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Some commutativity theorems for torsion free rings (**)

1 - Introduction

Throughout, R will denote an associative ring and for any x, y in R, as usual, $[x, y] = xy - yx$. Bell $[4]_1$ presented a simple alternate proof of a long standing result due to Herstein [5], which states that a ring R satisfying $(x+y)^n = x^n + y^n$ must have nil commutator ideal and the set of nilpotent elements of R form an ideal. The proof due to Bell depends on the observations that the rings satisfying $(x+y)^n = x^n + y^n$, satisfy the identity $[x^n, y] = [x, y^n]$ and also (at least in the absence of zero divisors) the identity $[x, y]^{n^2} = [x^n, y^n]$. Later in $[4]_3$ (Theorem 5) it was established that if R is an n-torsion free ring with unity 1 satisfying $[x^n, y] = [x, y^n]$, then R is commutative. Now our objective is to investigate the commutativity of rings satisfying $[x, y]^{n^2} = [x^n, y^n]$. In fact, we consider rather a wearker condition.

Theorem 1. *Let R be a ring with unity* 1 *in which for every x, y in R, $[x, y]^k = [x^m, y^n]$, where k, m, n are positive integers and atleast one of m, n is larger than one. Further, if commutators in R are $m!n!$-torsion free, then R is commutative.*

For $k = m = n$, we get the following corollary to our theorem which generalizes Theorem 2 (a) of [8] and at the same time extends Corollary 1 of $[4]_2$ (Theorem 1). This is also to pointout that in a paper [2] the authors have discussed an other weaker condition, namely $[[x, y]^n - [x^n, y^n], x] = 0$ but in the setting of semiprime rings.

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Corollary 1. *Let $n > 1$ be a fixed positive integer and R be a ring with unity 1 satisfying $[x, y]^n = [x^n, y^n]$ for all x, y in R. Further, if commutators in R are $n!$-torsion free, then R is commutative.*

2 - Proof of the main theorem.

First we state the following lemma due to Tong [9], which will be used extensively throughout the paper.

Lemma 1. *Let R be a ring with unity 1. Let $I_0^r(x) = x^r$. If $k \geqslant 1$ let $I_k^r(x) = I_{k-1}^r(1+x) - I_{k-1}^r(x)$. Then $I_{r-1}^r(x) = \frac{1}{2}(r-1)r! + r!\,x$, $I_r^r(x) = r!$ and $I_j^r(x) = 0$ for $j > r$.*

Proof of Theorem 1. By hypothesis, we have

$$[x, y]^k = [x^m, y^n] \quad \text{for all } x, y \text{ in } R. \tag{2.1}$$

Letting $I_p(x) = I_p^m(x)$ $(p = 0, 1, 2, \ldots, m)$ in Lemma 1, we get

$$[x, y]^k = [I_0(x), y^n] \quad \text{for all } x, y \text{ in } R. \tag{2.2}$$

Replace x by $1 + x$ in the above expression and use Lemma 1 to get $[x, y]^k = [I_0(1+x), y^n] = [I_0(x) + I_1(x), y^n]$ and hence in view of (2.2), we have

$$[I_1(x), y^n] = 0 \quad \text{for all } x, y \text{ in } R. \tag{2.3}$$

Again replacing x by $1 + x$ and using Lemma 1, we obtain

$$[I_1(1+x), y^n] = [I_1(x) + I_2(x), y^n] = 0.$$

This in view of (2.3), implies that $[I_2(x), y^n] = 0$. Thus, it is clear that on replacing x by $1 + x$ and iterating $(m-1)$-times, we have $m![x, y^n] = 0$. Finally, replacing y by $1 + y$ and using the same techniques as above, we get $m!n![x, y] = 0$. But since commutators in R are $m!n!$-torsion free, hence $[x, y] = 0$ and R is commutative.

3 - Some more commutativity theorems

Awtar [3], Bell [4]$_4$ and Nicholson and Yaqub [6] have studied rings with unity 1 satisfying the identity $[x^n, y^n] = 0$ for some positive integer n. In particular, Awtar showed that if R is $n!$-torsion free satisfying $[x^n, y^n] = 0$, then it must be commutative. Though Awtar proved the result for nonassociative rings, his proof involves very complicated combinatorial arguments. Now our aim is to provide a direct and much simpler proof of the result for associative rings. In the end of this section, we will also discuss some more conditions on commutators which imply commutativity. For the polynomial identities considered in Theorem 3 and Thorem 4, one may also refer to $[7]_1$ and $[7]_2$.

We begin with the following known lemma.

Lemma 2 (Nicholson and Yaqub [6]). *Let R be a ring with unity 1 and $f\colon R \to R$ be a function such that $f(x+1) = f(x)$ holds for every x in R. If for some positive integer k, $x^k f(x) = 0$ for all x in R or $f(x)x^k = 0$ for all x in R, then necessarily $f(x) = 0$ for all x in R.*

Theorem 2 (Awtar [3]). *Let $n \geqslant 1$ be a fixed positive integer and R be a ring with unity 1 satisfying $[x^n, y^n] = 0$ for all x, y in R. Further, if commutators in R are $n!$-torsion free, then R is commutative.*

Proof. R satisfies the identity $[x^n, y^n] = 0$ for all x, y in R. Now we shall apply the iteration technique to x^n in the given identity. Let $I_p(x) = I_p^n(x)$ for $p = 0, 1, 2, \ldots$ then by Lemma 1, $I_0(x) = x^n$. Thus our identity reduces to $[I_0(x), y^n] = 0$. After replacing x by $1 + x$ in the expression and iterating $(n-1)$ times, we get $n![x, y^n] = 0$. But since commutators in R are $n!$-torsion free, $[x, y^n] = 0$. Now using the same iteration technique to y^n, we find that $n![x, y] = 0$, which implies that $[x, y] = 0$. Hence, R is commutative.

Theorem 3. *Let $n > 1$, m be fixed positive integers and t be a non-negative integer. Let R be a ring with unity 1 satisfying $x^t[x^n, y] = [x^m, y]y^m$ for all x, y in R. If commutators in R are $m!$-torsion free, then R is commutative.*

Proof. We shall apply iteration method to y^m in the above polynomial identity. Let $I_p(y) = I_p^m(y)$ $(p = 0, 1, 2, \ldots)$. By Lemma 1, $I_0(y) = y^m$ and we have $x^t[x^n, y] = [x^m, y]I_0(y)$. Replacing y by $1 + y$ and using Lemma 1, we find that

$$x^t[x^n, y] = [x^m, y]I_0(1+y) = [x^m, y]I_0(y) + [x^m, y]I_1(y).$$

This implies that $[x^m, y]I_1(y) = 0$. Replacing y by $1 + y$ in the above relation and iterating $(m-1)$-times, we get $m![x^m, y]y = 0$. Again replace y by $1+y$, to get $m![x^m, y] = 0$, which yields that $[x^m, y] = 0$. Now using the same iteration technique to x^m, we ultimately find $[x, y] = 0$ and hence R is commutative.

If we apply iteration on y^m repeatedly for $(m-1)$-times in the identity $x^t[x^n, y] = [x, y^m]x^q$, then use of Lemma 2 yields us the following result proved in [1] also.

Theorem 4. *Let $n > 1$, m be fixed positive integers and t, q be non-negative integers. Let R be a ring with unity 1 satisfying*

$$x^t[x^n, y] = [x, y^m]x^q \quad \textit{for all } x, y \textit{ in } R.$$

If commutators in R are $m!$-torsion free, then R is commutative.

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Summary

In the present paper we have investigated the commutativity of associative rings under the following conditions: (i) $[x, y]^k = [x^m, y^n]$; (ii) $[x^n, y^n] = 0$; (iii) $x^t[x^n, y] = [x, y^m]x^q$; (iv) $x^t[x^n, y] = [x^m, y]y^m$. *We prove that every torsion-free ring satisfying any of* (i)-(iv) *is commutative. This allows us to extend several known results and to obtain a simple proof of a theorem of Awtar.*

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