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Stability of the exponential dichotomy
of linear impulsive differential equations (***)

1 - Introduction

In the present paper the investigations of [2] are continued. Theorems on stability of the exponential dichotomy on the real axis are proved. The work has influenced by the ideas of [3] and [5].

2 - Statement of the problem

Let $X$ be an arbitrary Banach space with identical operator $I$. By $L(X)$ we denote the space of all linear bounded operators acting in $X$. Consider the impulsive differential equation

$$\frac{dx}{dt} = A(t)x \quad t \neq t_n$$

(1)

$$x(t_n^+) = Q_n x(t_n) \quad t_n \in T = \{ t_j \}.$$  

(2)

We shall say that conditions (H) are satisfied if the following conditions hold:

H1. The function $A: R \rightarrow L(X)$ is continuous extendable on each interval $[t_n, t_{n+1}]$ ($n \in Z$).

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(***) The present investigation is supported by the Bulgarian Ministry of Science and Higher Education under Grant MM-7.
H2. \( Q = \{Q_n\} \subset L(X) \).

H3. The sequence of points of impulse effect meets the conditions

\[
\begin{align*}
  t_n &< t_{n+1} & (n \in \mathbb{Z}) \\
  \lim_{n \to \pm\infty} t_n & = \pm \infty.
\end{align*}
\]

Henceforth we shall often denote equation (1), (2) by \((A(t), Q_n, T)\).

Def. 1. The impulsive differential equation (1), (2) is said to belong to the class \( K \) if the operator-valued function \( A(t) \), the sequence of impulses operators \( Q = \{Q_n\} \) and the sequence of points \( T = \{t_n\} \) satisfy conditions (H).

Def. 2. Equation \((A(t), Q_n, T)\) is said to be exponentially dichotomous if there exist constants \( K \geq 1 \), \( \kappa > 0 \) and a projector \( P \) such that

\[
\begin{align*}
  \|U(t) P U^{-1}(\tau)\| &\leq K e^{-\kappa(t-\tau)} & \tau \leq t \\
  \|U(t)(I - P) U^{-1}(\tau)\| &\leq K e^{-\kappa(t-\tau)} & t \leq \tau
\end{align*}
\]

where \( U(t) \) is the evolutionary Cauchy operator of equation \((A(t), Q_n, T)\) (see [1], [6]).

Let equation \((A(t), Q_n, T)\) belong to the class \( K \) and let \( \delta > 0 \), \( H > 0 \) be constants. By \( N((A(t), Q_n, T), \delta, H) \) we shall denote the set of all impulsive equations \((B(t), R_n, \tilde{T})\) which belong to the class \( K \) and the fundamental operators \( V(t, \tau) = V(t) V^{-1}(\tau) \) of which satisfy the following condition: for any \( S \in R \) there exists \( \tau \in R \) such that the following inequality be valid

\[
\|V(t + s) V^{-1}(u + s) - U(t + \tau) U^{-1}(u + \tau)\| < \delta \quad t, u \in [-H, H].
\]

Def. 3. Equation \((A(t), Q_n, T)\) is said to be of bounded growth on \( R \) if there exist constants \( C \geq 1 \) and \( \mu \geq 0 \) for which the following inequality be valid

\[
\|U(t) U^{-1}(\tau)\| \leq C e^{\mu|\tau|} \quad t, \tau \in R.
\]

Remark 1. The impulsive equation \((A(t), Q_n, T)\) is of bounded growth if and only if there exist constants \( C \geq 1 \) and \( h > 0 \) such that for each solution \( x(t) \) the following estimate be valid

\[
\|x(t)\| \leq C\|x(s)\| \quad s, t \in R \quad s \leq t \leq s + h.
\]
Remark 2. The impulsive equation \((A(t), Q_n, t)\) is of bounded growth if, for instance, the operator-valued function \(A(t)\) is integrally bounded and the impulse operator \(Q_n\) satisfy the condition

\[
\prod_{\tau < t_i \leq t} \|Q_j\| \leq L \ e^{\lambda(t - \tau)}
\]

where \(L \geq 0, \lambda \in \mathbb{R}\) are constants.

Lemma 1. Let \(U(t, \tau) = U(t) U^{-1}(\tau)\) be the fundamental operator of equation \((A(t), Q_n, T)\). Then the fundamental operator \(V(t, \tau)\) of equation \((A(t + \alpha), Q_n, T_{\alpha})\), where \(T_{\alpha} = \{t_n - \alpha\}\), has the form

\[
V(t, \tau) = U(t + \alpha) U^{-1}(\tau + \alpha).
\]

Lemma 1 is proved by a straightforward verification taking into account the form of the operator \(U(t, \tau)\) (see [1], [6]).

Lemma 2. Let the impulsive equation \((A(t), Q_n, T)\) be exponentially dichotomous. Then the impulsive equation \((A(t + \alpha), Q_n, T_{\alpha})\), where \(T_{\alpha} = \{t_n - \alpha\}\), is also exponentially dichotomous with projector \(P = U(\alpha) PU^{-1}(\alpha)\).

Proof. Let \(V(t, \tau)\) be the evolutionary operator of equation \((A(t + \alpha), Q_n, T_{\alpha})\). Then by Lemma 1 the following equalities hold

\[
V(t) PV^{-1}(\tau) = U(t + \alpha) U^{-1}(\alpha) U(\alpha) PU^{-1}(\alpha) U(\alpha) U^{-1}(\tau + \alpha) = U(t + \alpha) U^{-1}(\tau + \alpha).
\]

The proof of Lemma 2 follows from estimates (3), (4).

Lemma 3. Let the impulsive equation \((A(t), Q_n, T)\) be exponentially dichotomous. Then for any number \(\theta \in (0, 1)\) there exists \(H > 0\) such that for \(s \geq H\) and for any solution \(x(t)\) the following inequality be valid

\[
\|x(s)\| \leq \theta \sup \{\|x(u)\|: |u - s| \leq H\}.
\]
Lemma 4. Let the following conditions hold:

1. the impulsive equation \((A(t), Q_n, T)\) is of bounded growth;
2. there exist constants \(H > 0\) and \(\delta \in (0,1)\) such that for \(t \geq H\) for any solution of equation \((A(t), Q_n, T)\) the following inequality be valid

\[
\|x(t)\| \leq \delta \sup \{\|x(u)\|: |u - t| \leq H\};
\]

3. \(\dim X < \infty\).

Then the impulsive equation \((A(t), Q_n, T)\) is exponentially dichotomous.


Remark 3. The condition \(\dim X < \infty\) is essential [4].

3 - Main results

Theorem 1. Let the following conditions hold:

1. conditions (H) are met;
2. the impulsive equation \((A(t), Q_n, T)\) is of bounded growth.

Then each impulsive equation \((B(t), R_n, \tilde{T}) \in N((A(t), Q_n, T), \delta, H)\) is of bounded growth.

Proof. Since \((B(t), R_n, \tilde{T}) \in N((A(t), Q_n, T), \delta, H)\), then for any \(S\) there exists \(\tau\) such that for \(-H \leq t, u \leq H\) the following inequality be valid

\[
\|V(t + s) V^{-1}(u + s) - U(t + \tau) U^{-1}(u + \tau)\| < \delta
\]

where \(V(t)\) is the evolutionary operator of equation \((B(t), R_n, \tilde{T})\). Condition 2 of Theorem 1 and inequality 10 imply the estimate

\[
\|V(t + s) V^{-1}(u + s)\| \leq C e^{\mu|t - s| + \delta} \quad -H \leq t, u \leq H.
\]

From the fact that the number \(s\) is arbitrary, there follows the inequality

\[
\|V(t) V^{-1}(u)\| \leq C e^{\mu|t - u| + \delta} \leq (C + \delta) e^{\mu|t - u|} \quad -H \leq t, u \leq H.
\]
Let $t \geq u$. Then there exists a positive integer $k$ such that $u + 2kH \leq t \leq u + 2(k + 1)H$. Inequality (12) implies the estimate

$$
\|V(t) V^{-1}(u)\| \leq \|V(t) V^{-1}(u + 2kH)\| \cdot \|V(u + 2kH) V^{-1}(u + 2(k - 1)H)\|
$$

$$
\cdots \|V(u + 2H) V^{-1}(u)\| \leq (C + \delta)^{k+1} e^{u(t-u)} = (C + \delta)(C + \delta)^k e^{u(t-u)}
$$

$$
= (C + \delta)(C + \delta)^{(1/2)H^{-1}2Hk} e^{u(t-u)} \leq (C + \delta)(C + \delta)^{(1/2)H^{-1}(t-u)} e^{u(t-u)}
$$

$$
\leq (C + \delta) e^{t\varepsilon_u + (1/2)H^{-1}\ln(C + \delta)(t-u)}
$$

The case $t < u$ is considered analogously.

**Theorem 2.** Let the following conditions hold:

1. the conditions of Theorem 1 are met;
2. the impulsive equation $(A(t), Q_n, T)$ is exponentially dichotomous.

Then for each impulsive equation $(B(t), R, \bar{T}) \in N((A(t), Q_n, T), \varepsilon, H)$ the only bounded solution is the trivial one $x(t) = 0$.

**Proof.** Set $h = x^{-1}(\sin h^{-1}H + \ln K)$, where $x$ and $K$, are the constants from inequalities (3), (4). Then

$$
K^{-1} e^{nh} - K e^{-nh} = 2 \sin h(xh - \ln K) = 8.
$$

Let $x(t)$ be a solution of the impulsive equation $(A(t + \tau - s), Q_n, T_{\tau-s})$ where $T_{\tau-s} = \{t_j - \tau + s\}$. By Lemma 2 equation $(A(t + \tau - s), Q_n, T_{\tau-s})$ is exponentially dichotomous with projector $U(\alpha) P U^{-1}(\alpha)$ ($\alpha = \tau - s$).

From equality (13) and from [3] (p. 14) there follows the estimate

$$
\|x(s)\| \leq \frac{1}{4} \sup \{\|x(t)\|: |t - \varepsilon| \leq h\}.
$$

For any $s$ there exists $\tau$ such that for $-H \leq t, u \leq H$ the following inequality be valid

$$
\|V(t+s) V^{-1}(u+s) - U(t+\tau) U^{-1}(u+\tau)\| < \delta
$$

where $V(t)$ is the evolutionary operator of equation $(B(t), R_n, \bar{T})$. From inequality (15) for $u = 0$ we obtain

$$
\|V(t+s) V^{-1}(s) - U(t+\tau) U^{-1}(\tau)\| < \delta.
$$
Let $y(t)$ be an arbitrary bounded solution of equation $(B(t), R_n, T)$ and let $x(t)$ be a solution of $(A(t + \tau - s), Q_n, T_{\tau - s})$ for which $x(s) = y(s)$. We shall prove that $y(t) \equiv 0$.

By Lemma 1 the following representation is valid

$$x(t) = U(t + \tau - s) U^{-1}(\tau) x(s).$$

Then for $\|y(t) - x(t)\|$ we obtain the estimate

$$\|y(t) - x(t)\| = \|V(t) V^{-1}(s) - U(t + \tau - s) U^{-1}(\tau) y(s)\|$$

$$\leq \|V(t) V^{-1}(s) - U(t + \tau - s) U^{-1}(\tau) \| y(s)\| \leq 3\|y(s)\|.$$

Let $H \geq h$. From the inequalities

$$\|y(s)\| = \|x(s)\| \leq \frac{1}{4} \sup \{\|x(t)\|: |t - s| \leq h\}$$

$$\leq \frac{1}{4} \sup \{\|y(t)\| + \|y(t) - x(t)\|: |t - s| \leq h\} < \frac{1}{4} \sup \{\|y(t)\|: |t - s| \leq h\} + \frac{3}{4} \|y(s)\|$$

there follows the estimate

$$\|y(s)\| < \sup \{\|y(t)\|: |t - s| \leq h\} \leq \sup_{t \in R} \|y(t)\| \quad s \in R.$$ 

Therefore $y(t) = 0$, $t \in R$.

**Theorem 3.** Let the conditions of Theorem 2 hold and, moreover, dim $X < \infty$.

Then each equation $(B(T), R_n, T) \in N((A(t), Q_n, T), \delta, H)$ for sufficiently small $\delta$ and sufficiently large $H$ is exponentially dichotomous.

**Proof.** From Theorem 1 it follows that equation $(B(t), R_n, T)$ is of bounded growth. The proof of Theorem 3 follows from inequality (18), Lemma 4, [5], Theorem 2 and [3].
References


Summary

Sufficient conditions for stability of the notion of exponential dichotomy for linear impulsive differential equations are found.

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