $v \in H^1(B(R,\ x_0), \xi)$ and $N(v) = \{x;\ x \in B(R,\ x_0),\ v(x) = 0\}$. We then have*

$$(23)\qquad \int_{B(R,\ x_0)} |v|^2\,dx \leqslant \frac{K|B(R,\ x_0)|}{\operatorname{cap}(N(v);\ B(c^*R,\ x_0))} \int_{B(c^*R,\ x_0)} \sum_{i=1}^{m} |X_i(v)|^2\,dx\,.$$

Proof. Let us prove Lemma 2. The demonstration is essentially the same as given in [1], the only slight difference being in the test function v.

Precisely we put $v = u - (u-d)^{\pm} G_\rho^z \varphi^2((G_\rho^z)^{-1})$ where G_ρ^z is the regularized Green function relative to L in $\mathfrak{R}^N$ with singularity in z and $\varphi \in C_0^\infty(\mathfrak{R})$ and such that $\varphi(t) = 1$ for $|t| < \lambda$, $\varphi(t) = 0$ for $|t| > \overline{C}\lambda$, $|\varphi'(t)| \leq \dfrac{2}{(\overline{C}-1)\lambda}$.

We choose λ and $\overline{C}$ such that

$$\{x | G^z \geq \frac{1}{\lambda}\} \supset B(2qr,\ z) \qquad \text{and} \qquad \{x | G^z \geq \frac{1}{\overline{C}\lambda}\} \subseteq B((1-q)r,\ z)$$

and then we take the supremum for $z \in B(qr,\ x_0)$.

Let us consider Lemma 3. Let $v \in H^1(B(R,\ x_0),\ \xi)$ and

$$\overline{v} = \frac{1}{|B(R,\ x_0)|} \int_{B(R,\ x_0)} v\, dx\,.$$

Then it is known from (5) that

$$\int_{B(R,\ x_0)} |v - \overline{v}|^2\, dx \leq CR^2 \int_{B(2R,\ x_0)} \sum_{i=1}^{m} |X_i(v)|^2\, dx\,. \tag{24}$$

We now define $N(v) = \{x;\ x \in B(R,\ x_0),\ v(x) = 0\}$ and let us suppose $\overline{v} \neq 0$. (If $\overline{v} = 0$ the result is evident).

Let us choose as test function $\eta(x) = \varphi((G_\rho^{x_0})^{-1})$, where φ is as specified above. We will clearly have $0 \leq \eta \leq 1$. We choose λ and $\overline{C}$ such that $\eta = 1$ in $B(R,\ x_0)$. Accordingly, recalling the estimates previously given for the Green function, we have $\eta \in H_0^1(B(c^* R,\ x_0))$, where c^* takes into account the different dimensions of the spheres which limit $G_\rho^{x_0}$.

Actually $G_\rho^{x_0} \in C_0^\infty(B(c^* R,\ x_0))$, as it is easy to show, but H_0^1 is enough for our purposes.

Let us show that $|X_i(\eta)| \leq K_1/R\ \forall i$. In fact we have

$$X_i(\eta) = X_i(\varphi((G_\rho^{x_0})^{-1})) = \varphi' \frac{1}{(G_\rho^{x_0})^2} X_i(G_\rho^{x_0})\,.$$

Therefore $|X_i(\eta)| = |\varphi'| \dfrac{1}{(G_\rho^{x_0})^2} |X_i(G_\rho^{x_0})|$. Recalling the definition of φ, we then obtain

$$|X_i(\eta)| \leqslant (K_0 \frac{G_\rho^{x_0}}{(G_\rho^{x_0})^2})|X_i(G_\rho^{x_0})| = K_0 \frac{|X_i(G_\rho^{x_0})|}{G_\rho^{x_0}}.$$

When $\rho \to 0$, as we have uniform convergence out of the singularity in x_0 and recalling the estimates for the Green function and its derivatives, we get $|X_i(\eta)| \leqslant \dfrac{K_1}{R}$.

The following function $\psi = \eta(\bar{v} - v)/\bar{v}$ is in $H_0^1(B(c^*R, x_0), \xi)$ and $\psi = 1$ on $N(v)$. We have then, thanks to the bilinearity of $a(u, v)$ on $H_0^1(\Omega, \xi)$,

$$\operatorname{cap}(N(v), B(c^*R, x_0)) \leqslant C \int \sum_{i=1}^{m} |X_i(\psi)|^2 dx$$

$$\leqslant \frac{C}{\bar{v}^2}[\int_{B(c^*R, x_0)} \sum_{i=1}^{m} |X_i(v)|^2 dx + \frac{K_1}{R^2} \int_{B(R, x_0)} |v - \bar{v}|^2 dx]$$

$$\leqslant \frac{K_2}{\bar{v}^2} \int_{B(c^*R, x_0)} \sum_{i=1}^{m} |X_i(v)|^2 dx.$$

Therefore

$$\bar{v}^2 \leqslant \frac{K_2}{\operatorname{cap}(N(v), B(c^*R, x_0))} \int_{B(c^*R, x_0)} \sum_{i=1}^{m} |X_i(v)|^2 dx. \tag{25}$$

From (24) and (25) we obtain

$$\int_{B(R, x_0)} |v|^2 dx \leqslant 2[\int_{B(R, x_0)} |\bar{v}|^2 dx + \int_{B(R, x_0)} |v - \bar{v}|^2 dx]$$

$$\leqslant 2[|\bar{v}|^2 |B(c^*R, x_0)| + CR^2 \int_{B(c^*R, x_0)} \sum_{i=1}^{m} |X_i(v)|^2 dx]$$

$$\leqslant 2[\frac{K_2 |B(R, x_0)|}{\operatorname{cap}(N(v), B(c^*R, x_0))} + \frac{C}{R^2}] \int_{B(c^*R, x_0)} \sum_{i=1}^{m} |X_i(v)|^2 dx.$$

Working along the lines of [6] we can prove that

$$\operatorname{cap}(B(R, x_0), B(c^*R, x_0)) \simeq |B(R, x_0)|/R^2 \tag{26}$$

where by $\simeq$ we mean the usual equivalence relation as in [3]. Finally, working as in Proposition 2 of [2]

$$\int_{B(R,\ x_0)} |v|^2 \,dx \leqslant \frac{K|B(R,\ x_0)|}{\operatorname{cap}(N(v),\ B(c^*R,\ x_0))} \int_{B(c^*R,\ x_0)} \sum_{i=1}^{m} |X_i(v)|^2 \,dx .$$

Let us now prove our main result, Theorem 1.

We choose as test function $v = u - (u-d)^{\pm} G^z_\rho \phi$, $\rho \leqslant sr/2$, where $G^z_\rho = G^z_{B(tr,\ z),\rho}$ is the regularized Green function of $G^z_{B(tr,\ z)}$ which we will denote in the following simply by G^z, and ϕ is the potential of $B(sr,\ z)$ with respect to $B(tr,\ z)$ and $d \geqslant \sup\limits_{B(tr,\ z)} \psi$.

Let us recall that

$$\langle L\phi,\ G^z_\rho \rangle = \frac{1}{|B(\rho,\ z)|} \int_{B(\rho,\ z)} \phi \,dx = 1 .$$

We then have

$$\sum_{i=1}^{m} \int X_i(u)(X_i((u-d)^{\pm})\phi G^z_\rho) \,dx + \sum_{i=1}^{m} \int b_i X_i(u)(u-d)^{\pm} \phi G^z_\rho \,dx$$
$$+ \sum_{i=1}^{m} \int c_{ij} [X_i,\ X_j](u)(u-d)^{\pm} \phi G^z_\rho \,dx + \int c_0 |(u-d)^{\pm}|^2 \phi G^z_\rho \,dx \leqslant 0 . \tag{27}$$

Therefore

$$\sum_{i=1}^{m} \int_{B(tr,\ z)} |X_i((u-d)^{\pm})|^2 \phi G^z_\rho \,dx + \frac{1}{2} \frac{1}{|B(\rho,\ z)|} \int_{B(\rho,\ z)} |(u-d)^{\pm}|^2 \,dx$$
$$\leqslant -2 \sum_{i=1}^{m} \int_{B(tr,\ z)} X_i((u-d)^{\pm}) X_i(\phi)(u-d)^{\pm} G^z_\rho \,dx$$
$$+ [\frac{1}{2} \sum_{i=1}^{m} \int_{B(tr,z)} X_i((u-d)^{\pm}) X_i(\phi)\, 2(u-d)^{\pm} G^z_\rho \,dx$$
$$+ \frac{1}{2} \sum_{i=1}^{m} \int_{B(tr,\ z)} X_i(\phi) X_i(G^z_\rho) |(u-d)^{\pm}|^2 \,dx$$
$$+ \frac{1}{2} \sum_{i=1}^{m} \int_{B(tr,z)} b_i X_i(\phi) |(u-d)^{\pm}|^2 G^z_\rho \,dx + \frac{1}{2} \int_{B(tr,\ z)} c_0 \phi |(u-d)^{\pm}|^2 G^z_\rho \,dx$$

$$+\frac{1}{2}\sum_{i=1}^{m}\int_{B(tr,z)} c_{ij}[X_i, X_j](\phi)|(u-d)^{\pm}|^2 G_\rho^z\,dx] + \frac{1}{2}\int_{B(tr,z)} c_0\phi|(u-d)^{\pm}|^2 G_\rho^z\,dx$$

$$\leqslant -2\sum_{i=1}^{m}\int_{B(tr,z)} X_i((u-d)^{\pm})X_i(\phi)(u-d)^{\pm} G_\rho^z\,dx$$

$$+\frac{1}{2}\langle \mathsf{L}\phi, |(u-d)^{\pm}|^2 G_\rho^z\rangle + K_1\int_{B(tr,z)} G_\rho^z\,dx.$$

As $\rho \Rightarrow 0$ we obtain

$$2\sum_{i=1}^{m}\int_{B(sr,z)} |X_i((u-d)^{\pm})|^2 G^z\,dx + |(u-d)^{\pm}|^2 \tag{28}$$

$$\leqslant \sup_{B(tr,z)} |(u-d)^{\pm}|^2 - 4\sum_{i=1}^{m}\int_{B(tr,z)} X_i((u-d)^{\pm})X_i(\phi)(u-d)^{\pm} G^z\,dx + K_1\int_{B(tr,z)} G^z\,dx.$$

Recalling that $G^z \in L^{1+\varepsilon}(B(tr, z))$ and the estimates on the measures of intrinsic balls [11], we can adjust the last term in the right hand side. Moreover, putting for short $B^{**} = B(tr, z) - B(sr, z)$, we have

$$\sum_{i=1}^{m}\int_{B(tr,z)} X_i((u-d)^{\pm})X_i(\phi)(u-d)^{\pm} G^z\,dx$$

$$=\sum_{i=1}^{m}\int_{B^{**}} X_i((u-d)^{\pm})X_i(\phi)(u-d)^{\pm} G^z\,dx$$

$$\leqslant \frac{1}{\eta}\sum_{i=1}^{m}\int_{B^{**}} |X_i((u-d)^{\pm})|^2 G^z\,dx + \eta\sum_{i=1}^{m}\int_{B^{**}} |X_i(\phi)|^2|(u-d)^{\pm}|^2 G^z\,dx$$

$$\leqslant \frac{1}{\eta}\sum_{i=1}^{m}\int_{B^{**}} |X_i((u-d)^{\pm})|^2 G^z\,dx$$

$$+\eta \sup_{B(tr,z)} |(u-d)^{\pm}|^2 \sup_{\partial B(sr,z)} G^z \sum_{i=1}^{m}\int_{B(tr,z)} |X_i(\phi)|^2\,dx.$$

We have also

$$\sum_{i=1}^{m}\int_{B(tr,z)} |X_i(\phi)|^2\,dx \leqslant K_3\langle \mathsf{L}\phi, \phi\rangle_{B(tr,z)}$$

$$\leqslant K_3\langle \mathsf{L}\phi, G^z\phi\rangle 1/\inf_{\partial B(sr,z)} G^z \leqslant K_4 1/\inf_{\partial B(sr,z)} G^z.$$

Therefore at the end we get

$$\sum_{i=1}^{m} \int_{B(sr,\ z)} |X_i((u-d)^{\pm})|^2 G^z \,\mathrm{d}x + |(u(z)-d)^{\pm}|^2 \tag{29}$$

$$\leqslant (1+K_5\eta) \sup_{B(tr,\ z)} |(u(z)-d)^{\pm}|^2$$

$$+\frac{K_6}{\eta} \sum_{i=1}^{m} \int_{B(tr,\ z)-B(sr,\ z)} |X_i((u-d)^{\pm})|^2 G^z \,\mathrm{d}x + K_7 r^{\alpha} .$$

Let us now take the supremum for $z \in B(qr,\ x_0)$ and let us choose $t = 1/m - q$, $s = 2q$, $q \in (0,\ 1/5mc^*)$. If we neglect the first term on the left hand side, we obtain

$$\sup_{B(qr,\ x_0)} |(u-d)^{\pm}|^2 \leqslant (1+K_5\eta) \sup_{B(r,\ x_0)} |(u-d)^{\pm}|^2$$

$$+\frac{K_6}{\eta} \sup_{B(qr,\ x_0)} \left(\frac{r^2}{B(sr,\ z)}\right) \sum_{i=1}^{m} \int_{B(r,\ x_0)-B(qr,\ x_0)} |X_i((u-d)^{\pm})|^2 \,\mathrm{d}x + K_7 r^{\alpha}$$

where $d > \sup_{B(r/c^* m,\ x_0)} \psi$.

Once again, recalling the estimates proved for the measures of intrinsic balls and for the Green function, we get

$$\sup_{B(qr,\ x_0)} |(u-d)^{\pm}|^2 \leqslant (1+K_5\eta) \sup_{B(r,\ x_0)} |(u-d)^{\pm}|^2$$

$$+\frac{K_8}{\eta} \sum_{i=1}^{m} \int_{B(r,\ x_0)-B(qr,\ x_0)} |X_i((u-d)^{\pm})|^2 G^{x_0}_{B(2r,\ x_0)} \,\mathrm{d}x + K_7 r^{\alpha} .$$

Now from our Caccioppoli inequality proved in Lemma 2 we have

$$\sup_{B(r,\ x_0)} |(u-d)^{\pm}|^2 > K_9 \sum_{i=1}^{m} \int_{B(qr,\ x_0)} |X_i((u-d)^{\pm})|^2 G^{x_0}_{B(2r,\ x_0)} \,\mathrm{d}x$$

where $K_9 \leqslant 1$ can be taken arbitrarily small. Therefore we obtain

$$K_9 \sum_{i=1}^{m} \int_{B(qr,\ x_0)} |X_i((u-d)^{\pm})|^2 G^{x_0}_{B(2r,\ x_0)} \,\mathrm{d}x + \sup_{B(qr,\ x_0)} |(u-d)^{\pm}|^2$$

$$\leqslant (2+K_5\eta) \sup_{B(r,\ x_0)} |(u-d)^{\pm}|^2$$

$$+\frac{K_8}{\eta} \sum_{i=1}^{m} \int_{B(r,\ x_0)-B(qr,\ x_0)} |X_i((u-d)^{\pm})|^2 G^{x_0}_{B(2r,\ x_0)} \,\mathrm{d}x + K_7 r^{\alpha} .$$

Coming to the usual «hole filling» argument, we get

$$(30)\quad (K_8+K_9\eta)\sum_{i=1}^{m}\int_{B(qr,\,x_0)}|X_i((u-d)^{\pm})|^2 G^{x_0}_{B(2r,\,x_0)}\,dx+\eta\sup_{B(qr,\,x_0)}|(u-d)^{\pm}|^2$$

$$\leq \eta(2+K_5\eta)\sup_{B(r,\,x_0)}|(u-d)^{\pm}|^2+K_8\sum_{i=1}^{m}\int_{B(r,\,x_0)}|X_i((u-d)^{\pm})|^2 G^{x_0}_{B(2r,\,x_0)}\,dx+\eta K_7 r^{\alpha}\,.$$

We now estimate the second term on the right hand side

$$(31)\quad \sum_{i=1}^{m}\int_{B(r,\,x_0)}|X_i((u-d)^{\pm})|^2 G^{x_0}_{B(2r,\,x_0)}\,dx=\sum_{i=1}^{m}\int_{B(qr,\,x_0)}|X_i((u-d)^{\pm})|^2 G^{x_0}_{B(2q^{-1}r,\,x_0)}\,dx$$

$$+\sum_{i=1}^{m}\int_{B(r,\,x_0)}|X_i((u-d)^{\pm})|^2 (G^{x_0}_{B(2r,\,x_0)}-G^{x_0}_{B(2q^{-1}r,\,x_0)})\,dx\,.$$

We take into account that the function $F=G^{x_0}_{B(2q^{-1}r,\,x_0)}-G^{x_0}_{B(2r,\,x_0)}$ is harmonic with respect to operator L and we have

$$\inf_{B(r,x_0)}F=\inf_{\partial B(r,x_0)}F\geq K_{10}\,\frac{r^2}{|B(r,\,x_0)|}\,.$$

Therefore we have

$$(32)\quad (K_8+K_9\eta)\sum_{i=1}^{m}\int_{B(qr,\,x_0)}|X_i((u-d)^{\pm})|^2 G^{x_0}_{B(2r,\,x_0)}\,dx+\eta\sup_{B(qr,\,x_0)}|(u-d)^{\pm}|^2$$

$$\leq \eta(2+K_5\eta)\sup_{B(r,\,x_0)}|(u-d)^{\pm}|^2+K_8\sum_{i=1}^{m}\int_{B(r,\,x_0)}|X_i((u-d)^{\pm})|^2 G^{x_0}_{B(2q^{-1}r,\,x_0)}\,dx$$

$$-K_{11}\,\frac{r^2}{|B(r,\,x_0)|}\sum_{i=1}^{m}\int_{B(r,\,x_0)}|X_i((u-d)^{\pm})|^2\,dx+\eta K_7 r^{\alpha}\,.$$

Now as in [3] let us choose $\hat{d}\in(\inf_{E(\varepsilon,\,r)}u,\ \sup_{E(\varepsilon,\,r)}u)$ such that

$$\operatorname{cap}(\{x\colon x\in B(r,\,x_0),\ (u(x)-\hat{d})^{\pm}=0\},\ B(c^*r,\,x_0))$$

$$\geq 1/4\ \operatorname{cap}(E(\varepsilon,\,r),\ B(c^*r,\,x_0))\,.$$

We have then $d = \hat{d} + \varepsilon > \sup_{B(r/c^* m,\ x_0)} \psi$. We also note that

$$\sup_{B(r,\ x_0)} |(u-d)^+|^2 + \sup_{B(r,\ x_0)} |(u-d)^-|^2$$

$$= (\operatorname*{osc}_{B(r,\ x_0)} u)^2 - 2 \sup_{B(r,\ x_0)} |(u-d)^-| \sup_{B(r,\ x_0)} |(u-d)^+|$$

whereas

$$\sup_{B(qr,\ x_0)} |(u-d)^+|^2 + \sup_{B(qr,\ x_0)} |(u-d)^-|^2$$

$$\geqslant (\operatorname*{osc}_{B(qr,\ x_0)} u)^2 - 2 \sup_{B(qr,\ x_0)} |(u-d)^-| \sup_{B(qr,\ x_0)} |(u-d)^+|.$$

If we apply (32) separately to $(u-d)^+$ and $(u-d)^-$ and then we add the two inequalities, we get

$$(K_8 + K_9 \eta) \sum_{i=1}^{m} \int_{B(qr,\ x_0)} |X_i(u)|^2 G^{x_0}_{B(2r,\ x_0)} \, dx + \eta (\operatorname*{osc}_{B(qr,\ x_0)} u)^2 \tag{33}$$

$$\leqslant \eta(2 + K_5 \eta)(\operatorname*{osc}_{B(r,\ x_0)} u)^2 + K_8 \sum_{i=1}^{m} \int_{B(r,\ x_0)} |X_i(u)|^2 G^{x_0}_{B(2q^{-1} r,\ x_0)} \, dx$$

$$-K_{11} \frac{r^2}{|B(r,\ x_0)|} \sum_{i=1}^{m} \int_{B(r,\ x_0)} |X_i(u)|^2 \, dx + \eta K_{12} r^\alpha + \eta K_{13} \varepsilon^2 .$$

We now recall Lemma 3. Accordingly we have

$$-\frac{K|B(r,\ x_0)|}{\operatorname{cap}(N(v),\ B(r,\ x_0))} \sum_{i=1}^{m} \int_{B(r,\ x_0)} |X_i(v)|^2 \, dx \leqslant - \int_{B(r/c^*,\ x_0)} |v|^2 \, dx$$

and also, recalling estimate (26),

$$-\frac{K_{11} r^2}{|B(r, x_0)|} \sum_{i=1}^{m} \int_{B(r,\ x_0)} |X_i(v)|^2 \, dx \leqslant - \frac{K_{14} p}{|B(r/c^*,\ x_0)|} \sum_{i=1}^{m} \int_{B(r/c^*,\ x_0)} |v|^2 \, dx$$

where

$$p = \frac{\operatorname{cap}(N(v),\ B(r,\ x_0))}{\operatorname{cap}(B(r/c^*,\ x_0),\ B(r,\ x_0))} .$$

Therefore, applying it to our inequality and taking into account the definition of

$\delta(R)$ we have

$$(K_8 + K_9 \eta) \sum_{i=1}^{m} \int_{B(qr,\ x_0)} |X_i(u)|^2 G^{x_0}_{B(2r,\ x_0)} \, dx + \eta(\operatorname*{osc}_{B(qr,\ x_0)} u)^2$$

$$\leqslant \eta(2 + K_5 \eta)(\operatorname*{osc}_{B(r,\ x_0)} u)^2 + K_8 \sum_{i=1}^{m} \int_{B(r,\ x_0)} |X_i(u)|^2 G^{x_0}_{B(2q^{-1}r,\ x_0)} \, dx$$

$$- \frac{K_{14}}{|B(r,\ x_0)|} \delta(r/c^*) \int_{B(r/c^*,\ x_0)} (|(u-d)^+|^2 + |(u-d)^-|^2) \, dx + \eta K_{12} r^\alpha + \eta K_{15} \varepsilon^2 .$$

The last integral on the right can be estimated thanks to Lemma 2. We have

$$(K_8 + K_9 \eta) \sum_{i=1}^{m} \int_{B(qr,\ x_0)} |X_i(u)|^2 G^{x_0}_{B(2r,\ x_0)} \, dx + (\eta + K_{16} \delta(r/c^*))(\operatorname*{osc}_{B(qr,\ x_0)} u)^2$$

$$\leqslant \eta(2 + K_5 \eta)(\operatorname*{osc}_{B(r,\ x_0)} u)^2 + K_8 \sum_{i=1}^{m} \int_{B(r,\ x_0)} |X_i(u)|^2 G^{x_0}_{B(2q^{-1}r,\ x_0)} \, dx + K_{17} r^\alpha + \eta K_{15} \varepsilon^2 .$$

Without any loss of generality, due to the many simplifications we did, we can assume $K_{16} = 5K_9$. Furthermore, if we add to both sides the term $6(K_8/K_9)(\operatorname*{osc}_{B(qr,\ x_0)} u)^2$ we get

$$(K_8 + K_9 \eta) \sum_{i=1}^{m} \int_{B(qr,\ x_0)} |X_i(u)|^2 G^{x_0}_{B(2r,\ x_0)} \, dx$$

$$+ [K_8 + K_9/6(\eta + 5K_9 \delta(r/c^*))] \, 6/K_9 (\operatorname*{osc}_{B(qr,\ x_0)} u)^2$$

$$\leqslant [K_8 + (K_9/6) \eta (2 + K_5 \eta)] \, 6/K_9 (\operatorname*{osc}_{B(r,\ x_0)} u)^2$$

$$+ K_8 \sum_{i=1}^{m} \int_{B(r,\ x_0)} |X_i(u)|^2 G^{x_0}_{B(2q^{-1}r,\ x_0)} \, dx + K_{17} r^\alpha + \eta K_{15} \varepsilon^2 .$$

We now put $\eta = K_9 \delta(r/c^*)$ and we get

(34)
$$(K_8 + K_9^2 \delta(r/c^*)) \sum_{i=1}^{m} \int_{B(qr,\ x_0)} |X_i(u)|^2 G^{x_0}_{B(2r,\ x_0)} \, dx$$

$$+ (K_8 + K_9^2 \delta(r/c^*)) \, 6/K_9 (\operatorname*{osc}_{B(qr,\ x_0)} u)^2$$

$$\leqslant [K_8 + K_9^2 \delta(r/c^*) \frac{2 + K_5 K_9}{6}] 6/K_9 (\operatorname*{osc}_{B(r,\ x_0)} u)^2$$

$$+K_8 \sum_{i=1}^{m} \int_{B(r,\ x_0)} |X_i(u)|^2 G^{x_0}_{B(2q^{-1}r,\ x_0)} \, dx + K_{17} r^\alpha + \eta K_{15} \varepsilon^2 .$$

Recalling that $K_9 < 1$ can be fixed arbitrarily small, let us suppose that $2 + K_5 K_9 < 3$ so that inequality (34) becomes

$$\sum_{i=1}^{m} \int_{B(qr,\ x_0)} |X_i(u)|^2 G^{x_0}_{B(2r,\ x_0)} \, dx + 6/K_9 (\operatorname*{osc}_{B(qr,\ x_0)} u)^2$$

$$\leqslant \frac{K_8 + (K_9^2/2)\, \delta(r/c^*)}{K_8 + K_9^2 \delta(r/c^*)} [\sum_{i=1}^{m} \int_{B(r,\ x_0)} |X_i(u)|^2 G^{x_0}_{B(2q^{-1}r,\ x_0)} \, dx$$

$$+6/K_9 (\operatorname*{osc}_{B(r,\ x_0)} u)^2] + K_{18} r^\alpha + K_{19} \varepsilon^2 ,$$

with obvious meanings for K_{18} and K_{19}. As in [3] we can suppose $K_9^2/K_8 < 1$ and therefore we get

$$\sum_{i=1}^{m} \int_{B(qr,\ x_0)} |X_i(u)|^2 G^{x_0}_{B(2r,\ x_0)} \, dx + 6/K_9 (\operatorname*{osc}_{B(qr,\ x_0)} u)^2$$

$$\leqslant \frac{1}{1 + (K_9^2/3K_8)\, \delta(r/c^*)} [\sum_{i=1}^{m} \int_{B(r,\ x_0)} |X_i(u)|^2 G^{x_0}_{B(2q^{-1}r,\ x_0)} \, dx$$

$$+6/K_9 (\operatorname*{osc}_{B(r,\ x_0)} u)^2] + K_{18} r^\alpha + K_{19} \varepsilon^2 .$$

We now define

$$\tilde{V}(r) = \sum_{i=1}^{m} \int_{B(r,\ x_0)} |X_i(u)|^2 G^{x_0}_{B(2q^{-1}r,\ x_0)} \, dx + 6/K_9 (\operatorname*{osc}_{B(r,\ x_0)} u)^2$$

and we observe that $\tilde{V}(r)$ is increasing in r.

If we reason as in [3], [1] and [9] we get at the end

$$V(r) \leqslant K_{20} \exp(-\beta \int_r^R \delta(\rho)\, d\rho/\rho)\, V(R) + K_{21} R^\alpha + K_{22} \varepsilon^2 .$$

Choosing $\varepsilon^2 = \sigma\omega_\sigma(r,\ R)$ we finally obtain

$$V(r) \leqslant K_{20}\, \omega_\sigma(r,\ R)^\beta V(R) + K_{22}\, \sigma\omega_\sigma(r,\ R) + K_{21} R^\alpha .$$

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Abstract

An obstacle problem relative to a sum of squares of vector fields satisfying the Hörmander condition is considered. A study of local behaviour of local solutions is carried out by giving a potential estimate as usually done in analogous elliptic problems.

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