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On direction dependent $f$-structures satisfying $f^8 + f^5 = 0$ (**)

Introduction

In papers [11], Kentaro Yano has unified the notions of almost complex structure and almost contact structure by considering a tensor field of type $(1, 1)$ on an $n$-dimensional manifold $M^n$ such that $f^3 + f = 0$ and such that the rank of $f$ is equal to a constant $k$ everywhere. This field $f$ defines an $f(3, 1)$-structure on the manifold $M^n$. In Finsler Geometry E. Heil, Y. Ichijyo, A. Benjancu, M. Matsumoto, R. Miron and others have studied almost complex structures depending on the direction. In 1988 B. Sinha and B. Yadov published a consideration of almost contact Finsler structure depending on the direction.

In this paper, we begin a study of $f$-structures depending on the direction which unified the notions of direction dependent almost complex and direction dependent almost contact structures. So, we prove some results on the existence conditions of a direction dependent $f(s, b)$-structures on the connections compatible with such structures and their properties.

2 - Preliminaries

Let $N$ be a real differentiable manifold of dimension $n$. Denote by $TN$ the tangent bundle over $N$ and by $\pi$ the canonical projection of $TN$ to $N$. Also, denote by $\text{d}\pi: TTN \to TN$ the differential of $\pi$ and define the vertical subbundle $VTN$ of $TTN$ as the kernel of $\text{d}\pi$. A complementary distribution $HTN$ to $VTN$ in $TTN$ is called a nonlinear connection on $TN$. Of course, the fibres of vector bundles $VTN$

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and $HTN$ are of the same dimension $n$. It is well known, $VTN$ is an integrable distribution on $TN$.

Let $(x^i, y^i)$ be a canonical coordinate system on $TN$ and $\{\delta/\delta x^i, \partial/\partial y^i\}$ be a local field of frames on $TN$ adapted to the decomposition $TTN = HTN \oplus VTN$, where

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^i_j(x, y) \frac{\partial}{\partial y^j}$$

and $N^i_j(x, y)$ are $n^2$ differentiable functions locally defined on $TN$. The automorphism

$$P: \Gamma(TTN) \rightarrow \Gamma(TTN)$$

which is defined by

$$PX = X^i_V(x, y) \frac{\delta}{\delta x^i} + X^j_H(x, y) \frac{\partial}{\partial y^j} \quad \text{for} \quad X = X^j_H(x, y) \frac{\delta}{\delta x^j} + X^i_V(x, y) \frac{\partial}{\partial y^i}$$

is the natural almost product structure on $TN$, i.e., $P^2 = I$ where $\Gamma(TTN)$ is the $\mathcal{C}(TN)$-module of all differentiable cross sections of $TTN$. We keep the same notation for any other bundle. If we denote by $v$ and $h$ the projection morphism of $TTN$ to $VTN$ and $HTN$ respectively, we have

$$P \circ h = v \circ P.$$ 

The automorphism

$$JX = -X^i_V(x, y) \frac{\delta}{\delta x^i} + X^j_H(x, y) \frac{\partial}{\partial y^j} \quad \text{for} \quad X = X^j_H(x, y) \frac{\delta}{\delta x^j} + X^i_V(x, y) \frac{\partial}{\partial y^i}$$

is the natural almost complex structure on $TN$.

3 - Direction dependent $f(s, t)$-structures

Def. 3.1. We call Finsler $f(s, t)$-structure of rank $r$ on $N$, a non-null Finsler tensor field $f$ of type $(1, 1)$ and of class $C^\infty$ such that $f^s + f^t = 0$, $s, t \in \mathbb{N}$, $s \geq 2t$, and $t \geq 1$, and rank $f = r$, where $r$ is constant everywhere.

Def. 3.2. We call horizontal $f_h(s, t)$-structure of rank $r$ on $N$ a non-null horizontal tensor field $f_h$ on $TN$ of type $(1, 1)$ satisfying $f^s_h + f^t_h = 0$, $s, t \in \mathbb{N}$, $s \geq 2t$, $t \geq 1$ and rank $f_h = r$, where $r$ is constant everywhere.
Of course, an $F(s, t)$-structure on $TN$ of rank $r$ is a non null tensor field $F$ of type $(1, 1)$ such that $F^s + F^t = 0$, $s, t \in \mathbb{N}$, $s \geq 2t$, $t \geq 1$ and rank $F = r$, where $r$ is constant everywhere.

For our study it is very convenient to consider $f$ and $f_h$ as morphisms of vector bundles

$$f : I(VTN) \to I(VTN) \quad f_h : I(HTN) \to I(HTN).$$

Let $f$ be a Finsler $f(s, t)$-structure of rank $r$ on $N$. We define the morphisms

$$l = -f^{s-t} \quad m = f^{s-t} + I_{I(VTN)}$$

where $I_{I(VTN)}$ denotes the indentity morphism on $I(VTN)$. It is clear that $l + m = I$. Also we have

$$lm = ml = -f^{2s-2t} - f^{s-t} = -f^{s-t}(f^s + f^t) = 0.$$ 

Hence the morphisms $l$, $m$ applied to the vertical vector bundle on $TN$ are complementary projection morphisms. Then there exist complementary distributions $VL$ and $VM$ corresponding to the projection morphisms $l$ and $m$ respectively such that dim $VL = r$ and dim $VM = n - r$.

For the tensor field $f$ and the morphisms $l$, $m$ we have

$$(3.1) \quad f'tl = lft = f' \quad f'm = mf' = 0 \quad f'^{-1}tl = lft^{s-t} = -l$$

because of

$$f^{s-t}l = -f^{s-t}f^{s-t} = -f^s f^{2s-2t} = f' f^{s-2t} = f^{s-t} = -l.$$

In the case $s = 3$, $t = 1$ and rank $f = n$ the tensor field $f$ defines a direction dependent almost complex Finsler structure on $N$ and $n$ must be even. If $s = 3$, $t = 1$ and rank $f = n - 1$, the tensor field $f$ defines a direction dependent almost contact Finsler structure on $N$.

If $s = 4$, $t = 2$ and rank $f = n$ the $f(4, 2)$-structure is a direction dependent almost complex Finsler structure. A Finsler $f(4, 2)$-structure of rank $n/2$ is a direction dependent almost tangent Finsler structure on $N$. 
Proposition 3.1. If a Finsler \( f(s, t) \)-structure of rank \( r \) is defined on \( N \), then an horizontal \( f_h \)(s, t)-structure of rank \( r \) is defined on \( N \) by the natural almost product structure of \( TN \) or the natural complex structure of \( TN \).

Proof. If we put

\[
(3.2) \quad f_p X = PfPX \quad f_j X = -Jf_j X \quad \forall X \in \Gamma(HTN)
\]

it is easy to see that \( f_p^{s}X = Pf^{s}PX \), \( f_j^{s}X = -Jf_j^{s}X \). Hence

\[
(3.3) \quad f_p^{s} + f_p^{t} = 0 \quad f_j^{s} + f_j^{t} = 0.
\]

Of course, \( \text{rank } f_p = \text{rank } f_j = r \).

Proposition 3.2. If a Finsler \( f(s, t) \)-structure of rank \( r \) is defined on \( N \), then an \( F(s, t) \)-structure of rank \( 2r \) is defined on \( TN \) by the natural almost product or the natural almost complex structure of \( TN \).

Proof. We put

\[
(3.4) \quad F_p = f_p h + f_v \quad F_j = f_j h + f_v
\]

where \( f_p \), \( f_j \) are defined by (3.2), and \( h \), \( v \) are the projection morphisms of \( TTN \) to \( HTN \) and \( VTN \). Then it is easy to check that \( F_p^{s} = f_p^{s}h + f^{s}v \) and finally \( F_p^{s} = f_p^{s}h + f^{s}v \). Thus \( F_p^{s} + F_p^{t} = 0 \) and similarly \( F_j^{s} + F_j^{t} = 0 \). It is clear that \( \text{rank } F_p = \text{rank } F_j = 2r \).

If \( l_p \), \( m_p \) are the complementary projection morphisms of the horizontal \( f_p \)(s, t)-structure \( f_p \), which is defined by the natural almost product structure of \( TN \), then \( \forall X \in \Gamma(HTN) \) we have

\[
(3.5) \quad l_p X = -f_p^{s-1}X = -Pf^{s-1}PX = PlPX
\]

\[
(3.6) \quad m_p X = f_p^{s-1} + I_{\Gamma(HTN)} X = Pf^{s-1}PX + Pl_{\Gamma(VTN)} PX = PmPX.
\]

If \( l_j \), \( m_j \) are the complementary projection morphisms of the horizontal \( f_j \)(s, t)-structure, which is defined by the natural almost complex structure of \( TN \), we have

\[
(3.7) \quad l_j X = -Jl_j X \quad m_j X = -Jm_j X \quad \forall X \in \Gamma(HTN).
\]

If \( L_p \), \( M_p \) and \( L_j \), \( M_j \) are the complementary projection morphisms of the
$F_p(s, t)$ and $F_j(s, t)$-structures on $TN$ respectively, then we have

$$(3.8) \quad L_p = - F_p^s - t = -f_p^{s - t} h - f_p^{s - t} v = l_p h + lv$$

$$(3.9) \quad M_p = F_p^s - t + I_{(TN)} = f_p^{s - t} h + f_p^{s - t} v + I_{(TN)} = m_p h + mv.$$ 

Similarly

$$(3.10) \quad L_j = l_j h + lv \quad M_j = m_j h + mv.$$ 

Thus, if there is given a Finsler $f(s, t)$-structure on $N$ of rank $r$, then there exist complementary distributions $HL_p$, $HM_p$ or $HL_j$, $HM_j$ of $HTN$, corresponding to the morphisms $l_p$, $m_p$ and $l_j$, $m_j$ respectively such that

$$(3.11) \quad HL_p = PVL \quad HM_p = PVM \quad HL_j = - JV L \quad HM_p = - JVM.$$ 

Thus we have the decompositions

$$TTN = HTN \oplus VTN = PVL \oplus PVM \oplus VL \oplus VM \quad TTN = (-JVL) \oplus (-JVM) \oplus VL \oplus VM.$$ 

If $TL_p$, $TM_p$ and $TL_j$, $TM_j$ denote the complementary distributions corresponding to the morphisms $L_p$, $M_p$, $L_j$, $M_j$ respectively, then from (3.10) and (3.11) we have

$$TL_p = PVL \oplus VL \quad TM_p = PVM \oplus VM \quad TL_j = - JV L \oplus VL \quad TM_j = - JVM \oplus VM.$$ 

4 - Existence theorems

Let $CVTN$ be the complexified vertical bundle on $TN$ and $\Pi$ be a complex sub-bundle of $CVTN$ such that $\Pi \cap \overline{\Pi} = \{0\}$, where $\overline{\Pi}$ is the complex conjugate of $\Pi$.

Let $K$ be the complementary distribution of $Re \Pi$ to $VTN$, that is $VTN = Re \Pi \oplus K$. We can define a morphism of vector bundles by the relations

$$f(z) = 0 \quad \forall z \in \Gamma(K) \quad f(\bar{z}) = \frac{i}{2} (U - \overline{U})$$
where $U = X + iY$ is a cross section of $\Pi$ and $i = \sqrt{-1}$. Then $f(X) = -Y f^2(X) = -X$. Hence

\[(4.1) \quad f^{t+2} + f^t = 0 \quad t = 1, 2.\]

Also, it is easy to check that

\[(4.2) \quad f^{4k-1} + f = 0 \quad k = 1, 2, 3, \ldots.\]

It is clear that rank $f = \dim \text{Re} \Pi_q = r$, where $\Pi_q$ is the fibre of $\Pi$ over $q \in TN$. Thus a Finsler $f(t + 2, t)$-structure of a Finsler $f(4k - 1, 1)$-structure is defined on $N$ by the subbundle $\Pi$.

Conversely, let $f$ be a Finsler $f(t + 2, t)$-structure of $f$ on $N$. We can define a subbundle of $CVTN$ by

$$\Pi = \{X - if^t X, \; t = 1, 2, \; X \in \Gamma(VL)\}$$

where $VTN = VL \oplus VM$ is the decomposition of $VTN$ with respect to Finsler $f(t + 2, t)$-structure $f$ on $N$.

Similarly a Finsler $f(4k-1, 1)$-structure $f$ on $N$ can define a subbundle of $CVTN$ by $\Pi_1 = \{X - ifX, \; X \in \Gamma(L1)\}$ where $VTN = VL \oplus VM$ is the decomposition of $VTN$ with respect to Finsler $f(4k - 1, 1)$-structure $f$ on $N$.

It is clear that $\Pi \cap \Pi = \{0\}$ and $\Pi_1 \cap \Pi_1 = \{0\}$.

Thus we have

**Theorem 4.1.** A necessary and sufficient condition for an $n$-dimensional manifold $N$ to admit a Finsler $f(s, t)$-structure in cases $s - t = 2$, $t = 1, 2$ or $s = 4k - 1$, $t = 1, \forall k = 1, 2, 3, \ldots$ is that there exists a complex subbundle $\Pi$ of $CVTN$ such that $\Pi \cap \Pi = \{0\}$. Then rank $f = \dim \text{Re} \Pi$.

**Remark 4.1.** The previous existence theorem does not depend on any Finsler or pseudo-riemannian metric on $TN$.

**Remark 4.2.** If there exists a complex subbundle of $CVTN$ such that $\Pi \cap \Pi = \{0\}$, then according to Propositions 3.1 and 3.2, an horizontal $f(s, t)$-structure on $N$ and an $f(s, t)$-structure on $TN$ are defined in cases $s - t = 2$, $t = 1, 2$ and $s = 4k - 1$, $t = 1, k = 1, 2, \ldots$.

Now, we introduce in the manifold a pseudo-riemannian structure i.e. a mapping $h$: $\Gamma(VTN) \times \Gamma(VTN) \rightarrow \mathcal{F}(TN)$, which is symmetric, $\mathcal{F}(TN)$-bilinear and non-degenerate on each fibre of $VTN$. Of course, a Finsler structure, or a Lagrange structure are examples of a pseudo-riemannian structure $[1]$. 
The mapping $\alpha: I(VTN) \times I(VTN) \rightarrow \mathcal{F}(TN)$ which is defined by

$$
\alpha(X, Y) = \frac{1}{2} [h(lX, lY) + h(mX, mY)] \quad \forall X, Y \in I(VTN)
$$

is a pseudo-riemannian structure on $TN$ such that:

$$
\alpha(X, Y) = 0 \quad \forall X \in I(VL) \quad Y \in I(VM).
$$

Proposition 4.1. If a Finsler $f(2k + 1, 1)$-structure $k \geq 1$ of rank $r$ is defined on $N$, then there exist a pseudo-riemannian structure on $TN$ with respect to which the complementary distributions $VL$ and $VM$ are orthogonal and the $f$ is an isometry on $VL$.

Proof. If we put

$$
g(X, Y) = \frac{1}{2k} [\alpha(X, Y) + \alpha(fX, fY) + \ldots + \alpha(f^{2k-1}X, f^{2k-1}Y)]
$$

it is easy to see that

$$
g(X, Y) = 0 \quad \forall X \in I(VL) \quad Y \in I(VM).
$$

Also, using the relation 3.1, when $s = 2k + 1, t = 1$ we get

$$
g(fX, fY) = \frac{1}{2k} [\alpha(fX, fY) + \alpha(f^2, f^2Y) + \ldots + \alpha(X, Y)].
$$

Thus $f$ is an isometry with respect to $g$.

Let $X$ be a differentiable section of $VL$. The sections $fX, f^2X, \ldots, f^{2k}X$ are also differentiable sections of $VL$, which satisfy the relation

$$
g(X, f^kX) = g(fX, f^{k+1}X) = \ldots = g(f^kX, f^{2k}X) = -g(f^kX, X).
$$

Consequently

$$
g(X, f^kX) = g(fX, f^{k+1}X) = \ldots = g(f^{k-1}X, f^{2k-1}X) = 0
$$

and $r = 2km$.

Thus we can choose in $I(VL)$ $r = 2km$ mutuay orthogonal unit vector fields such that

$$
f(X_\alpha) = X_{\alpha + m} \quad \alpha = 1, 2, \ldots, 2km - m = 1, 2, \ldots, (2k - 1)m.
$$

An adapted frame of the Finsler $f(2k + 1, 1)$-structure is the orthogonal fra-
me $X_b, X_B, b = 1, \ldots, 2km, B = 2km + 1, \ldots, n$, where $f(X_a) = X_{a+m}$, $\alpha = 1, 2, \ldots, (2k - 1)m$ and $X_b$ is an orthogonal frame of $VM$.

Now if we take two different adapted frames $R$ and $R'$, we can easily see that $R' = AR$, where the orthogonal matrix $A$ is an element of the group $U(km) \times O(n - 2km)$.

Thus we have

**Theorem 4.2.** A necessary and sufficient condition for an $n$-dimensional manifold, whose vertical bundle has a pseudo-riemannian structure, to admit a Finsler $f(2k + 1, 1)$-structure $k \geq 1$ of rank $r$ is that $r = 2km$ and the structure group of the vertical bundle of the manifold be reduced to the group $U(km) \times O(n - 2km)$.

By means of the pseudo-riemannian structure $g$ on $VTN$, we can define a mapping $g_p : \Gamma(HTN) \times \Gamma(HTN) \rightarrow \mathcal{F}(TN)$ such that

$$g_p(X, Y) = g(PX, PY) \quad \forall X, Y \in \Gamma(HTN).$$

The mapping $g_p$ has the properties of $g$ and defines a metric structure on $HTN$. Then, using (3.11), the distributions $HL_p, HM_p$ are orthogonal with respect to $g_p$ and the horizontal $f_p(2k + 1, 1)$-structure, defined by (3.2)$_1$, is an isometry on $HL_p$.

Similarly if we defined a metric on $HTN$ by the relation

$$g_J(X, Y) = g(JX, JY) \quad \forall X, Y \in \Gamma(HTN)$$

then the distributions $HL_J, HM_J$ are orthogonal with respect to $g_J$ and the horizontal $f_J(2k + 1, 1)$-structure which is defined by (3.2)$_2$ is an isometry on $HL_J$ with respect to $g_J$.

If $(X_b, X_B), b = 1, \ldots, 2km, B = 2km + 1, \ldots, n$ is an adapted frame of a given Finsler $f(2k + 1, 1)$-structure on $N$ we have

$$g_p(PX_a, f_p^k PX_a) = g(P^2 X_a, Pf_p^k PX_a).$$

Using (3.2)$_1$ we have

$$Pf_p = P^2 fP = fP \Rightarrow Pf_p P = f \quad Pf_p^k P = f^k.$$

Thus we have the relation

$$g_p(PX_a, f_p^k PX_a) = g(X_a, f^k X_a) = 0 \quad \forall \alpha = 1, \ldots, (2k - 1)m.$$
Similarly
\[ g_p(f_p^k PX_a, f_p^{k-1} PX_a) = \cdots = g_p(f_p^{k-1} PX_a, f_p^{2k-1} PX_a) = 0. \]

Similarly
\[ g_J(JX_a, f_J^k JX_a) = g_J(f_J JX_a, f_J^{k-1} JX_a) = \cdots = g_J(f_J^{k-1} JX_a, f_J^{2k-1} JX_a) = 0. \]

Thus we have

Proposition 4.2. If \((X_b, X_B)\) is an adapted frame of a given Finsler \(f(2k + 1, 1)\)-structure \(f\) on \(N\) with respect to \(g\), then the frame \((PX_b, PX_B)\) is an adapted frame of the horizontal \(f(2k + 1, 1)\)-structure \(f_p\) with respect to \(g_p\) and the frame \((JX_b, JX_B)\) is an adapted frame of the horizontal \(f(2k + 1, 1)\)-structure \(f_J\) with respect to \(g_J\).

It is clear that the frames \((PX_b, PX_B, X_b, X_B)\) and \((JX_b, JX_B, X_b, X_B)\) are adapted frames to the decompositions \(TTN = HL_p \oplus HM_p \oplus VL \oplus VM\) and \(TTN = HL_J \oplus HM_J \oplus VL \oplus VM\) respectively.

Thus we have

Theorem 4.3. If a Finsler \(f(2k + 1, 1)\)-structure is defined on an \(n\)-dimensional manifold whose vertical bundle has endowed with a pseudo-riemannian structure, then the structure group of the tangent bundle on \(TN\) is reduced to
\[ U(km) \times 0(n - 2km) \times U(km) \times 0(n - 2km). \]

5 - Linear connections compatible with direction dependent \(f(s, t)\)-structures satisfying \(f^s + f^t = 0\)

It is well known that an arbitrary distribution \(D\) is parallel with respect to a linear connection \(\nabla\), if for any tangent field \(Y\), \(\nabla_Y \) is a transformation of \(D\).

Def. 5.1. An \(f(s, t)\)-connection on \(VTN\) (or a linear connection compatible with a Finsler \(f(s, t)\)-structure satisfying \(f^s + f^t = 0\)) is a linear connection \(\nabla\) on \(VTN\) with the property that the distributions \(VL\) and \(VM\) are parallel with respect to \(\nabla\).

Of course, there are \(f(s, t)\)-connections on \(VTN\).
Example. Let $\nabla$ be an arbitrary linear connection on $VTN$. It is easy to check that the operators

\begin{align}
(5.1) \quad \bar{\nabla}_X Y &= l\nabla_X (lY) + m\nabla_X (mY) \\
(5.2) \quad \bar{\nabla}_X Y &= l\nabla_{lX} (lY) + m\nabla_{mX} (mY) + l[mX, lY] + m[lX, mY]
\end{align}

are $f(s, t)$-connections on $VTN$.

**Theorem 5.1.** Let $l, m$ be the complementary projection morphisms of a Finsler $f(s, t)$-structure on $VTN$. A linear connection on $VTN$ is an $f(s, t)$-connection, if and only if $\nabla_X l = 0$.

**Proof.** Since $l$ is a morphism on $VTN$ and $\nabla$ is a linear connection on $VTN$, the covariant derivative of $l$ is defined as usually

\begin{equation}
(5.3) \quad (\nabla_X l)(Y) = \nabla_X lY - l\nabla_X Y \quad \forall X \in \Gamma(TTN) \quad \forall Y \in \Gamma(VTN).
\end{equation}

If $\nabla_X l = 0$ then from $l + m = 1$ and (5.3) we have

\begin{align}
(\nabla_X m)(Y) &= 0 \\
\nabla_X lY &= l\nabla_X Y \\
\nabla_X mY &= m\nabla_X Y.
\end{align}

Since $ml = lm = 0$ we have

\begin{align}
m\nabla_X Y &= 0 \quad \forall Y \in \Gamma(VL) \\
l\nabla_X Y &= 0 \quad \forall Y \in \Gamma(VM) \quad X \in \Gamma(TTN).
\end{align}

Thus $\nabla_X Y \in \Gamma(VL)$ for every $Y \in \Gamma(VL)$ and $\nabla_X Y \in \Gamma(VM)$ for every $Y \in \Gamma(VM)$.

Conversely, using the decomposition $Y = lY + mY$ and the relation (5.3) we get

\begin{equation}
(\nabla_X l)(Y) = \nabla_X lY - l\nabla_X lY - l\nabla_X mY.
\end{equation}

Since $\nabla_X$ is an $f(s, t)$-connection $\nabla_X mY \in \Gamma(VM)$. Consequently

\begin{align}
l\nabla_X mY &= 0 \\
\nabla_X lY &= l\nabla_X lY.
\end{align}

Thus

\begin{align}
\nabla_X l &= 0 \quad \forall X \in \Gamma(TTN).
\end{align}
Theorem 5.2. If $\nabla_X$ is an arbitrary linear connection on $VTN$, then the operator

\[(5.4) \quad \nabla_X = f^i \nabla_X f^i \quad \forall X \in \Gamma(TTN)\]

is an $f(s, t)$-connection on $VTN$.

Proof. Applying Theorem 5.1 we have

\[(\nabla_X l) Y = f^i \nabla_X (f^i l) Y - l f^i \nabla_X f^i Y \quad \forall Y \in \Gamma(VTN).\]

Since $f^i l = l f^i = f^i$ we have $\Delta_X^k l = 0 \quad \forall X \in \Gamma(TTN)$.

Consequently the connection $\nabla^i$ is an $f(s, t)$-connection.

Theorem 5.3. If $\nabla$ is an $f(s, t)$-connection on $VTN$ and $A$ a Finsler tensor field of type $\begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}$ of the vector bundles $TTN$ over $TN$, then the operator

\[(5.5) \quad \tilde{\nabla}_X = \nabla_X + f^i A_X f^i \quad \forall X \in \Gamma(TTN)\]

is an $f(s, t)$-connection on $VTN$. Then the set of all $f(s, t)$-connections on $VTN$ is given by (5.5).

Proof. Using (5.3) and (3.1), we have $\tilde{\nabla}_X l = 0 \quad \forall X \in \Gamma(TTN)$. So, according to Theorem 5.1, $\tilde{\nabla}$ is an $f$-connection.

Corollary 5.1. If $\nabla$ is an arbitrary linear connection on $VTN$, then for any Finsler tensor field of type $\begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}$ on $TN$ the operator

\[(5.6) \quad \tilde{\nabla}_X = \nabla_X + f^i A_X f^i\]

is an $f(s, t)$-connection on $VTN$.

Remark 5.1. The Relation (5.5) defines, for any Finsler tensor field of type $\begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} A$, a transformation of the set of all $f(s, t)$-connections on $VTN$.

Remark 5.2. It is clear that the previous Theorems 5.1, 5.2, 5.3 are valid in the case of $f(s, t)$-connections on a manifold $N$.

Let $\nabla$ be a linear connection on the vertical vector bundle. We define the linear
connection $\nabla'$ on the horizontal vector bundle by

\begin{equation}
    \nabla'_X Y = P \nabla_X P Y \quad \forall X \in \Gamma(TTN) \quad \forall Y \in \Gamma(HTN).
\end{equation}

Next, by means of $\nabla$ and $\nabla'$ we define the mapping

\[ D: \Gamma(TTN) \times \Gamma(TTN) \rightarrow \Gamma(TTN) \]

such that

\begin{equation}
    D_X Y = \nabla'_X h Y + \nabla_X v Y.
\end{equation}

It is easy to see that $D$ is a Finsler connection on $TTN$ according to the R. Miron's definition [6], i.e. the distributions $HTN$ and $VTN$ are parallel with respect to $D$. We call $\nabla'$ and $D$ associate horizontal connection and associate Finsler connection of $\nabla$, respectively.

Theorem 5.4. If $\nabla$ is an $f(s, t)$-connection on $VTN$, then its associate horizontal connection $\nabla'$ is a connection compatible with the horizontal $f_h(s, t)$-structures, which are defined by Proposition 3.1.

Proof. Using (3.5), (5.3) and (5.7), $\forall Y \in \Gamma(HTN)$ we have

\[ (\nabla'_{l_p}) Y = \nabla'_X l_p Y - l_p \nabla'_X Y = P \nabla_X PP^1 Y - PI PP \nabla'_X Y = P (\nabla_X l P Y - l \nabla_X P Y). \]

According to Theorem 5.1 $\nabla_X l = 0$. Thus $\nabla'_X l_p = 0$. Similarly $\nabla'_X l_t = 0$.

Consequently the associate horizontal connection $\nabla'$ of $\nabla$ is compatible with the $f_h(s, t)$-structures of Proposition 3.1.

Theorem 5.5. If $\nabla$ is an $f(s, t)$-connection on $VTN$, then its associate Finsler connection $D$ is compatible with the $F(s, t)$-structures, which are defined by Proposition 3.2.

Proof. It is enough to prove that $(D_X L_p) Y = 0$, $\forall X, Y \in \Gamma(TTN)$.

Using (3.8), (5.3) and (5.8) we get

\[ (D_X L_p)(Y) = \nabla'_X h l_p Y + \nabla'_X v l_p Y - l_p \nabla'_X h Y - l_p \nabla'_X v L \]

\[ = \nabla'_X h (l_p h + l v) Y + \nabla'_X v (l_p h + l v) Y - (l_p h + l v) \nabla'_X h Y - (l_p h + l v) \nabla'_X v Y. \]

Consequently $(D_X L_p)(Y) = (\nabla'_X l_p)(h Y) + (\nabla'_X l)(v Y)$.
According to Theorems 5.1 and 5.4 \( D_X L_p = 0, \forall X \in \mathcal{I}(TTN) \). Similarly \( D_X l_p = 0, \forall X \in \mathcal{I}(TTN) \).

The proof is complete.

Let \( \nabla \) be an \( f(s, t) \)-connection on \( VTN \) and \( R(X, Y) \) its curvature tensor, then \( \forall X, Y \in \mathcal{I}(TTN) \) and \( \forall Z \in \mathcal{I}(VTN) \) we have

\[
R(X, Y) lZ = \nabla_X \nabla_Y lZ - \nabla_Y \nabla_X lZ - \nabla_{[X,Y]} lZ
\]

\[
= l\nabla_X \nabla_Y Z - l\nabla_Y \nabla_X Z - l\nabla_{[X,Y]} Z = lR(X, Y) Z.
\]

Similarly

\[
R(X, Y) mZ = mR(X, Y) Z.
\]

Thus we have

Proposition 5.1. If \( R(X, Y) \) is the curvature tensor of an \( f(s, t) \)-connection, then the \( R(X, Y) \) is an endomorphism of the complementary distributions \( VL \) and \( VM \), which are defined by the Finsler \( f(s, t) \)-structure \( f \) on \( N \).

In the same way, we can prove that

Proposition 5.2. If \( R'(X, Y), \bar{R}(X, Y) \) are the curvature tensors of the associate horizontal and Finsler connection of an \( f(s, t) \)-connection \( \nabla \) on \( VTN \) respectively, then we have the properties:

\[
\bar{R}(X, Y) = R'(X, Y) h + R(X, Y) v
\]

\[
\]

\[
R'(X, Y) m_p = m_p R'(X, Y) = P m P R(X, Y) = P m R(X, Y) P = P R(X, Y) m P
\]

\[
\bar{R}(X, Y) L_p = l_p R'(X, Y) h + l R(X, Y) v
\]

\[
\bar{R}(X, Y) M_p = m_p R'(X, Y) h + m R(X, Y) v
\]

for any \( X, Y \in \mathcal{I}(TTN) \).

The complementary morphisms \( l_j, m_j, L_j, M_j \) satisfy relations, analogous to (5.10), (5.11), (5.12) and (5.13).
From the previous study of $f(s, t)$-connections we see that all properties of the $f(s, t)$-connections are the same for any $s, t \in \mathbb{N}, s \geq 2t$.

References


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[12] K. YANO, G. S. HOUG and B. Y. CHEN, Structures defined by a tensor field $\varphi$ of type $(1, 1)$ satisfying $\varphi^2 + \varphi^3 = 0$, Tensor N.S. 23 (1972), 81-87.

Summary

We consider tensor fields \( f \neq 0 \) of type \((1, 1)\) satisfying \( f^s f^t = 0 \) depending on both point and direction on a manifold. These fields define the direction-dependent \( f(s, t)\)-structures. Starting with a Finsler \( f(s, t)\)-structure on a manifold \( N \) we introduce an horizontal \( f(s, t)\)-structure on \( N \) and an \( f(s, t)\)-structure on \( TN \). Next we prove two necessary and sufficient conditions for a manifold to admit a Finsler \( f(s, t)\)-structure in cases \( s = t + 2, s = 4k - 1 \) and \( t = 1, s = 2k + 1 \) and \( t = 1 \). Also we define connections compatible with \( f(s, t)\)-structures and we consider their properties.

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