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Evolution of the geometric Hahn-Banach theorem (**)

1 - Introduction

If a theorem has a geometric, analytic or algebraic form, this form is usually a restatement of the original theorem with appropriate changes in terminology. In the case of the Hahn-Banach theorem, however, it can be demonstrated that the geometric form of this theorem evolved independently of the original.

Initially the Hahn-Banach theorem was proved by H. Hahn in 1927 [7] (p. 215) for normed linear spaces. The analytic version of the theorem, which was proved by S. Banach in 1932 [2] (p. 27), may be stated as follows [10] (p. 156).

Theorem 1. *Let L be a vector space, p a seminorm on L and M a subspace of L. If f is a linear form on M such that $|f(x)| \leq p(x)$ for all $x \in M$, then there exists a linear form F extending f to L such that $|F(x)| \leq p(x)$ for all $x \in L$.*

The following theorem is commonly referred to as the geometric form of the Hahn-Banach theorem [10] (p. 160).

Theorem 2. *Let L be a topological vector space, M a linear variety in L and A a nonempty convex open subset of L which does not intersect M. Then there exists a closed hyperplane in L containing M and not intersecting A.*

The similarity between these two theorems is easy to see. A seminorm p gives rise to a natural topology on a vector space, i.e., the topology defined by the pseudometric $\mathrm{d}(x, y) = p(x - y)$. Conversely, seminorms are readily obtainable in to-

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pological vector spaces, e.g., the gauge of an absorbent or radial disk. Since a hyperplane H may be represented in terms of a nonzero linear form F, i.e., $H = \{x|F(x) = \alpha\}$, the existence of a linear form is equivalent to the existence of a hyperplane. These observations form the basis of the proof that Theorem 1 is equivalent to Theorem 2 [10] (p. 159). However, in spite of this equivalence between the two theorems, it can be demonstrated that the geometric form of the Hahn-Banach theorem, i.e., (Theorem 2) evolved independently of the analytic form (Theorem 1) through an analysis of the works of H. Minkowski [9], G. Ascoli [1] and S. Mazur [8].

2 - Minkowski's results

In the complete works of Minkowski, published in 1911, there is a paper entitled «Theorie der konvexen Körper» («The Theory of convex bodies») which was previously unpublished. The editor's footnote [9] (p. 131) indicates that this paper was probably written before Minkowski's «Volumen und Oberfläsche» which appeared in 1903. In the «Theory of convex bodies» Minkowski proves the following theorem in three dimensions [9] (p. 139).

Theorem 3. *Through each point of the boundary of a convex body, there passes at least one support plane of the body.*

Minkowski's definition of a convex body is «a point set which has in common with an arbitrary line either a finite line segment, one point or no points and it does not lie wholly in a plane» [9] (p. 131). His definition of a support plane for a closed point set Γ is a plane of the form $ax + by + cz = d(a^2 + b^2 + c^2 = 1)$ which contains at least one point of Γ and for all points (x, y, z) in Γ the inequality $ax + by + cz \leq d$ is satisfied [9] (p. 136).

Theorem 3 is proved in two stages, first for convex polyhedra and then for arbitrary convex bodies. Minkowski did not attach any special significance to this result but used it to prove the following [9] (p. 141).

Theorem 4. *Let Γ and Γ^* be two distinct convex bodies which do not have an interior point in common. Then it is always possible to construct a plane which has the interior of Γ lying wholly on one side and the interior of Γ^* lying wholly on the other side.*

Theorems 3 and 4 are the original versions of the separation theorems for convex sets. In fact the geometric form of the Hahn-Banach theorem (Theorem 2) is a separation theorem. It asserts that every nonempty open convex set A and linear variety M not intersecting A can be separated by a closed hyperplane.

Although Minkowski's results are related geometrically to the Hahn-Banach theorem, this theorem, as noted above, was not proved until 1927 by Hahn [7] and generalized in 1932 by Banach $[2]_1$. However, neither Hahn nor Banach makes any mention of a connection with Minkowski's results given above.

3 - Ascoli's results

It was until 1932, when G. Ascoli's paper [1] appeared, that any significant results related to the geometric Hahn-Banach theorem were published. This can be determined by the reviews of publications in Analysis and Geometry found in both the «Jahrbuch über die Fortschritte der Mathematik» and the «Revue Semestrielle des Publications Mathématiques».

In the introduction to his paper, Ascoli states that his intention is to synthesize the known results on abstract spaces in a systematic exposition and embellish these results with the aid of geometric intuition [1] (p. 33). In the course of this exposition he states and proves the following theorem which he calls a fundamental theorem for convex sets [1] (p. 205).

Theorem 5. *Every convex body Γ in a separable space S admits at each point of its boundary at least one «touching» hyperplane.*

Ascoli defines a *convex body* as a closed convex set with a nonempty interior. A hyperplane is a maximal linear variety, i.e., a translate of a maximal subspace. By a *touching* hyperplane Ascoli means a supporting hyperplane in the context of current terminology. The restriction to separable spaces is by his own choice. Ascoli wants to develop the material independent of the «Principle of Zermelo» [1] (p. 59).

Theorem 5 is proved by constructing a sequence of functionals which converges to a bounded linear functional. Ascoli then shows that this limit functional leads to the desired hyperplane. As a fundamental theorem for convex sets, Ascoli uses Theorem 5 to establish some geometric results about separable spaces.

These results, however, are not primarily concerned with the geometric Hahn-Banach theorem. In fact, Ascoli does not mention Minkowski's «Theorie der konvexen Körper» although he does refer to his paper «Volumen und Oberfläche» [1] (p. 50). It would seem that Ascoli's fundamental theorem for convex sets (Theorem 5) evolved independently of Minkowski's result (Theorem 3). In spite of this, it is clear that Theorem 5 is essentially a generalization of Theorem 3 to arbitrary separable spaces.

Immediately after proving Theorem 5, Ascoli gives an analytic version of this fundamental result [1] (p. 206). This is done primarily for the sake of completeness.

Theorem 6. *Let $p(x)$ be a functional defined on a separable space S with the following properties:*

(a) $p(x+y) \leq p(x) + p(y)$.
(b) $p(\lambda x) = \lambda p(x)$ *for* $\lambda > 0$.
(c) *There is a* $K > 0$ *such that* $p(x) < K\|x\|$ *for all* x *in* S.

If x_0 is any point in S, then there is a continuous linear functional $F(x)$ such that for each x, $F(x) \leq p(x)$ and $F(x_0) = p(x_0)$.

This result is related to a theorem proved by Banach in 1929 $[2]_2$ (p. 226) Ascoli, however, seems unaware of that result although footnotes indicate he was aware of some of Banach's earlier results [1] (p. 34, 38, 43, 46).

Banach's classic text «Opérationes Linéaires», in which the original analytic version of the Hahn-Banach theorem appears, was published in 1932. It is this theorem which is readily transformed to a geometric form. Since Ascoli's results were also published in 1932, one would not expect him to be aware of the connection between his fundamental result and Banach's version of the Hahn-Banach theorem. However, Ascoli was belatedly aware of Hahn's work with regard to the Hahn-Banach theorem.

Ascoli's paper appears in two parts. At the end of the first part is an additional note submitted after the paper and before its publication [1] (p. 80). In this note Ascoli states that he just became aware of the results published by Helly in 1921 and by Hahn in 1927.

It's Hahn's work [7] that is of interest here. As Ascoli explains in the note, the title of Hahn's paper, «Über lineare Gleichungssysteme in lineare Raümen» (On systems of linear equations in linear spaces) [7], does not indicate any geometric ramifications. Also, Ascoli did not have timely access to the journal in which Hahn's paper appeared. In fact a review of Hahn's work did not appear in the «Jahrbuch über die Fortschritte der Mathematik» until 1931 [6]. A shorter review (5 lines) appeared in the «Revue Semestrielle des Publications Mathématiques» [4].

Even if he had read these reviews, it is doubtful that Ascoli would have inferred any geometric ramifications from Hahn's work. The geometric underpinnings of this work appear in the first part of Hahn's paper. The reviews concentrate on the results in the later part of that paper.

Ascoli specifically notes that there are some similarities between Hahn's Theorem III, which is the original form of the Hahn-Banach theorem, and some of his own results on linear varieties. Hahn's Theorem III may be stated as follows [7] (p. 215).

Theorem 7. *Let M be a complete linear subspace of the complete normed linear space L. Let $f(x)$ be a bounded linear functional defined on M with norm K. Then there exists a bounded linear functional F on L with norm K which agrees with f on M.*

The fact that there is some common ground between Hahn's and Ascoli's papers does not diminish the importance of Ascoli's work. In particular, his exposition on touching hyperplanes is original and not found anywhere else prior to this publication. However, even after he became aware of Hahn's work and notes some similarities to his own work, Ascoli does not make any connection between his fundamental result and Hahn's Theorem III, i.e., Hahn's form of the Hahn-Banach theorem.

4 - Mazur's results

The original statement of the geometric form of the Hahn-Banach theorem appeared in 1933 in a paper by S. Mazur [8]. In the introductory remarks of this paper Mazur refers to the results of Minkowski and Ascoli. He specifically refers to the versions of Theorem 3 and Theorem 5 given above and notes that the following theorem generalizes these results [8] (p. 73).

Theorem 8. *Let L be a normed linear space and M be a linear manifold which does not contain an interior point of the convex body A. Then there exists a hyperplane H such that $M \subset H$ and A lies on one side of H.*

The definition of a convex body is the same as Ascoli's. *A set A is said to lie on one side of the hyperplane H* if for any two distinct points $x, y \in A - H$, the line segment xy does not contain a point of H.

As might be expected, Mazur's proof is independent of Ascoli's work. In fact he uses the analytic form of the Hahn-Banach theorem, i.e., Theorem 1, to obtain the extension of a linear functional. However, he does not discuss the relationship between the Hahn-Banach theorem and his work.

To prove Theorem 7, Mazur shows that the linear manifold M gives rise to a hyperplane. The linear functional representing this hyperplane can be extended, by Theorem 1, to a linear functional on L. The latter linear functional represents the desired hyperplane H and proves the theorem. Mazur's remarks before proving this theorem indicate his awareness of the connection between linear functionals and hyperplanes. A full page is devoted to a discussion of the relation between these two concepts [8] (p. 71). His remarks also show that he was aware of the geometric characteristics of seminorms through the Minkowski functional [8] (p. 72). In spite of this, however, Mazur does not discuss the similarities between his result and Banach's, i.e., Theorem 1. After proving Theorem 7, Mazur generalizes some of the other results established by Minkowski and Ascoli.

5 - Dieudonné's results

In 1941 J. Dieudonné published a paper [5] in which he gives a proof of the Hahn-Banach theorem using methods from the theory of ordered groups. He refers to the Hahn-Banach theorem in Banach's «Opérationes Linéaires», i.e., the original version of Theorem 1, then defines the concepts needed for the geometric form of the theorem. Dieudonné then states and proves the following theorem which he calls the Hahn-Banach theorem [5] (p. 642).

Theorem 9. *Let E be a real vector space and K a subset of E. If K is a convex set which is balanced with respect to one of its points, and does not contain the origin of E, then there exists a hyperplane with respect to which all of the points of K lie on the same side.*

Although the statement of the Hahn-Banach theorem is given in a geometric form, Dieudonné does not elaborate on it. In fact, he makes no mention of the works of Minkowski, Ascoli, or Mazur regarding this theorem. However, his paper [5] marks the first time any connection is made between the geometric and analytic forms of the theorem.

6 - Conclusion

The above analysis clearly shows that the geometric form of the Hahn-Banach theorem evolved independently of the analytic form. It began with Minkowski, was given in a more general setting (although restricted to separable spaces) by Ascoli, and in the form for general linear spaces by Mazur. Although Mazur was aware of both forms of the theorem, he did not make the connection between them.

From the remarks at the end of his book [2] (p. 246), it is evident that Banach was aware of the results due to Ascoli [1] and Mazur [8]. Even though he devotes almost a full page to a discussion of the geometry of normed vector space, however, Banach also does not mention the connection between the geometric and analytic forms of this fundamental result.

Although Dieudonné does not refer to the geometric ideas of Minkowski, Ascoli, or Mazur, he does make the first connection between the geometric and analytic forms of the theorem. Finally, in the 1953 Bourbaki publication «Espaces Vectoriels Topologiques», Theorem 8 is called the geometric form of the Hahn-Banach theorem.

References

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Abstract

Di solito, la forma geometrica, analitica, o algebrica d'un teorema è replica del teorema originale nella quale i vecchi termini vengono sostituiti da una terminologia appropriata. Le opere di H. Minkowski, G. Ascoli e S. Mazur indicano che questo non è vero per quanto riguarda la forma geometrica del teorema di Hahn-Banach. Un'analisi dei loro risultati e vari saggi posteriori riferendosi al teorema di Hahn-Banach approfondiscono le nostre conoscenze sull'evoluzione della forma geometrica e per di più mostrano la sua indipendenza dal risultato originale.

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