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Stability of the exponential dichotomy of linear impulsive differential equations (**)

1 - Introduction

In the present paper the investigations of [2] are continued. Theorems on stability of the exponential dichotomy on the real axis are proved. The work has influenced by the ideas of [3] and [5].

2 - Statement of the problem

Let X be an arbitrary Banach space with identical operator I. By $L(X)$ we denote the space of all linear bounded operators acting in X. Consider the impulsive differential equation

(1) $$\frac{dx}{dt} = A(t)\,x \qquad t \neq t_n$$

(2) $$x(t_n^+) = Q_n\, x(t_n) \qquad t_n \in T = \{t_j\}\,.$$

We shall say that *conditions* (H) *are satisfied* if the following conditions hold:

H1. The function A: $R \to L(X)$ is continuons extendable on each interval $[t_n,\ t_{n+1}]$ $(n \in Z)$.

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H2. $Q = \{Q_n\} \subset L(X)$.

H3. The sequence of points of impulse effect meets the conditions

$$t_n < t_{n+1} \qquad (n \in Z) \qquad \lim_{n \to \pm\infty} t_n = \pm\infty .$$

Henceforth we shall often denote equation (1), (2) by $(A(t),\ Q_n,\ T)$.

Def. 1. The impulsive differential equation (1), (2) is said to *belong to the class* K if the operator-valued function $A(t)$, the sequence of impulsive operators $Q = \{Q_n\}$ and the sequence of points $T = \{t_n\}$ satisfy conditions (H).

Def. 2. Equation $(A(t),\ Q_n, T)$ is said to be *exponentially dichotomous* if there exist constants $K \geqslant 1$, $\varkappa > 0$ and a projector P such that

$$\|U(t)PU^{-1}(\tau)\| \leqslant K\, e^{-\varkappa(t-\tau)} \qquad \tau \leqslant t \tag{3}$$

$$\|U(t)(I-P)\,U^{-1}(\tau)\| \leqslant K\, e^{-\varkappa(\tau - t)} \qquad t \leqslant \tau \tag{4}$$

where $U(t)$ is the evolutionary Cauchy operator of equation $(A(t),\ Q_n,\ T)$ (see [1], [6]).

Let equation $(A(t),\ Q_n,\ T)$ belong to the class K and let $\delta > 0$, $H > 0$ be constants. By $N((A(t),\ Q_n,\ T),\ \delta,\ H)$ we shall denote the set of all impulsive equations $(B(t),\ R_n,\ \tilde{T})$ which belong to the class K and the fundamental operators $V(t,\ \tau) = V(t)\,V^{-1}(\tau)$ of which satisfy the following condition: *for any* $S \in R$ *there exists* $\tau \in R$ *such that the following inequality be valid*

$$\|V(t+s)\,V^{-1}(u+s) - U(t+\tau)\,U^{-1}(u+\tau)\| < \delta \qquad t, u \in [-H,\ H]. \tag{5}$$

Def. 3. Equation $(A(t),\ Q_n,\ T)$ is said to be of *bounded growth* on R if there exist constants $C \geqslant 1$ and $\mu \geqslant 0$ for which the following inequality be valid

$$\|U(t)\,U^{-1}(\tau)\| \leqslant C\, e^{\mu|t-\tau|} \qquad t, \tau \in R. \tag{6}$$

Remark 1. The impulsive equation $(A(t),\ Q_n,\ T)$ is of bounded growth if and only if there exist constants $C \geqslant 1$ and $h > 0$ such that for each solution $x(t)$ the following estimate be valid

$$\|x(t)\| \leqslant C\|x(s)\| \qquad s, t \in R \qquad s \leqslant t \leqslant s+h. \tag{7}$$

Remark 2. The impulsive equation $(A(t),\ Q_n,\ t)$ is of bounded growth if, for instance, the operator-valued function $A(t)$ is integrally bounded and the impulse operator Q_n satisfy the condition

$$\prod_{\tau < t_j \leq t} \|Q_j\| \leq L\ e^{\lambda(t-\tau)} \tag{8}$$

where $L \geq 0$, $\lambda \in R$ are constants.

Lemma 1. *Let $U(t,\ \tau) = U(t)\,U^{-1}(\tau)$ be the fundamental operator of equation $(A(t),\ Q_n,\ T)$. Then the fundamental operator $V(t,\ \tau)$ of equation $(A(t+\alpha),\ Q_n,\ T_\alpha)$, where $T_\alpha = \{t_n - \alpha\}$, has the form*

$$V(t,\ \tau) = U(t+\alpha)\,U^{-1}(\tau+\alpha).$$

Lemma 1 is proved by a straighforward verification taking into account the form of the operator $U(t,\ \tau)$ (see [1], [6]).

Lemma 2. *Let the impulsive equation $(A(t), Q_n,\ T)$ be exponentially dichotomous. Then the impulsive equation $(A(t+\alpha),\ Q_n,\ T_\alpha)$, where $T_\alpha = \{t_n - \alpha\}$, is also exponentially dichotomous with projector $\tilde{P} = U(\alpha)\,P\,U^{-1}(\alpha)$.*

Proof. Let $V(t,\ \tau)$ be the evolutionary operator of equation $(A(t+\alpha),$ $Q_n,\ T_\alpha)$. Then by Lemma 1 the following equalities hold

$$V(t)\tilde{P}V^{-1}(\tau) = U(t+\alpha)\,U^{-1}(\alpha)\,U(\alpha)P\,U^{-1}(\alpha)\,U(\alpha)\,U^{-1}(\tau+\alpha) = U(t+\alpha)\,U^{-1}(\tau+\alpha).$$

The proof of Lemma 2 follows from eastimates (3), (4).

Lemma 3. *Let the impulsive equation $(A(t),\ Q_n,\ T)$ be exponentially dichotomous. Then for any number $\theta \in (0,\ 1)$ there exists $H > 0$ such that for $s \geq H$ and for any solution $x(t)$ the following inequality be valid*

$$\|x(s)\| \leq \theta\ \sup\{\|x(u)\|:\ |u-s| \leq H\}.$$

Lemma 4. *Let the following conditions hold:*

(1) *the impulsive equation* $(A(t),\ Q_n,\ T)$ *is of bounded growth;*

(2) *there exist constants* $H > 0$ *and* $\theta \in (0,\ 1)$ *such that for* $t \geq H$ *for any solution of equation* $(A(t),\ Q_n,\ T)$ *the following inequality be valid*

$$\|x(t)\| \leq \theta \sup \{\|x(u)\|:\ |u-t| \leq H\}; \tag{9}$$

(3) $\dim X < \infty$.

Then the impulsive equation $(A(t),\ Q_n,\ T)$ *is exponentially dichotomous.*

The proofs of Lemma 3 and Lemma 4 follow from the argumens in [3] (p. 14-16).

Remark 3. The condition $\dim X < \infty$ is essential [4].

3 - Main results

Theorem 1. *Let the following conditions hold:*

(1) *conditions* (H) *are met;*

(2) *the impulsive equation* $(A(t),\ Q_n,\ T)$ *is of bounded growth.*

Then each impulsive equation $(B(t),\ R_n,\ \tilde{T}) \in N((A(t),\ Q_n,\ T),\ \delta,\ H)$ *is of bounded growth.*

Proof. Since $(B(t),\ R_n,\ \tilde{T}) \in N((A(t),\ Q_n,\ T),\ \delta,\ H)$, then for any S there exists τ such that for $-H \leq t,\ u \leq H$ the following inequality be valid

$$\|V(t+s)\,V^{-1}(u+s) - U(t+\tau)\,U^{-1}(u+\tau)\| < \delta \tag{10}$$

where $V(t)$ is the evolutionary operator of equation $(B(t),\ R_n,\ \tilde{T})$. Condition 2 of Theorem 1 and inequality 10 imply the estimate

$$\|V(t+s)\,V^{-1}(u+s)\| \leq C\ e^{\mu|t-u|} + \delta \qquad -H \leq t \quad u \leq H. \tag{11}$$

From the fact that the number s is arbitrary, there follows the inequality

$$\|V(t)\,V^{-1}(u)\| \leq C\ e^{\mu|t-u|} + \delta \leq (C+\delta)\ e^{\mu|t-u|} \qquad -H \leq t \quad u \leq H. \tag{12}$$

Let $t \geqslant u$. Then there exists a positive integer k such that $u + 2kH \leqslant t \leqslant u + 2(k+1)H$. Inequality (12) implies the estimate

$$\|V(t)V^{-1}(u)\| \leqslant \|V(t)V^{-1}(u+2kH)\| \cdot \|V(u+2kH)V^{-1}(u+2(k-1)H\|$$

$$\ldots \|V(u+2H)V^{-1}(u)\| \leqslant (C+\delta)^{k+1} e^{\mu(t-u)} = (C+\delta)(C+\delta)^k e^{\mu(t-u)}$$

$$= (C+\delta)(C+\delta)^{(1/2)H^{-1}2Hk} e^{\mu(t-u)} \leqslant (C+\delta)(C+\delta)^{(1/2)H^{-1}(t-u)} e^{\mu(t-u)}$$

$$\leqslant (C+\delta)\, e^{[\mu + (1/2)H^{-1}\ln(\delta + C)](t-u)}.$$

The case $t < u$ is considered analogously.

Theorem 2. *Let the following conditions hold:*

(1) *the conditions of Theorem* 1 *are met;*

(2) *the impulsive equation* $(A(t),\ Q_n,\ T)$ *is exponentially dichotomous.*

Then for each impulsive equation $(B(t),\ R,\ \tilde{T}) \in N((A(t),\ Q_n,\ T),\ \delta,\ H)$ *the only bounded solution is the trivial one* $x(t) \equiv 0$.

Proof. Set $h = \varkappa^{-1}(\sin h^{-1}H + \ln K)$, where $\varkappa$ and K, are the constants from inequalities (3), (4). Then

$$K^{-1} e^{\varkappa h} - K\, e^{-\varkappa h} = 2 \sin h(\varkappa h - \ln K) = 8. \tag{13}$$

Let $x(t)$ be a solution of the impulsive equation $(A(t+\tau-s),\ Q_n,\ T_{\tau-s})$ where $T_{\tau-s} = \{t_j - \tau + s\}$. By Lemma 2 equation $(A(t+\tau-s),\ Q_n,\ T_{\tau-s})$ is exponentially dichotomous with projector $U(\alpha)PU^{-1}(\alpha)$ $(\alpha = \tau - s)$.

From equality (13) and from [3] (p. 14) there follows the estimate

$$\|x(s)\| \leqslant \frac{1}{4} \sup\{\|x(t)\|:\ |t-s| \leqslant h\}. \tag{14}$$

For any s there exists τ such that for $-H \leqslant t,\ u \leqslant H$ the following inequality be valid

$$\|V(t+s)V^{-1}(u+s) - U(t+\tau)U^{-1}(u+\tau)\| < \delta \tag{15}$$

where $V(t)$ is the evolutionary operator of equation $(B(t),\ R_n,\ \tilde{T})$. From inequality (15) for $u = 0$ we obtain

$$\|V(t+s)V^{-1}(s) - U(t+\tau)U^{-1}(\tau)\| < \delta. \tag{16}$$

Let $y(t)$ be an arbitrary bounded solution of equation $(B(t),\ R_n\ \tilde{T})$ and let $x(t)$ be a solution of $(A(t+\tau-s),\ Q_n,\ T_{\tau-s})$ for which $x(s)=y(s)$. We shall prove that $y(t)\equiv 0$.

By Lemma 1 the following representation is valid

$$x(t) = U(t+\tau-s)\,U^{-1}(\tau)\,x(s). \tag{17}$$

Then for $\|y(t)-x(t)\|$ we obtain the estimate

$$\begin{aligned}\|y(t)-x(t)\| &= \|[V(t)\,V^{-1}(s) - U(t+\tau-s)\,U^{-1}(\tau)]\,y(s)\| \\ &\leqslant \|V(t)\,V^{-1}(s) - U(t+\tau-s)\,U^{-1}(\tau)\|\,\|y(s)\| < 3\|y(s)\|.\end{aligned} \tag{18}$$

Let $H \geqslant h$. From the inequalities

$$\|y(s)\| = \|x(s) \leqslant \frac{1}{4}\ \sup\{\|x(t)\|:\ |t-s| \leqslant h\}$$

$$\leqslant \frac{1}{4}\ \sup\{\|y(t)\| + \|y(t)-x(t)\|:\ |t-s| \leqslant h\} < \frac{1}{4}\ \sup\{\|y(t)\|:\ |t-s| \leqslant h\} + \frac{3}{4}\|y(s)\|$$

there follows the estimate

$$\|y(s)\| < \sup\{\|y(t)\|:\ |t-s| \leqslant h\} \leqslant \sup_{t\in R}\|y(t)\| \qquad s \in R.$$

Therefore $y(t)=0,\ t\in R$.

Theorem 3. *Let the conditions of Theorem 2 hold and, moreover,* $\dim X < \infty$.

Then each equation $(B(T),\ R_n,\ \tilde{T}) \in N((A(t),\ Q_n,\ T),\ \delta,\ H)$ *for sufficiently small* δ *and sufficiently large* H *is exponentially dichotomous.*

Proof. From Theorem 1 it follows that equation $(B(t),\ R_n,\ \tilde{T})$ is of bounded growth. The proof of Theorem 3 follows from inequality (18), Lemma 4, [5], Theorem 2 and [3].

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Summary

Sufficient conditions for stability of the notion of exponential dichotomy for linear impulsive differential equations are found.

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