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On direction dependent f-structures satisfying $f^s + f^t = 0$ (**)

Introduction

In papers [11]$_{1,2}$ Kentaro Yano has unified the notions of almost complex structure and almost contact structure by considering a tensor field of type (1, 1) on an n-dimensional manifold M^n such that $f^3 + f = 0$ and such that the rank of f is equal to a constant k everywhere. This field f defined an $f(3, 1)$-structure on the manifold M^n. In Finsler Geometry E. Heil, Y. Ichijyo, A. Benjancu, M. Matsumoto, R. Miron and others have studied almost complex structures depending on the direction. In 1988 B. Sinha and B. Yadov published a cosideration of almost contact Finsler structure depending on the direction.

In this paper, we begin a study of f-structures depending on the direction which unified the notions of direction dependent almost complex and direction dependent almost contact structures. So, we prove some results on the existence conditions of a direction dependent $f(s, t)$-structures on the connections compatible whith such structures and their properties.

2 - Preliminaries

Let N be a real differentiable manifold of dimension n. Denote by TN the tangent bundle over N and by π the canonical projection of TN to N. Also, denote by $d\pi$: $TTN \to TN$ the differential of π and define the vertical subbundle VTN of TTN as the kernel of $d\pi$. A complementary distribution HTN to VTN in TTN is called *a nonlinear connection on TN*. Of course, the fibres of vector bundles VTN

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and HTN are of the same dimension n. It is well known, VTN is an integrable distribution on TN.

Let (x^i, y^i) be a canonical coordinate system on TN and $\{\delta/\delta x^i, \partial/\partial y^i\}$ be a local field of frames on TN adapted to the decomposition $TTN = HTN \oplus VTN$, where

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j(x, y)\frac{\partial}{\partial y^j}$$

and $N_i^j(x, y)$ are n^2 differentiable functions locally defined on TN. The automorphism

$$P: \Gamma(TTN) \to \Gamma(TTN)$$

which is defined by

$$PX = X_V^i(x, y)\frac{\delta}{\delta x^i} + X_H^i(x, y)\frac{\partial}{\partial y^i} \qquad \text{for} \qquad X = X_H^i(x, y)\frac{\delta}{\delta x^i} + X_V^i(x, y)\frac{\partial}{\partial y^i}$$

is the natural almost product structure on TN, i.e., $P^2 = I$ where $\Gamma(TTN)$ is the $\mathscr{F}(TN)$-modulo of all differentiable cross sections of TTN. We keep the same notation for any other bundle. If we denote by v and h the projection morphism of TTN to VTN and HTN respectively, we have

$$P \circ h = v \circ P.$$

The automorphism

$$JX = -X_V^i(x, y)\frac{\delta}{\delta x^i} + X_H^i(x, y)\frac{\partial}{\partial y^i} \quad \text{for} \quad X = X_H^i(x, y)\frac{\delta}{\delta x^i} + X_V^i(x, y)\frac{\partial}{\partial y^i}$$

is the natural almost complex structure on TN.

3 - Direction dependent $f(s, t)$-structures

Def. 3.1. We call *Finsler $f(s, t)$-structure of rank r on N*, a non-null Finsler tensor field f of type (1, 1) and of class C^∞ such that $f^s + f^t = 0$, $s, t \in \mathbb{N}$, $s \geqq 2t$ and $t \geqq 1$, and rank $f = r$, where r is constant everywhere.

Def. 3.2. We call *horizontal $f_h(s, t)$-structure of rank r on N* a non-null horizontal tensor field f_h on TN of type (1, 1) satisfying $f_h^s + f_h^t = 0$, $s, t \in \mathbb{N}$, $s \geqq 2t$, $t \geqq 1$ and rank $f_h = r$, where r is constant everywhere.

Of course, an *$F(s, t)$-structure on TN of rank r* is a non null tensor field F of type $\binom{1\ \ 1}{1\ \ 1}$ such that $F^s + F^t = 0$, $s, t \in \mathbb{N}$, $s \geqq 2t$, $t \geqq 1$ and rank $F = r$, where r is constant everywhere.

For our study it is very convenient to consider f and f_h as morphisms of vector bundles

$$f\colon \Gamma(VTN) \to \Gamma(VTN) \qquad f_h\colon \Gamma(HTN) \to \Gamma(HTN).$$

Let f be a Finsler $f(s, t)$-structure of rank r on N. We define the morphisms

$$l = -f^{s-t} \qquad m = f^{s-t} + I_{\Gamma(VTN)}$$

where $I_{\Gamma(VTN)}$ denotes the indentity morphism on $\Gamma(VTN)$. It is clear that $l + m = I$. Also we have

$$lm = ml = -f^{2s-2t} - f^{s-t} = -f^{s-2t}(f^s + f^t) = 0.$$

Hence the morphisms l, m applied to the vertical vector bundle on TN are complementary projection morphisms. Then there exist complementary distributions VL and VM corresponding to the projection morphisms l and m respectively such that dim $VL = r$ and dim $VM = n - r$.

For the tensor fied f and the morphisms l, m we have

$$(3.1) \qquad f^t l = l f^t = f^t \qquad f^t m = m f^t = 0 \qquad f^{s-t} l = l f^{s-t} = -l$$

because of

$$f^{s-t} l = -f^{s-t} f^{s-t} = -f^s f^{s-2t} = f^t f^{s-2t} = f^{s-t} = -l.$$

In the case $s = 3$, $t = 1$ and rank $f = n$ the tensor field f defines a direction dependent almost complex Finsler structure on N and n must be even. If $s = 3$, $t = 1$ and rank $f = n - 1$, the tensor field f defines a direction dependent almost contact Finsler structure on N.

If $s = 4$, $t = 2$ and rank $f = n$ the $f(4, 2)$-structure is a direction dependent almost complex Finsler structure. A Finsler $f(4, 2)$-structure of rank $n/2$ is a direction dependent almost tangent Finsler structure on N.

Proposition 3.1. *If a Finsler $f(s,\ t)$-structure of rank r is defined on N, then an horizontal $f_h(s,\ t)$-structure of rank r is defined on N by the natural almost product structure of TN or the natural complex structure of TN.*

Proof. If we put

$$f_p X = PfPX \qquad f_j X = -JfJX \qquad \forall X \in \Gamma(HTN) \tag{3.2}$$

it is easy to see that $f_p^s X = Pf^s PX$, $f_j^s X = -Jf^s JX$. Hence

$$f_p^s + f_p^t = 0 \qquad f_j^s + f_j^t = 0. \tag{3.3}$$

Of course, rank f_p = rank $f_j = r$.

Proposition 3.2. *If a Finsler $f(s,\ t)$-structure of rank r is defined on N, then an $F(s,\ t)$-structure of rank $2r$ is defined on TN by the natural almost product or the natural almost complex structure of TN.*

Proof. We put

$$F_p = f_p h + fv \qquad F_j = f_j h + fv \tag{3.4}$$

where f_p, f_j are defined by (3.2), and h, v are the projection morphisms of TTN to HTN and VTN. Then it is easy to check that $F_2^2 = f_p^2 h + f^2 v$ and finally $F_p^s = f_p^s h + f^s v$. Thus $F_p^s + F_p^t = 0$ and symilarly $F_j^s + F_j^t = 0$. It is clear that rank F_p = rank $F_j = 2r$.

If l_p, m_p are the complementary projection morphisms of the horizontal $f_p(s,\ t)$-structure f_p, which is defined by the natural almost product structure of TN, then $\forall X \in \Gamma(HTN)$ we have

$$l_p X = -f_p^{s-t} X = -Pf^{s-t} PX = PlPX \tag{3.5}$$

$$m_p X = f_p^{s-t} + I_{\Gamma(HTN)} X = Pf^{s-t} PX + PI_{\Gamma(VTN)} PX = PmPX. \tag{3.6}$$

If l_j, m_j are the complementary projection morphisms of the horizontal $f_j(s,\ t)$-structure, which is defined by the natural almost complex structure of TN, we have

$$l_j X = -JlJX \qquad m_j X = -JmJX \qquad \forall X \in \Gamma(HTN). \tag{3.7}$$

If L_p, M_p and L_j, M_j are the complementary projection morphisms of the

$F_p(s,\ t)$ and $F_j(s, t)$-structures on TN respectively, then we have

$$L_p = -F_p^{s-t} = -f_p^{s-t}h - f^{s-t}v = l_p h + lv \tag{3.8}$$

$$M_p = F_p^{s-t} + I_{\Gamma(TTN)} = f_p^{s-t}h + f^{s-t}v + I_{\Gamma(TTN)} = m_p h + mv. \tag{3.9}$$

Similarly

$$L_j = l_j h + lv \qquad M_j = m_j h + mv. \tag{3.10}$$

Thus, if there is given a Finsler $f(s,\ t)$-structure on N of rank r, then there exist complementary distributions HL_p, HM_p or HL_j, HM_j of HTN, corresponding to the morphisms l_p, m_p and l_j, m_j respectively such that

$$HL_p = PVL \qquad HM_p = PVM \qquad HL_j = -JVL \qquad HM_p = -JVM. \tag{3.11}$$

Thus we have the decompositions

$$TTN = HTN \oplus VTN = PVL \oplus PVM \oplus VL \oplus VM$$

$$TTN = (-JVL) \oplus (-JVM) \oplus VL \oplus VM.$$

If TL_p, TM_p and TL_j, TM_j denote the complementary distributions correponding to the morphisms L_p, M_p, L_j, M_j respectively, then from (3.10) and (3.11) we have

$$TL_p = PVL \oplus VL \qquad TM_p = PVM \oplus VM$$

$$TL_j = -JVL \oplus VL \qquad TM_j = -JVM \oplus VM.$$

4 - Existence theorems

Let $CVTN$ be the complexified vertical bundle on TN and Π be a complex subbundle of $CVTN$ such that $\Pi \cap \overline{\Pi} = \{0\}$, where $\overline{\Pi}$ is the complex conjugate of Π.

Let K be the complementary distribution of $\operatorname{Re}\Pi$ to VTN, that is $VTN = \operatorname{Re}\Pi \oplus K$. We can define a morphism of vector bundles by the relations

$$f(z) = 0 \quad \forall z \in \Gamma(K) \qquad f(X) = \frac{i}{2}(U - \overline{U})$$

where $U = X + iY$ is a cross section of Π and $i = \sqrt{-1}$. Then $f(X) = -Y$ $f^2(X) = -X$. Hence

$$f^{t+2} + f^t = 0 \qquad t = 1,\ 2. \tag{4.1}$$

Also, it is easy to check that

$$f^{4k-1} + f = 0 \qquad k = 1,\ 2,\ 3,\ \ldots . \tag{4.2}$$

It is clear that rank $f = \dim \operatorname{Re} \Pi_q = r$, where Π_q is the fibre of Π over $q \in TN$. Thus a Finsler $f(t+2,\ t)$-structure of a Finsler $f(4k-1,\ 1)$-structure is defined on N by the subbundle Π.

Conservely, let f be a Finsler $f(t+2,\ t)$-structure f on N. We can define a subbundle of $CVTN$ by

$$\Pi = \{X - if^t X,\ t = 1,\ 2,\ X \in \Gamma(VL)\}$$

where $VTN = VL \oplus VM$ is the decomposition of VTN with respect to Finsler $f(t+2,\ t)$-structure f on N.

Similarly a Finsler $f(4k-1,\ 1)$-structure f on N can define a subbundle of $CVTN$ by $\Pi_1 = \{X - ifX,\ X \in \Gamma(L_1)\}$ where $VTN = VL_1 \oplus VM_1$ is the decomposition of VTN with respect to Finsler $f(4k-1,\ 1)$-structure f on N.

It is clear that $\Pi \cap \Pi = \{0\}$ and $\Pi_1 \cap \overline{\Pi}_1 = \{0\}$.

Thus we have

Theorem 4.1. *A necessary and sufficient condition for an n-dimensional manifold N to admit a Finsler $f(s,\ t)$-structure in cases $s - t = 2$, $t = 1,\ 2$ or $s = 4k - 1$, $t = 1$, $\forall k = 1,\ 2,\ 3, \ldots$ is that there exists a complex subbundle Π of $CVTN$ such that $\Pi \cap \overline{\Pi} = \{0\}$. Then* rank $f = \dim \operatorname{Re} \Pi$.

Remark 4.1. The previous existence theorem does not depend on any Finsler or pseudo-riemannian metric on TN.

Remark 4.2. If there exists a complex subbundle of $CVTN$ such that $\Pi \cap \overline{\Pi} = \{0\}$, then according to Propositions 3.1 and 3.2, an horizontal $f(s,\ t)$-structure on N and an $f(s,\ t)$-structure on TN are defined in cases $s - t = 2$, $t = 1,\ 2$ and $s = 4k - 1$, $t = 1$, $k = 1,\ 2, \ldots$.

Now, we introduce in the manifold a pseudo-riemannian structure i.e. a mapping h: $\Gamma(VTN) \times \Gamma(VTN) \to \mathscr{F}(TN)$, which is symmetric, $\mathscr{F}(TN)$-bilinear and non-degenerate on each fibre of VTN. Of course, a Finsler structure, or a Lagrange structure are examples of a pseudo-riemannian structure $[1]_1$.

The mapping α: $\Gamma(VTN) \times \Gamma(VTN) \to \mathscr{F}(TN)$ which is defined by

$$\alpha(X, Y) = \frac{1}{2}[h(lX, lY) + h(mX, mY)] \qquad \forall X, Y \in \Gamma(VTN)$$

is a pseudo-riemannian structure on TN such that

$$\alpha(X, Y) = 0 \qquad \forall X \in \Gamma(VL) \quad Y \in \Gamma(VM).$$

Proposition 4.1. *If a Finsler $f(2k+1, 1)$-structure $k \geqq 1$ of rank r is defined on N, then there exist a pseudo-riemannian structure on TN with respect to which the complementary distributions VL and VM are orthogonal and the f is an isometry on VL.*

Proof. If we put

$$g(X, Y) = \frac{1}{2k}[\alpha(X, Y) + \alpha(fX, fY) + \ldots + \alpha(f^{2k-1}X, f^{2k-1}Y)]$$

it is easy to see that

$$g(X, Y) = 0 \qquad \forall X \in \Gamma(VL) \qquad Y \in \Gamma(VM).$$

Also, using the relation 3.1, when $s = 2k+1$, $t = 1$ we get

$$g(fX, fY) = \frac{1}{2k}[\alpha(fX, fY) + \alpha(f^2, f^2 Y) + \ldots + \alpha(X, Y)].$$

Thus f is an isometry with respect to g.

Let X be a differentiable section of VL. The sections $fX, f^2X, \ldots, f^{2k}X$ are also differentiable sections of VL, which satisfy the relation

$$g(X, f^kX) = g(fX, f^{k+1}X) = \ldots = g(f^kX, f^{2k}X) = -g(f^kX, X).$$

Consequently

$$g(X, f^kX) = g(fX, f^{k+1}X) = \ldots = g(f^{k-1}X, f^{2k-1}X) = 0$$

and $r = 2km$.

Thus we can choose in $\Gamma(VL)$ $r = 2km$ mutualy orthogonal unit vector fields such that

$$f(X_\alpha) = X_{\alpha+m} \qquad \alpha = 1, 2, \ldots, 2km - m = 1, 2, \ldots, (2k-1)m.$$

An adapted frame of the Finsler $f(2k+1, 1)$-structure is the orthogonal frame (X_b, X_B), $b=1, \ldots, 2km$, $B=2km+1, \ldots, n$, where $f(X_\alpha)=X_{\alpha+m}$, $\alpha=1, 2, \ldots, (2k-1)m$ and X_B is an-orthogonal frame of VM.

Now if we take two different adapted frames R and R', we can easily see that $R'=AR$, where the orthogonal matrix A is an element of the group $U(km)\times O(n-2km)$.

Thus we have

Theorem 4.2. *A necessary and sufficient condition for an n-dimensional manifold, whose vertical bundle has a pseudo-riemannian structure, to admit a Finsler $f(2k+1, 1)$-structure $k\geqq 1$ of rank r is that $r=2km$ and the structure group of the vertical bundle of the manifold be reduced to the group $U(km)\times O(n-2km)$.*

By means of the pseudo-riemannian structure g on VTN, we can define a mapping g_p: $\Gamma(HTN)\times\Gamma(HTN)\to\mathscr{F}(TN)$ such that

$$g_p(X, Y)=g(PX, PY) \qquad \forall X, Y\in\Gamma(HTN).$$

The mapping g_p has the properties of g and defines a metric structure on HTN. Then, using (3. 11), the distributions HL_p, HM_p are orthogonal with respect to g_p and the horizontal $f_p(2_{k+1}, 1)$-structure, defined by $(3.2)_1$, is an isometry on HL_p.

Similarly if we defined a metric on HTN by the relation

$$g_J(X, Y)=g(JX, JY) \qquad \forall X, Y\in\Gamma(HTN)$$

then the distributions HL_J, HM_J are orthogonal with respect to g_J and the horizontal $f_J(2k+1, 1)$-structure which is defined by $(3.2)_2$ is an isometry on HL_J with respect to g_J.

If (X_b, X_B), $b=1, \ldots, 2km$, $B=2km+1, \ldots, n$ is an adapted frame of a given Finsler $f(2k+1, 1)$-structure on N we have

$$g_p(PX_\alpha, f_p^k PX_\alpha)=g(P^2X_\alpha, Pf_p^kPX_\alpha).$$

Using $(3.2)_1$ we have

$$Pf_p=P^2fP=fP\Rightarrow Pf_pP=f \qquad Pf_p^kP=f^k.$$

Thus we have the relation

$$g_p(PX_\alpha, f_p^kPX_\alpha)=g(X_\alpha, f^kX_\alpha)=0 \qquad \forall\alpha=1, \ldots, (2k-1)m.$$

Similarly

$$g_p(f_p PX_\alpha, f_p^{k-1} PX_\alpha) = \dots = g_p(f_p^{k-1} PX_\alpha, f_p^{2k-1} PX_\alpha) = 0.$$

Similarly

$$g_J(JX_\alpha, f_J^k JX\alpha) = g_J(f_J JX_\alpha, f_J^{k-1} JX_\alpha) = \dots = g_J(f_J^{k-1} JX_\alpha, f_J^{2k-1} JX_\alpha) = 0.$$

Thus we have

Proposition 4.2. *If (X_b, X_B) is an adapted frame of a given Finsler $f(2k+1, 1)$-structure f on N with respet to g, then the frame (PX_b, PX_B) is an adapted frame of the horizontal $f(2k+1, 1)$-structure f_p with respect to g_p and the frame (JX_b, JX_B) is an adapted frame of the horizontal $f(2k+1, 1)$-structure f_J with respect to g_J.*

It is clear that the frames (PX_b, PX_B, X_b, X_B) and (JX_b, JX_B, X_b, X_B) are adapted frames to the decompositions $TTN = HL_p \oplus HM_p \oplus VL \oplus VM$ and $TTN = HL_J \oplus HM_J \oplus VL \oplus VM$ respectively.

Thus we have

Theorem 4.3. *If a Finsler $f(2k+1, 1)$-structure is defined on an n-dimensional manifold whose vertical bundle has endowed with a pseudo-riemannian structure, then the structure group of the tangent bundle on TN is reduced to*

$$U(km) \times 0(n-2km) \times U(km) \times 0(n-2km).$$

5 - Linear connections compatible with direction dependent $f(s, t)$-structures satisfying $f^s + f^t = 0$

It is well known that an arbitrary distribution D is parallel with respect to a linear connection ∇, if for any tangent field Y, ∇_Y is a transformation of D.

Def. 5.1. An *$f(s, t)$-connection on VTN* (or *a linear connection compatible with a Finsler $f(s, t)$-structure satisfying $f^s + f^t = 0$*) is a linear connection ∇ on VTN with the property that the distributions VL and VM are parallel with respect to ∇.

Of course, there are $f(s, t)$-connections on VTN.

Example. Let ∇ be an arbitrary linear connection on VTN. It is easy to check that the operators

$$\bar{\nabla}_X Y = l\nabla_X(lY) + m\nabla_X(mY) \tag{5.1}$$

$$\tilde{\nabla}_X Y = l\nabla_{lX}(lY) + m\nabla_{mX}(mY) + l[mX,\ lY] + m[lX,\ mY] \tag{5.2}$$

are $f(s,\ t)$-connections on VTN.

Theorem 5.1. *Let l, m be the complementary projection morphisms of a Finsler $f(s,\ t)$-structure on VTN. A linear connection on VTN is an $f(s,\ t)$-connection, if and only if $\nabla_X l = 0$.*

Proof. Since l is a morphism on VTN and ∇ is a linear connection on VTN, the covariant derivative of l is defined as usually

$$(\nabla_X l)(Y) = \nabla_X lY - l\nabla_X Y \qquad \forall X \in \Gamma(TTN) \qquad \forall Y \in \Gamma(VTN). \tag{5.3}$$

If $\nabla_X l = 0$ then from $l + m = 1$ and (5.3) we have

$$(\nabla_X m)(Y) = 0 \qquad \nabla_X lY = l\nabla_X Y \qquad \nabla_X mY = m\nabla_X Y.$$

Since $ml = lm = 0$ we have

$$m\nabla_X Y = 0 \qquad \forall Y \in \Gamma(VL) \qquad \forall X \in \Gamma(TTN)$$

$$l\nabla_X = 0 \qquad \forall Y \in \Gamma(VM) \qquad X \in \Gamma(TTN).$$

Thus $\nabla_X Y \in \Gamma(VL)$ for every $Y \in \Gamma(VL)$ and $\nabla_X Y \in \Gamma(VM)$ for every $Y \in \Gamma(VM)$.

Conversely, using the decomposition $Y = lY + mY$ and the relation (5.3) we get

$$(\nabla_X l)(Y) = \nabla_X lY - l\nabla_X lY - l\nabla_X mY.$$

Since ∇_X is an $f(s,\ t)$-connection $\nabla_X mY \in \Gamma(VM)$. Consequently

$$l\nabla_X mY = 0 \qquad \nabla_X lY = l\nabla_X lY.$$

Thus
$$\nabla_X l = 0 \qquad \forall X \in \Gamma(TTN).$$

Theorem 5.2. *If ∇_X is an arbitrary linear connection on VTN, then the operator*

$$\nabla^t_X = f^t \nabla_X f^t \qquad \forall X \in \Gamma(TTN) \tag{5.4}$$

is an $f(s, t)$-connection on VTN.

Proof. Applying Theorem 5.1 we have

$$(\nabla^t_X l)\, Y = f^t \nabla_X f^t l Y - l f^t \nabla_X f^t Y \qquad \forall Y \in \Gamma(VTN).$$

Since $f^t l = l f^t = f^t$ we have $\Delta^k_X l = 0 \quad \forall X \in \Gamma(TTN)$.

Consenquently the connection ∇^t is an $f(s, t)$-connection.

Theorem 5.3. *If ∇ is an $f(s, t)$-connection on VTN and A a Finsler tensor field of type $\binom{0\ 1}{0\ 2}$ of the vector bundles TTN over TN, then the operator*

$$\overset{\circ}{\nabla}_X = \nabla_X + f^t A_X f^t \qquad \forall X \in \Gamma(TTN) \tag{5.5}$$

is an $f(s, t)$-connection on VTN. Then the set of all $f(s, t)$-connections on VTN is given by (5.5).

Proof. Using (5.3) and (3.1), we have $\overset{\circ}{\nabla}_X l = 0\ \forall X \in \Gamma(TTN)$. So, according to Theorem 5.1, $\overset{\circ}{\nabla}$ is an f-connection.

Corollary 5.1. *If ∇ is an arbitrary linear connection on VTN, then for any Finsler tensor field of type $\binom{0\ 1}{0\ 2}$ on TN the operator*

$$\overset{\circ}{\nabla}_X = \nabla^t_X + f^t A_X f^t \tag{5.6}$$

is an $f(s, t)$-connection on VTN.

Remark 5.1. The Relation (5.5) defines, for any Finsler tensor field of type $\binom{0\ 1}{0\ 2}$ A, a transformation of the set of all $f(s, t)$-connections on VTN.

Remark 5.2. It is clear that the previous Theorems 5.1, 5.2, 5.3 are valid in the case of $f(s, t)$-connections on a manifold N.

Let ∇ be a linear connection on the vertical vector bundle. We define the linear connection ∇' on the horizontal vector bundle by

$$(5.7) \qquad \nabla'_X Y = P\nabla_X PY \qquad \forall X \in \Gamma(TTN) \qquad \forall Y \in \Gamma(HTN).$$

Next, by means of ∇ and ∇' we define the mapping

$$D\colon \Gamma(TTN) \times \Gamma(TTN) \to \Gamma(TTN)$$

such that

$$(5.8) \qquad D_X Y = \nabla'_X hY + \nabla_X vY.$$

It is easy to see that D is a Finsler connection on TTN according to the R. Miron's definiton [**6**]$_1$ i.e. the distributions HTN and VTN are parallel with respect to D. We call ∇' and D *associate horizonal connection* and *associate Finsler connection* of ∇, respectively.

Theorem 5.4. *If ∇ is an $f(s, t)$-connection on VTN, then its associate horizontal connection ∇' is a connection compatible with the horizontal $f_h(s, t)$-structures, which are defined by Proposition* 3.1.

Proof. Using (3.5), (5.3) and (5.7), $\forall Y \in \Gamma(HTN)$ we have

$$(\nabla'_X l_p)Y = \nabla'_X l_p Y - l_p \nabla'_X Y = P\nabla_X PPlPY - PlPP\nabla_X PY = P(\nabla_X lPY - l\nabla_X PY).$$

According to Theorem 5.1 $\nabla_X l = 0$. Thus $\nabla'_X lp = 0$. Similarly $\nabla'_X l_J = 0$.
Consequently the associate horizontal connection ∇' of ∇ is compatible with the $f_h(s, t)$-structures of Proposition 3.1.

Theorem 5.5. *If ∇ is an $f(s, t)$-connection on VTN, then its associate Finsler connection D is compatible with the $F(s, t)$- structures, which are defined by Proposition* 3.2

Proof. It is enough to prove that $(D_X L_p)Y = 0$, $\forall X, Y \in \Gamma(TTN)$.

Using (3.8), (5.3) and (5.8) we get

$$(D_X L_p)(Y) = \nabla'_X hL_p Y + \nabla_X vL_p Y - L_p \nabla'_X hY - L_p \nabla_X vL$$

$$= \nabla'_X h(l_p h + lv)Y + \nabla_X v(l_p h + lv)Y - (l_p h + lv)\nabla'_X hY - (l_p h + lv)\nabla_X vY.$$

Consequently $(D_X L_p)(Y) = (\nabla'_X l_p)(hY) + (\nabla_X l)vY$.

According to Theorems 5.1 and 5.4 $D_X L_p = 0$, $\forall X \in \Gamma(TTN)$. Similarly $D_X L_j = 0$ $\forall X \in \Gamma(TTN)$.

The proof is complete.

Let ∇ be an $f(s, t)$-connection on VTN and $R(X, Y)$ its curvature tensor, then $\forall X, Y \in \Gamma(TTN)$ and $\forall Z \in \Gamma(VTN)$ we have

$$R(X, Y)lZ = \nabla_X \nabla_Y lZ - \nabla_Y \nabla_X lZ - \nabla_{[X,y]} lZ$$

$$= l\nabla_X \nabla_Y Z - l\nabla_Y \nabla_X Z - l\nabla_{[X,Y]} Z = lR(X, Y)Z.$$

Similarly

$$R(X, Y)mZ = mR(X, Y)Z.$$

Thus we have

Proposition 5.1. *If $R(X, Y)$ is the curvature tensor of an $f(s, t)$-connection, then the $R(X, Y)$ is an endomorphism of the complementary distributions VL and VM, which are defined by the Finsler $f(s, t)$-structure f on N.*

In the same way, we can prove that

Proposition 5.2. *If $R'(X, Y)$, $\tilde{R}(X, Y)$ are the curvature tensors of the associate horizontal and Finsler connection of an $f(s, t)$-connection ∇ on VTN respectively, then we have the properties:*

$$\tilde{R}(X, Y) = R'(X, Y)h + R(X, Y)v \tag{5.9}$$

$$R'(X, Y)l_p = l_p R'(X, Y) = PlPR(X, Y) = PlR(X, Y)P = PR(X, Y)lP \tag{5.10}$$

$$R'(X, Y)m_p = m_p R'(X, Y) = PmPR(X, Y) = PmR(X, Y)P = PR(X, Y)mP \tag{5.11}$$

$$\tilde{R}(X, Y)L_p = l_p R'(X, Y)h + lR(X, Y)v \tag{5.12}$$

$$\tilde{R}(X, Y)M_p = m_p R'(X, Y)h + mR(X, Y)v \tag{5.13}$$

for any $X, Y \in \Gamma(TTN)$.

The complementary morphisms l_j, m_j, L_j, M_j satisfy relations, analogous to (5.10), (5.11), (5.12) and (5.13).

From the previous study of $f(s, t)$-connections we see that all properties of the $f(s, t)$-connections are the same for any $s, t \in \mathbb{N}$, $s \geqq 2t$.

References

[1] A. BEJANCU: [•]$_1$ *Geometry of vertical vector bundle and its applications*, Math. Rep. Toyama Univ. **10** (1987), 133-168; [•]$_2$ *Complex Finsler space and CR-structures*, Anal. St. Univ. «Al I Cuza» Din Iasi **33** (1987), S. Ia Matematica.

[2] D. DEMETROPOULOU-PSOMOPOULOU and F. GOULI-ANDREOU, *On necessary and sufficient conditions for an n-dimensional manifold to admit a tensor field* $F \neq 0$ *of type* (1, 1) *satisfying* $F^{2v+3}+F=0$, Tensor N.S. 42 (1985).

[3] S. ISHIHARA and K. YANO, *On integrability conditions of a structure satisfying* $f^3+f=0$, Quart. J. Math. **15** (1964), 217-222.

[4] Y. ICHIJYO, *Almost complex structures of tangent bundle and Finsler metrics*, J. Math. Kyoto Univ. **6-3** (1967), 419-452.

[5] M. MATSUMOTO: [•]$_1$ *Connections, metric and almost complex structures of tangent bundles*, J. Math. Kyto Univ. **5** (1966), 251-278; [•]$_2$ *Foundations of Finsler geometry and special Finsler spaces*, Kaiseisha Press 1986.

[6] R. MIRON: [•]$_1$ *Almost complex Finsler structures*, Anal. Univ. «Al. I. Cuza» (Iasi), **29** (1983); [•]$_2$ *Techniques of Finsler geometry in the theory of vector bundles*, Acta Sci. Math. (1985), 119-129.

[7] R. MIRON and F. C. KLEPP, *Almost complex Finsler structures on the tangent bundle*, Topics in Differential Geometry Pap., Colloq. Hajduszobszlo (Hung 1984), Colloq. Math. Soc. Janos Bolyai **46**, V. 2 (1988), 853-881.

[8] B. SINHA and B. YADAV, *On almost contact Finsler structure on vector bundles*, Indian J. Pure Appl. Math. 1988, 27-35.

[9] N. SOARE, *Sur quelques f-connexions linéaire*, Riv. Mat. Univ. Parma (4) **14** (1988), 143-150.

[10] P. TAMIA-DIMOPOULOU, *Direction dependent f-structures satisfying* $f^3+f=0$, To appear in Anal. St. Univ. «Al I Cuza» Din Iasi.

[11] K. YANO: [•]$_1$ *On structure f satisfying* $f^3+f=0$, Univ. of Washington Technical Report 12 (1961); [•]$_2$ *On a structure defined by a tensor field f of type* (1, 1) *satisfying* $f^3+f=0$, Tensor N.S. 14 (1963), 99-109; [•]$_3$ *Structures on manifolds*, World Scientific Publishing Co. Ltd. Singapore (1984).

[12] K. YANO, G. S. HOUH and B. Y. CHEN, *Structures defined by a tensor field* φ *of type* (1, 1) *satisfying* $\varphi^4 \pm \varphi^2 = 0$, Tensor N.S. 23 (1972), 81-87.

[13] K. YANO and S. ISHIHARA, *Tangent and cotangent bundles*, Dekker, New York 1973.

Summary

We consider tensor fields $f \neq 0$ *of type* (1, 1) *satysfying* $f^s + f^t = 0$ *depending on both point and direction on a manifold. These fields define the direction-dependent* $f(s, t)$*-structures. Starting with a Finsler* $f(s, t)$*-structure on a manifold* N *we introduce an horizontal* $f(s, t)$*-structure on* N *and an* $f(s, t)$*-structure on* TN. *Next we prove two necessary and sufficient conditions for a manifold to admit a Finsler* $f(s, t)$*-structure in cases* $s = t + 2$, $s = 4k - 1$ *and* $t = 1$, $s = 2k + 1$ *and* $t = 1$. *Also we define connections compatible with* $f(s, t)$*-structures and we consider their properties.*

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