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Locally conformal Kähler manifolds with pointwise constant antiholomorphic sectional curvature (**)

1 - Introduction

In [5]$_1$ and [10]$_2$ G. T. Ganĉev and O. T. Kassabov classify the nearly-Kähler and the almost Kähler manifolds whose antiholomorphic sectional curvature is (pointwise) constant.

The aim of this paper is the study of the locally conformal Kähler (l.c.K.) manifolds whose antiholomorphic sectional curvature ν is pointwise constant.

To this purpose, the authors use the decomposition in suitable subbundles $\{\mathcal{W}_k\}$ of the vector bundle $\mathcal{R}(M)$ on an almost hermitian manifold (M, g, J), whose sections are the algebraic curvature tensor fields on M (see [13]).

It is well known that a $2n$-dimensional Kähler manifold, with $n \geqq 3$, has pointwise constant holomorphic sectional curvature H iff it has pointwise constant antiholomorphic sectional curvature $\nu = (1/4) H$.

Moreover, such a manifold satisfies the Schür's lemma of holomorphic (antiholomorphic) type.

These results dont's hold for a $2n$-dimensional l.c.K. manifold, $n \geqq 3$.

Indeed, when the antiholomorphic sectional curvature is (pointwise) constant, then the holomorphic sectional curvature is pointwise constant iff the manifold is a generalized complex space form. Moreover, the l.c.K. manifolds satisfy the Schür's lemma of antiholomorphic type; in [3]$_2$ it is proved that they dont' satisfy the Schür's lemma of holomorphic type.

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In the section 3, it is proved that, if $n \geqq 3$, a $2n$-dimensional l.c.K. manifold with ν (pointwise) constant is either a globally conformal Kähler manifold or a flat Kähler manifold or, possibly, its sectional curvature is constant and negative.

Finally, a classification of the 4-dimensional l.c.K. manifolds with ν (pointwise) constant and parallel Ricci tensor is obtained.

1 - Preliminaries

Let (M, g, J) be an *almost hermitian* (C^∞-differentiable) *manifold*, with real dimension $2n$ and fundamental 2-form Ω, such that $\Omega(X, Y) = g(X, JY)$.

According to [13], (M, g, J) is said to be a $\mathcal{R}_3$-manifold (RK-manifold, AH_3-manifold) if its riemannian curvature R satisfies

$$R(X, Y, Z, W) = R(JX, JY, JZ, JW), X, Y, Z, W \in X(M).$$

Moreover, $C(R)$, ρ, ρ^*, τ, τ^* denote *the Weyl tensor, the Ricci tensor, the *-Ricci tensor, the scalar and the *-scalar curvature of* R.

For a given $(0, 2)$-tensor field S on M, ϕS and ψS are the $(0, 4)$-tensor fields defined by

$$\begin{aligned}(1.1)\qquad \phi S(X, Y, Z, W) &= g(X, Z)S(Y, W) + g(Y, W)S(X, Z)\\ &\quad - g(X, W)S(Y, Z) - g(Y, Z)S(X, W);\\ \psi S(X, Y, Z, W) &= 2\Omega(X, Y)S(Z, JW) + 2\Omega(Z, W)S(X, JY)\\ &\quad + \Omega(X, Z)S(Y, JW) + \Omega(Y, W)S(X, JZ)\\ &\quad - \Omega(X, W)S(Y, JZ) - \Omega(Y, Z)S(X, JW).\end{aligned}$$

Moreover, π_1 and π_2 are the tensor fields such that $2\pi_1 = \phi g$, $2\pi_2 = \psi g$.

A $(0, 2)$-tensor field S on M is said to be J-invariant (J-anti-invariant) if

$$S(x, Y) = S(JX, JY) \qquad (S(X, Y) = -S(JX, JY)).$$

Let ∇ be the Levi-Civita connection on (M, g). Then, for any $(0, 2)$-tensor field S, by a direct calculation, one has

$$(1.2)\qquad (\nabla_V(\phi S))(X, Y, Z, W) = \phi(\nabla_V S)(X, Y, Z, W)$$

$$(1.3)\quad (\nabla_V(\psi S))(X, Y, Z, W) = \psi(\nabla_V S)(X, Y, Z, W) + 2g(X, JY)S(Z, (\nabla_V J)W)$$
$$+2g(X, (\nabla_V J)Y)S(Z, JW) + 2g(Z, JW)S(X, (\nabla_V J)Y) + 2g(Z, (\nabla_V J)W)S(X, JY)$$
$$+g(X, JZ)S(Y, (\nabla_V J)W) + g(X, (\nabla_V J)Z)S(Y, JW) + g(Y, JW)S(X, (\nabla_V J)Z)$$
$$+g(Y, (\nabla_V J)W)S(X, JZ) - g(X, JW)S(Y, (\nabla_V J)Z) - g(X, (\nabla_V J)W)S(Y, JZ)$$
$$-g(Y, JZ)S(X, (\nabla_V J)W) - g(Y, (\nabla_V J)Z)S(X, JW).$$

Moreover, when S is J-invariant, one has

$$(1.4)\quad (\nabla_V S)(JX, JY) = (\nabla_V S)(X, Y) - S((\nabla_V J)X, JY) - S(JX, (\nabla_V J)Y).$$

Let now (M, g, J) be a l.c.K.-*manifold*; $\{U_j\}_{j\in J}$ denotes an open covering of M such that

$$(1.5)\qquad \tilde g_j = e^{-\sigma_j} g|_{U_j} \qquad j \in J$$

is a local Kähler metric. The Lee form $\omega = \dfrac{1}{n-1}\, i(\Omega)(d\Omega)$ satisfies

$$(1.6)\qquad \omega|_{U_j} = d\sigma_j \qquad j\in J \qquad d\Omega = \omega\wedge\Omega \qquad d\omega = 0.$$

(M, g, J) is a globally conformal Kähler (g.c.K.) *manifold* iff ω is exact; (M, g, J) is a *Kähler manifold* iff $\omega = 0$.

For any $j\in J$, let $\tilde\phi_j$, $\tilde\psi_j$ be the operators defined as in (1.1) with respect to $\tilde g_j$. Then, (1.5) implies $\tilde\phi_j = e^{-\sigma_j}\Phi|_{\mathfrak{T}^0_2(U_j)}$, $\tilde\psi_j = e^{-\sigma_j}\Psi|_{\mathfrak{T}^0_2(U_j)}$.

The Weyl connection $\tilde\nabla$ on (M, g, J) is defined by

$$(1.7)\qquad \tilde\nabla_X Y = \nabla_X Y - \frac{1}{2}(\omega(X)Y + \omega(Y)X) + \frac{1}{2} g(X, Y)B$$

where B is the vector field associated with ω by means of g.

The following formulas hold

$$(1.8)\quad (\nabla_X J)Y = -\frac{1}{2}(\omega(Y)JX - \omega(JY)X + \Omega(X, Y)B - g(X, Y)JB)$$

$$(1.9)\qquad \lambda(X, Y, Z, W) = (\phi-\psi)P(X, Y, Z, W)$$
$$+2\Omega(X, Y)P(Z, JW) + 2\Omega(Z, W)P(X, JY)$$

$$(1.10)\qquad \rho - \rho^* = 2(n-2)P + \overline{P} + \operatorname{tr} P g$$

$$\tau - \tau^* = 4(n-1)\ \mathrm{tr}\ P = 4(n-1)\left(\frac{n-1}{4}\ \|\omega\|^2 - \frac{1}{2}\ \mathrm{div}\ B\right) \tag{1.11}$$

where

$$\lambda(X,\ Y,\ Z,\ W) = R(X,\ Y,\ Z,\ W) - R(X,\ Y,\ JZ,\ JW) \tag{1.12}$$

$$P = -\frac{1}{2}\left(\nabla\omega + \frac{1}{2}\ \omega \otimes \omega\right) + \frac{1}{8}\ \|\omega\|^2 g \tag{1.13}$$

and $\overline{P}$ is the J-anti-invariant part of P ([3]$_2$, [9]).

Then, by means of (1.9), $(M,\ g,\ J)$ turns out to be a $\mathcal{R}_3$-*manifold*, iff P is J-invariant, iff ρ is J-invariant ([9], [14]).

Moreover, for any $j \in J$, the riemannian curvature $\tilde{R}_j$ such that

$$\tilde{R}_j(X,\ Y,\ Z,\ W) = \tilde{g}_j(\tilde{R}(X,\ Y)Z,\ W)$$

where $\tilde{R}$ is the curvature of $\tilde{\nabla}$, satisfies

$$\mathrm{e}^{\sigma_j}\tilde{R}_j = (R - \phi P)|_{U_j}\,. \tag{1.14}$$

This implies

$$\tilde{\rho} = \rho - 2(n-1)P - \mathrm{tr}\ Pg \qquad \mathrm{e}^{-\sigma_j}\tilde{\tau}|U_j = (\tau - 2(2n-1)\ \mathrm{tr}\ P)|_{U_j} \tag{1.15}$$

where $\tilde{\rho}$, $\tilde{\tau}$ are the Ricci tensor and the scalar curvature of $\tilde{R}$ ([3]$_2$, [14]).

Finally, the explicit expression of the projections $\{p_k(R)\}$ of R on $\{\mathcal{W}_k\}$ is given in [3]$_2$.

2 - On the antiholomorphic sectional curvature of a l.c.k. manifold

Let $(M,\ g,\ J)$ be an almost hermitian manifold. A 2-plane α in T_xM, $x \in M$, is said to be *antiholomorphic* if α and $J\alpha$ are orthogonal. The manifold M has pointwise constant antiholomorphic sectional curvature if the sectional curvature $K(x;\ \alpha)$ relative to α does not depend on the antiholomorphic 2-plane α in T_xM, $x \in M$.

In this case, we put $\nu(x) = K(x;\ \alpha)$, for any $x \in M$. In particular, M has constant antiholomorphic sectional curvature if the function ν is constant.

Moreover, we recall that, when $n \geqq 2$, a $2n$-dimensional $\mathcal{R}_3$-manifold has pointwise constant antiholomorphic sectional curvature ν, iff its riemannian curvature is given by

$$(2.1)\qquad R = \frac{1}{6}\,\psi\rho + \nu\pi_1 - \frac{2n-1}{3}\,\nu\pi_2 \qquad \text{(see [5]}_1\text{)}.$$

This formula implies

$$(2.2)\qquad 3\rho^* - (n+1)\,\rho = \frac{1}{2n}\,(3\tau^* - (n+1)\,\tau)\,g;$$

$$(2.3)\qquad (2n+1)\,\tau - 3\tau^* = 8n(n^2-1)\,\nu;$$

Proposition 2.1. *Any l.c.K. manifold $(M,\ g,\ J)$ with pointwise constant antiholomorphic sectional curvature is a $\mathcal{R}_3$-manifold.*

Let $\{X,\ Y\}$ be an orthogonal basis of an antiholomorphic 2-plane α in T_xM, $x \in M$. Then, in the hypothesis of the statement, using (1.12), (1.9) and (1.1), one has

$$0 = \lambda(X,\ Y) - \lambda(JX,\ JX) = \phi\overline{P}(X,\ Y,\ X,\ Y) = \|X\|^2\overline{P}(Y,\ Y) + \|Y\|^2\overline{P}(X,\ X)$$

where $\lambda(X,\ Y) = \lambda(X,\ Y,\ X,\ Y)$.

Therefore, one has $\|X\|^2\,\overline{P}(Y,\ Y) + \|Y\|^2\,\overline{P}(X,\ X) = \|X\|^2\,\overline{P}(JY,\ JY) + \|Y\|^2\,\overline{P}(X,\ X) = 0$

Hence, $\overline{P}$ is J-invariant; this, together with the J-anti-invariance of $\overline{P}$, gives $\overline{P} = 0$.

Then, as a consequence of the Theorem 2 in **[10]**$_1$, one obtains

Corollary 2.1. *If $n \geqq 3$, any connected $2n$-dimensional l.c.K. manifold with pointwise constant antiholomorphic sectional curvature has constant antiholomorphic sectional curvature.*

Propositions 2.2. *Let $(M,\ g,\ J)$ be a l.c.K. manifold, with* $\dim M \geqq 6$. *The following statements are equivalent:*

(a) *(M, g, J) has pointwise constant antiholomorphic sectional curvature ν;*

(b) $p_3(R) = p_8(R) = 0,\ \rho + 6P = 2(n+1)\,\nu g$.

In the hypothesis (a), the Proposition 2.1 implies that M is a $\mathcal{R}_3$-manifold, so

$p_8(R) = 0$. Moreover, (1.11) and (2.3) give

$$\tau + 6 \operatorname{tr} P = 4n(n+1)\nu. \tag{2.4}$$

Since $n \geqq 3$, (1.10) and (2.2) imply

$$\rho + 6P = \frac{\tau + 6 \operatorname{tr} P}{2n}\, g = 2(n+1)\nu g\,. \tag{2.5}$$

Finally, using (2.1), (2.4), (2.5), since we have [3]$_2$

$$\begin{aligned} p_3(r) = R &- \frac{1}{4(n+2)}(\phi + \psi)(2\rho - 3(n-2)P) \\ &- \frac{1}{4}(3\phi - \psi)P + \frac{\tau + 6 \operatorname{tr} P}{4(n+1)(n+2)}(\pi_1 + \pi_2) \end{aligned} \tag{2.6}$$

we get $p_3(R) = 0$.

A direct computation, together with the hypothesis (b), given, for any $x \in M$ and for any antiholomorphic 2-plane α in $T_x M$

$$\begin{aligned} K(x;\alpha) &= \frac{1}{2(n+2)}\{(\rho + 6P)(X,\ X) + (\rho + 6P)(Y,\ Y) - \frac{\tau + 6 \operatorname{tr} P}{2(n+1)}\}_x \\ &= \frac{\tau + 6 \operatorname{tr} P}{4n(n+1)}(x) \end{aligned}$$

where $\{X,\ Y\}$ is an orthonormal basis of α.

Proposition 2.3. *Let $(M,\ g,\ J)$ be a 4-dimensional l.c.K. manifold. The following conditions are equivalent:*

(a) *$(M,\ g,\ J)$ has pointwise constant antiholomorphic sectional curvature;*

(b) *$(M,\ g,\ J)$ is a self-dual $\mathcal{R}_3$-manifold.*

We recall that, when $\dim M = 4$, any symmetric J-invariant $(0,\ 2)$-tensor field S, with $\operatorname{tr} S = 0$, satisfies see [13])

$$(3\phi - \psi)S = 0\,. \tag{2.7}$$

In the hypothesis (a), M is a $\mathcal{R}_3$-manifold; moreover, (1.11) and (2.3) imply

$\tau + 6 \operatorname{tr} P = 24\nu$. So, (2.1) and (2.6) give

$$p_3(R) = \frac{1}{6}\psi\rho + \frac{\tau + 6 \operatorname{tr} P}{24}(\pi_1 - \pi_2) - \frac{1}{8}(\phi + \psi)\rho - \frac{1}{4}(3\phi - \psi)P$$

$$+ \frac{\tau + 6 \operatorname{tr} P}{48}(\pi_1 + \pi_2) = -\frac{1}{24}(3\phi - \psi)(\rho - \frac{\tau}{4} g) = 0$$

since $P - \frac{1}{4} \operatorname{tr} Pg$ and $\rho - \frac{\tau}{4}$ satisfy (2.7).

This condition means that M is self-dual, since $p_3(R) = W_-$ is the anti-self-dual component of $C(R)$ [**13**].

In the hypothesis (b), the condition $p_3(R) = 0$ and (2.7) imply

$$R = \frac{1}{6}\psi\rho + \frac{\tau + 6 \operatorname{tr} P}{24}(\pi_1 + \pi_2).$$

So, the $\mathcal{R}_3$-manifold M has pointwise constant antiholomorphic sectional curvature.

Remark 2.1. For a $2n$-dimensional l.c.K. manifold (M, g, J) with pointwise constant antiholomorphic sectional curvature ν, these formulas hold

$$\rho + 6P = 2(n+1)\nu g \qquad n \geqq 3 \tag{2.8}$$

$$\tau + 6 \operatorname{tr} P = 4n(n+1)\nu \qquad n \geqq 2. \tag{2.9}$$

Proposition 2.4. *For a l.c.K. manifold (M, g, J) with* $\dim M \geqq 6$ *and constant antiholomorphic sectional curvature ν, the following conditions are equivalent:*

(a) *the local metrics $\{\tilde{g}_j\}_{j \in J}$ have pointwise constant antiholomorphic sectional curvatures;*

(b) $P = \frac{1}{2n} \operatorname{tr} Pg$;

(c) *(M, g, J) is a generalized complex space form.*

In the hypothesis of the statement, by means of (2.8), (2.1) reduces to

$$R = -\psi P + \nu(\pi_1 + \pi_2). \tag{2.10}$$

Then, (1.14) gives

$$(2.11)\qquad \tilde{R}_j = -(\tilde{\phi}+\tilde{\psi})(P - \frac{1}{2n}\ \mathrm{tr}\ Pg) + e^{\sigma_j}(\nu - \frac{1}{n}\ \mathrm{tr}\ P)|_{U_j}(\tilde{\pi}_1 + \tilde{\pi}_2) \qquad j \in J.$$

Since any $\tilde{g}_j$ is a Kähler metric, the condition (a) is equivalent to the request that any $\tilde{g}_j$ has pointwise constant holomorphic sectional curvature. The last condition is equivalent to the vanishing of $p_2(\tilde{R}_j)$, $p_3(\tilde{R}_j)$ [3]$_2$. So, the equivalence between (a) and (b) is a consequence of the relations

$$p_2(\tilde{R}_j) = -(\tilde{\phi}+\tilde{\psi})(P - \frac{1}{2n}\ \mathrm{tr}\ Pg) \qquad p_3(\tilde{R}_j) = 0.$$

Finally, the equivalence between (b) and (c) is a consequence of (2.10) [**13**].

Proposition 2.5. *Let (M, g, J) be a 4-dimensional l.c.K. manifold, with pointwise constant antiholomorphic sectional curvature. Then the local metrics $\{\tilde{g}_j\}_{\psi\in J}$ have pointwise constant antiholomorphic sectional curvatures. Moreover, such metrics have pointwise constant holomorphic sectional curvatures iff $\rho - 2P = \frac{\tau - 2\ \mathrm{tr}\ P}{4}\, g$.*

In the hypothesis of the statement, the formulas (2.1), (1.14), (1.15) and (2.9) give

$$(2.12)\qquad \tilde{R}_j = \frac{1}{6}\,\tilde{\psi}\tilde{\rho} + e^{\sigma_j}\frac{\tau - 6\ \mathrm{tr}\ P}{24}(\tilde{\pi}_1 + \tilde{\pi}_2) = \frac{1}{6}\,\tilde{\psi}\tilde{\rho} + \frac{1}{24}\,\tilde{\tau}(\tilde{\pi}_1 + \tilde{\pi}_2).$$

Thus, any $\tilde{g}_j$ has pointwise constant antiholomorphic sectional curvature $\tilde{\nu}_j = \frac{1}{24}\tilde{\tau}$.

Moreover, the holomorphic sectional curvature of $\tilde{g}_j$ is given by

$$\tilde{H}_j(x, X) = (\frac{\tilde{\rho}(X, X)}{\tilde{g}_j(X, X)} + \frac{\tilde{\tau}}{6})_x \qquad (x, X) \in TU_j.$$

So, $\tilde{H}_j$ is constant on the fibres of TU_j, iff $\tilde{g}_j$ is an Einstein metric, iff $\rho - 2P = \frac{\tau - 2\ \mathrm{tr}\ P}{4} g$ (see (1.15)).

3 - A classification theorem in the case $n \geqq 3$

Lemma 3.1. *Let (M, g, J) be a l.c.K. manifold with pointwise constant antiholomorphic sectional curvature ν. For any local orthonormal vector fields*

X, Y, with $g(X, JY)=0$, one has

$$(3.1)\qquad (\nabla_Y\rho)(X, Y)$$

$$=\frac{1}{2}\{\omega(JY)\rho(JX, Y)-\omega(X)\rho(Y, Y)-3\omega(Y)\rho(X, Y)-\omega(X)\rho(X, X)\}$$

$$+(2n-1)\nu\omega(X)+2X(\nu).$$

When X, Y satisfy the hypothesis of the statement, the J-invariance of ρ, together with (1.3), (1.4), (1.8), gives

$$(3.2)\qquad (\nabla_X(\psi\rho))(JX, Y, JX, Y)+(\nabla_{JX}(\psi\rho))(Y, X, JX, Y)+(\nabla_Y(\psi\rho))(X, JX, JX, Y)$$

$$=\frac{3}{2}\{\omega(JY)\rho(JX, Y)-\omega(X)\rho(X, X)-\omega(X)\rho(Y, Y)-3\omega(Y)\rho(X, Y)\}-3(\nabla_Y\rho)(X, Y).$$

Analogously, one has

$$(3.3)\qquad (\nabla_X\pi_2)(JX, Y, JX, Y)+(\nabla_{JX}\pi_2)(Y, X, JX, Y)$$

$$+(\nabla_Y\pi_2)(X, JX, JX, Y)=-\frac{3}{2}\omega(X).$$

Therefore, (3.1) is a consequence of the second Bianchi's identity and of (2.1), (3.2), (3.3).

Corollary 3.1. *In the hypothesis of the Lemma* 3.1, *for any unit local vector field X, one has*

$$(3.4)\qquad (\nabla_X\rho)(X, X)=\frac{1}{2}\{X(\tau)+\omega(X)\tau+(2n-7)\omega(X)\rho(X, X)+3\rho(X, B)\}$$

$$-2(2n-1)(n-1)\nu\omega(X)-4(n-1)X(\nu).$$

Let $\{X, E_2, \ldots, E_n, JX, JE_2, \ldots, JE_n\}$ be an orthonormal family of local vector fields on M. The formula (3.1) implies

$$(3.5)\qquad \sum_{i=2}^{n}\{(\nabla_{E_i}\rho)(X, E_i)+(\nabla_{JE_i}\rho)(X, JE_i)\}$$

$$=-2\rho(X, B)-(n-4)\omega(X)\rho(X, X)-\frac{1}{2}\omega(X)\tau$$

$$+2(n-1)(2n-1)\nu\omega(X)+4(n-1)X(\nu).$$

Since

$$\sum_{i=2}^{n}\{(\nabla_{E_i}\rho)(X,\ E_i)+(\nabla_{JE_i}\rho)(X,\ JE_i)\}+(\nabla_X\rho)(X,\ X)+(\nabla_{JX}\rho)(X,\ JX)=\frac{1}{2}X(\tau)$$

by means of (1.4) and (1.8) one has

$$(3.6)\qquad \sum_{i=2}^{n}\{(\nabla_{E_i}\rho)(X,\ E_i)+(\nabla_{JE_i}\rho)(X,\ JE_i)\}$$

$$=\frac{1}{2}X(\tau)-(\nabla_X\rho)(X,\ X)+\frac{1}{2}(\omega(X)\rho(X,\ X)-\rho(X,\ B)).$$

Then (3.4) is obtained comparing (3.5) with (3.6).

Lemma 3.2. *In the hypothesis of the Lemma* 3.1, *for any local orthonormal vector fields* $X,\ Y$, *with* $g(X,\ JY)=0$, *one has*

$$(3.7)\quad (\nabla_X\rho)(Y,\ Y)=-\omega(Y)\rho(X,\ Y)-\omega(X)\rho(Y,\ Y)-\frac{1}{2}(\rho(X,\ B)+\omega(X)\rho(X,\ X))$$

$$+2(2n-1)\nu\omega(X)+2nX(\nu).$$

By means of (1.3), (1.4) and (1.8), one obtains

$$(\nabla_X(\psi\rho))(Y,\ JY,\ Y,\ JY)+(\nabla_Y(\psi\rho))(JY,\ X,\ Y,\ JY)+(\nabla_{JY}(\psi\rho))(X,\ Y,\ Y,\ JY)$$

$$=3\{2(\nabla_X\rho)(Y,\ Y)-2\omega(JY)\rho(X,\ JY)-(\nabla_Y\rho)(X,\ Y)$$

$$-(\nabla_{JY}\rho)(X,\ JY)+\rho(X,\ B)+\omega(X)\rho(Y,\ Y)\}.$$

Moreover, one has

$$(\nabla_X\pi_2)(Y,\ JY,\ Y,\ JY)+(\nabla_Y\pi_2)(JY,\ X,\ Y,\ JY)+(\nabla_{JY}\pi_2)(X,\ Y,\ Y,\ JY)=3\omega(X).$$

Then, (2.1) and the second Bianchi's identity imply

$$(\nabla_X\rho)(Y,\ Y)=\frac{1}{2}\{(\nabla_Y\rho)(X,\ Y)+(\nabla_{JY}\rho)(X,\ JY)-\omega(X)\rho(Y,\ Y)-\rho(X,\ B)\}$$

$$+\omega(JY)\rho(X,\ JY)+(2n-1)\nu\omega(X)+2(n-1)X(\nu).$$

Then, the Lemma 3.1 leads to the required result.

Corollary 3.2. *In the hypothesis of the Lemma* 3.1, *for any local unit vector field* X, *one has*

$$(3.8)\quad (\nabla_X \rho)(X,\ X) = \frac{1}{2}\{X(\tau) + \omega(X)\tau - 3\omega(X)\rho(X,\ X) + (2n-1)\rho(X,\ B)\}$$
$$-2(n-1)(2n-1)\nu\omega(X) - 4(n-1)^2 X(\nu).$$

The proof is a consequence of the Lemma 3.1 and is carried out as in the Corollary 3.1.

Corollary 3.3. *If* $(M,\ g,\ J)$ *is a l.c.K. manifold with* $\dim M \geqq 6$ *and constant antiholomorphic sectional curvature* ν, *one has*

$$(3.9)\qquad \omega(X)\rho(X,\ X) = \|X\|^2\rho(X,\ B) \qquad \textit{for any } X \in \mathcal{X}(M).$$

In fact, the comparison of (3.4) with (3.8) gives: $(n-2)$ $\{\omega(X)\rho(X,\ X) -\rho(X,\ B) + 4(n-1)X(\nu)\} = 0$ for any local unit vector field X.

Theorem 3.1. *Let* $(M,\ g,\ J)$ *be a connected l.c.K. manifold, with* $\dim M \geqq 6$ *and constant antiholomorphic sectional curvature* ν. *If* $\|\omega\|^2$ *is constant, then either* $(M,\ g,\ J)$ *is a Kähler manifold or* $-\frac{1}{4}\|\omega\|^2$ *is the constant value of its sectional curvature.*

In fact, taking account of (2.8), the formula (3.9) is equivalent to

$$(3.10)\qquad \omega(X)P(X,\ X) = \|X\|^2 P(X,\ B) \qquad X \in \mathcal{X}(M)$$

that is

$$(3.11)\qquad \underset{(X,Y,Z)}{\sigma}\omega(X)P(Y,\ Z) = \underset{(X,Y,Z)}{\sigma}P(X,\ B)g(Y,\ Z)$$

since P and g are symmetric. Here σ denotes the cyclic sum.

Putting in (3.11) $Z = B$, one has

$$(3.12)\qquad \|\omega\|^2 P = P(B,\ B)g.$$

Then, if $\|\omega\|^2$ is constant, by means of (1.13), (3.12) reduces to

$$\|\omega\|^2(P + \frac{1}{8}\|\omega\|^2 g) = 0.$$

Therefore, $(M,\ g,\ J)$ is Kähler manifold or $P = -\frac{1}{8}\|\omega\|^2 g$. In the last case,

$\frac{1}{n}$ tr $P = -\frac{1}{4}\|w\|^2$ and $R = \nu\pi_1 + (\nu + \frac{1}{4}\|\omega\|^2)\pi_2$, that is (M, g, J) is a generalized complex space form. The statement is a consequence of the Theorem 12.7 in [13].

Lemma 3.3. *If* (M, g, J) *is a l.c.K. manifold, with* $\dim M \geqq 6$ *and constant antiholomorphic section curvature* ν, *then*

$$\rho(X, B) = -\frac{1}{6}X(\tau) - \frac{1}{2}\{(n-1)X(\|\omega\|^2) + \operatorname{div} B\omega(X)\} \qquad X \in \mathcal{X}(M). \tag{3.13}$$

In fact, (2.8) and (1.13) imply

$$\rho = 3(\nabla\omega + \frac{1}{2}\omega \otimes \omega) + (2(n+1)\nu - \frac{3}{4}\|\omega\|^2)g. \tag{3.14}$$

Therefore, for any family of local orthonormal vector fields $\{E_h\}_{1 \leqq h \leqq 2n}$, one has

$$\frac{1}{2}X(\tau) = \sum_{h=1}^{2n}(\nabla_{E_h}\rho)(X, E_h) = 3\{\sum_{h=1}^{2n}(\nabla_{E_h}(\nabla\omega))(X, E_h) + \frac{1}{2}\omega(X) \operatorname{div} B\}.$$

Moreover, since ω is closed, then ([6]): $\rho(X, B) = -X(\operatorname{div} B) + \sum_{h=1}^{2n}(\nabla_{E_h}(\nabla\omega))(X, E_h)$, that is $\rho(X, B) = -X(\operatorname{div} B) + \frac{1}{6}X(\tau) - \frac{1}{2}\omega(X)\operatorname{div} B$. The formulas (2.9) and (1.11) imply (3.13).

Lemma 3.4. *In the hypothesis of the Lemma* 3.3, *one has*

$$n\, d\tau = (-2(n+1)\tau + 4n(n+1)(2n-1)\nu)\omega \tag{3.15}$$

$$6n\, d(\|\omega\|^2) = (-3n\|\omega\|^2 + 2\tau - 8n(n+1)\nu)\omega. \tag{3.16}$$

In fact, (3.14) implies

$$\rho(X, B) = \frac{3}{2}X(\|\omega\|^2) + (\frac{3}{4}\|\omega\|^2 + 2(n+1)\nu)\omega(X). \tag{3.17}$$

Therefore, by means of (3:17), (3.13), (1.11), (2.9), one has

$$2\, d\tau + 6(n+2)\, d(\|\omega\|^2) = (-3(n+2)\|\omega\|^2 - 2\tau + 8(n+1)(n-3)\nu)\omega. \tag{3.18}$$

Moreover, the Lemma 5.3 in $[3]_1$ gives

$$\rho(X, B) = \frac{1}{4}(X\|\omega\|^2) - 3 \operatorname{div} B\omega(X)) - \frac{1}{2}X(\operatorname{div} B) + \frac{2n-1}{8}\|\omega\|^2\omega(X). \tag{3.19}$$

Therefore, (3.16) is a consequence of (3.19), (3.17), (1.11) and (2.9). Finally, (3.16) and (3.18) imply (3.15).

Theorem 3.2. *Let (M, g, J) be a connected l.c.K. manifold, with* $\dim M \geqq 6$ *and constant antiholomorphic sectional curvature* ν. *Then,* (M, g, J) *is either a g.c.K. manifold of a flat Kähler manifold or, possibly, its constant sectional curvature is* $-\frac{1}{4}\|\omega\|^2$.

In the hypothesis of the statement, since ν is constant, (3.15) is equivalent to

$$d(\tau - 2n(2n-1)\nu) = -\frac{2(n+1)}{n}(\tau - 2n(2n-1)\nu)\,\omega .$$

Therefore, since M is connected and ω is locally exact, then ω is exact, iff there exists $x \in M$ such that $(\tau - 2n(2n-1)\nu)(x) \neq 0$. Otherwise, $\tau = 2n(2n-1)\nu$, and then (3.16) reduces to $2\,d(\|\omega\|^2 + 4\nu) = -(\|\omega\|^2 + 4\nu)\,\omega$.

Hence, if $\nu \neq -\frac{1}{4}\|\omega\|^2$, ω is exact. When $\nu = -\frac{1}{4}\|\omega\|^2$, $\|\omega\|^2$ is constant and the Theorem 3.1 can be applied. Then, M has constant sectional curvature $\nu = -\frac{1}{4}\|\omega\|^2$ or M is a flat Kähler manifold, since $\nu = 0$.

Remark. In the hypothesis of the Theorem 3.2, when M is compact, (M, g, J) is either a g.c.K. manifold or a flat Kähler manifold.

Indeed, when $\nu = -\frac{1}{4}\|\omega\|^2$, one has $\nu = \frac{1}{n}\operatorname{tr} P$, and so $(2n-1)\|\omega\|^2 = 2\operatorname{div} B$. Then, the divergence theorem leads to $\omega = 0$.

We conclude this section proving a lemma useful in the following

Lemma 3.5. *Let (M, g, J) be a l.c.K. manifold, with* $\dim M \geqq 6$ *and constant antiholomorphic sectional curvature. Then, one has*

$$(3.20)\quad (\nabla_X \rho)(Y, Z) - (\nabla_Y \rho)(X, Z) = 3\omega(R(X, Y)Z) + \frac{3}{2}(\omega(Y)(\nabla_X \omega)Z - \omega(X)(\nabla_Y \omega)Z)$$

$$-\frac{3}{4}(X(\|\omega\|^2)\,g(Y, Z) - Y(\|\omega\|^2)\,g(X, Z)) \qquad X, Y, Z, \in \mathfrak{X}(M).$$

Indeed, (3.14) implies

$$(\nabla_X \rho)(Y,\ Z) - (\nabla_Y \rho)(X,\ Z) = 3\{(\nabla_X(\nabla\omega))(Y,\ Z) - (\nabla_Y(\nabla\omega))(X,\ Z)\}$$

$$+\frac{3}{2}\{\omega(Y)(\nabla_X\omega)Z - \omega(X)(\nabla_Y\omega)Z\} - \frac{3}{4}\{X(\|\omega\|^2)g(Y,\ Z) - Y(\|\omega\|^2)g(X,\ Z)\}.$$

This formula leads to (3.20), since $(\nabla_X(\nabla\omega))(Y,\ Z) - (\nabla_Y(\nabla\omega))(X,\ Z) = \omega(R(X,\ Y)Z)$.

4 - Classification theorems in the case $\nabla\rho = 0$

Let $(M,\ g)$ be a n-dimensional riemannian manifold. $\mathcal{A}$ denotes the vector bundle on M whose sections are the (0, 3)-tensor fields η, symmetric with respect to the last two variables, such that

$$\sum_{h=1}^{n} \eta(X,\ E_h, E_h) = 2\sum_{h=1}^{n} \eta(E_h,\ E_h,\ X) \qquad X \in \mathcal{X}(M).$$

Here $\{E_h\}_{1\leqq h\leqq n}$ is an orthonormal family of local vector fields.

It is well known [7]$_2$ that $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \mathcal{A}_3$, where, for any $i \in \{1, 2, 3\}$, $\mathcal{A}_i$ is the subbundle of $\mathcal{A}$ whose sections satisfy the condition (i), with

(1) $\eta(X,\ X,\ X) = 0$

(2) $\eta(X,\ Y,\ Z) = \eta(Y,\ X,\ Z)$

(3) $$\eta(X,\ Y,\ Z) = \frac{1}{(n+2)(n-1)}\{n \operatorname{tr}\eta(X)\, g(Y,\ Z) + \frac{1}{2}(n-2)(\operatorname{tr}\eta(Y)g(X,\ Z) + \operatorname{tr}\eta(Z)g(X,\ Y))\}.$$

The symbol $\operatorname{tr}\eta$ stands for the 1-form defined by $\operatorname{tr}\eta(X) = \sum_{h=1}^{n}\eta(X,\ E_h,\ E_h)$. Moreover, the sections of $\mathcal{A}_1 \oplus \mathcal{A}_2$ are the sections η such that $\operatorname{tr}\eta = 0$.

In particular, when M is connected, $\nabla\rho$ is section of $\mathcal{A}_1 \oplus \mathcal{A}_2$, iff τ is constant.

Theorem 4.1. *Let $(M,\ g,\ J)$ be a connected l.c.K. manifold, with* $\dim M \geqq 6$ *and constant antiholomorphic sectional curvature ν. The following conditions are equivalent:*

(a) *τ is constant;*

(b) $\nabla\rho = 0$;

(c) *(M, g) is an Einstein manifold.*

In fact, when τ is constant, (3.15) and (3.16) become

$$(\tau - 2n(2n-1)\nu)\,\omega = 0; \tag{4.1}$$

$$d(\|\omega\|^2) = -\frac{1}{2}(\|\omega\|^2 + 4\nu)\,\omega. \tag{4.2}$$

Moreover, (3.17), (4.1), (4.2) give

$$\rho(X, B) = \frac{\tau}{2n}\,\omega(X). \tag{4.3}$$

Applying the Corollaries 3.2, 3.3 and the formulas (4.1), (4.3), $\nabla\rho$ turns out to be a section of $\mathcal{C}_1$. Moreover, since τ and ν are constant, the equation (4.1) leads to the following cases

$$\text{(I)}\quad \omega = 0 \qquad\qquad \text{(II)}\quad \tau = 2n(2n-1)\,\nu.$$

In the case (I), (M, g, J) is a Kähler manifold with constant holomorphic sectional curvature, and so $\nabla\rho = 0$.

In the case (II), the Corollary 3.3 and (4.3) imply $\omega(X)\rho(X, X) = \frac{\tau}{2n}\|X\|^2\omega(X)$, $X \in \mathcal{X}(M)$, which is equivalent to

$$\underset{(X,Y,Z)}{\sigma}\omega(X)\rho(Y, Z) = \underset{(X,Y,Z)}{\sigma}\frac{\tau}{2n}\,\omega(X)g(Y, Z). \tag{4.4}$$

Putting in (4.4) $Y = Z$ orthogonal to $X = B$, one has

$$\omega \otimes \rho = \frac{\tau}{2n}\,\omega \otimes g. \tag{4.5}$$

The condition (b) is achieved proving that $\nabla\rho$ is a section of $\mathcal{C}_2$; the Lemma 3.5 is applied to this end. In fact, (4.3), (4.5), (2.1) give

$$\omega(R(X, Y)Z) = R(X, Y, Z, B) = \nu(\omega(Y)g(X, Z) - \omega(X)g(Y, Z)).$$

By means of (1.13), (2.8), (4.5) one has

$$\omega(Y)(\nabla_X\omega)Z - \omega(X)(\nabla_Y\omega)Z = \left(\frac{1}{4}\|\omega\|^2 - \nu\right)(\omega(Y)g(X, Z) - \omega(X)g(Y, Z)).$$

These formulas, together with (3.20) and (4.2), lead to $(\nabla_X\rho)(Y, Z) - (\nabla_Y\rho)(X, Z) = 0$.

In the hypothesis (*b*), (4.1) holds. So, when $\omega = 0$, (M, g) is an Einsten manifold.

Suppose now that $\tau = 2n(2n-1)\nu$. In the case $\nu = 0$, the discussion of (4.2), combined with the formula (4.5), implies that (M, g) is an Einstein manifold. If $\nu \neq 0$, suppose that (M, g) is not an Einstein manifold and fix $x \in M$. Since $\nabla\rho = 0$, there exists an open neighbourhood U of x such $(U, g|_U)$ is isometric to the product $(M_1 \times M_2 \times \ldots \times M_k,\ g_1 + g_2 + \ldots + g_k)$ of Einstein manifolds, such that $\rho|_U = \mu_1 g_1 + \ldots + \mu_k g_k$, where $\mu_1, \ldots, \mu_k$ are different constants [12].

In this way, any M_h is a totally geodesic submanifold of U and $\rho(X, Y) = 0$, for any $X \in TM_h$, $Y \in TM_j$, with $h \neq j$. For any $h \in \{1, \ldots, k\}$, $y \in M_h$, $T_y M_h$ is the eigenspace of Q_y corresponding to the eigenvalue μ_h, where Q_y is the (1, 1)-tensor associated with ρ_y. The J-invariance of ρ implies that, for any $X \in T_y M_h$, $Q_y(JX) = \mu_h JX$; therefore $JX \in T_y M_h$. So M_h is an almost hermitian submanifold of U. Then, using (2.1), one has

$$0 = R(X, Y, X, Y) = \nu\|X\|^2\|Y\|^2 \qquad X \in TM_h \qquad Y \in TM_j \qquad h \neq j.$$

Therefore, $\nu = 0$, which contradicts the hypothesis.

Corollary 4.1. *For a connected l.c.K manifold (M, g, J), with* $\dim M \geqq 6$, *the following conditions are equivalent:*

(a) *(M, g, J) is a generalized complex space form;*

(b) *(M, g, J) is an Einstein manifold with constant antiholomorphic sectional curvature;*

(c) *(M, g, J) has constant scalar curvature and constant antiholomorphic sectional curvature.*

Moreover, if one of the previous conditions holds, (M, g, J) is either a Kähler manifold or its sectional curvature is constant. In the last case, M admits a flat Hähler metric or has sectional curvature $-\frac{1}{4}\|\omega\|^2$.

Infact, in the hypothesis (a), since $R = p_1(R) + p_4(R)$, one has: $p_3(R) = p_8(R) = 0$, $P = \frac{1}{2n}$ tr Pg and (M, g) is an Einstein manifold. Therefore, the Proposition 2.2 implies (b). The implication (b) $\Rightarrow$ (a) is a consequence of the Propositions 2.2 and 2.4. The equivalence (b) $\Leftrightarrow$ (c) follows immediately applying the Theorem 4.1.

The proof of the last part of the statement is carried out in the Theorem 4.1, where it is proved that: $R = \nu(\pi_1 + \pi_2)$, when $\omega = 0$; $R = \nu\pi_1$, when $\tau = 2n(2n-1)\nu$.

Note that, if $R = \nu\pi_1$, then $\nu = \frac{1}{n}$ tr P. So, M admits a flat global Kähler metric, if $\nu \neq -\frac{1}{4}\|\omega\|^2$ ([3]$_2$, [14]).

Theorem 4.2. *Let (M, g, J) be a connected, 4-dimensional l.c.K. manifold, with pointwise constant antiholomorphic sectional curvature ν. The following conditions are equivalent:*

(a) *$\nabla\rho$ is a section of $\mathcal{A}_1$;*

(b) *$\nabla\rho = 0$;*

(c) *$\nabla\rho$ is a section of $\mathcal{A}_2$.*

First of all, the formula (2.9) and Lemma 5.3 in [3]$_1$ imply

$$\rho(X, B) = 4X(\nu) + 6\nu\omega(X) - \frac{1}{6}X(\tau) - \frac{1}{4}\tau\omega(X). \tag{4.6}$$

Let T be the (0, 3)-tensor field defined by

$$T(X, Y, Z) = (\nabla_X\rho)(Y, Z) + \frac{3}{2}\omega(X)\rho(Y, Z) - (2\,\mathrm{d}\nu + 3\nu\omega + \frac{1}{8}\tau\omega)(X)\,g(Y, Z).$$

In the hypothesis (a), τ is constant: then (4.6), (3.8) give $T(X, X, X) = 0$, $X \in \mathcal{X}(M)$, that is

$$\underset{(X,Y,Z)}{\sigma} T(X, Y, Z) = 0. \tag{4.7}$$

Putting in (4.7) $Y = Z$ orthogonal to $X = B$ and using (4.6), one obtains

$$(\rho(B, B) + \frac{1}{2}\tau\|\omega\|^2)\,g(Y, Y) - 3\|\omega\|^2\rho(Y, Y) = 0. \tag{4.8}$$

Moreover, putting in (4.7) $X = Y = Z = B$, one has

$$\|\omega\|^2(\rho(B, B) - \frac{\tau}{4}\|\omega\|^2) = 0. \tag{4.9}$$

Then, (4.8) and (4.9) imply

$$\|\omega\|^4\rho(Y, Y) = \frac{\tau}{4}\|\omega\|^4 g(Y, Y) \qquad \text{for any Y orthogonal to B.} \tag{4.10}$$

Since $\rho - \frac{\tau}{4}g$ is symmetric, (4.9) and (4.10) lead to

$$\omega \otimes \rho = \frac{\tau}{4}\omega \otimes g. \tag{4.11}$$

Now, we are going to prove the following formula

$$\mathrm{d}(12\nu - \tau) = -\frac{3}{2}(12\nu - \tau)\,\omega. \tag{4.12}$$

To this end, consider $x \in M$ such that $\omega_x \neq 0$. Since $\rho_x = \frac{\tau}{4} g_x$, putting in (4.7) $Y = Z \neq 0$ orthogonal to X, one has $2(\mathrm{d}\nu)_x = (\frac{1}{4}\tau - 3\nu)(x)\,\omega_x$.

Moreover, for a fixed $x \in M$ such that $\omega_x = 0$, (4.7) reduces to

$$\underset{(X,Y,Z)}{\sigma}(\mathrm{d}\nu)_x(X)\, g_x(Y,Z) = 0\,,$$

which leads to $(\mathrm{d}\nu)_x = 0$. This completes the proof of (4.12).

Moreover, (4.12) and (4.6) give

$$\rho(X,\ B) = \frac{\tau}{4}\omega_x\,. \tag{4.13}$$

Let X be a local unit vector field and complete it to a local orthonormal frame $\{X,\ JX,\ Y,\ JY\}$. Then (3.1), (3.7), (1.4), (4.11), (4,13) imply that $\nabla_X \rho$ vanishes for any choice of two vector fields in the given frame. Therefore, ρ is ∇-parallel

In the hypothesis (c), τ is constant. Let $X,\ Y$ be local orthonormal vector fields such that $g(X,\ JY) = 0$. Then, the condition $(\nabla_X\rho)(Y,\ Y) = (\nabla_Y\rho)(X,\ Y)$ together with (3.1), (3.7), (4.6) implies

$$\omega(Y)\rho(X,\ Y) - \omega(X)\rho(Y,\ Y) + \omega(JY)\rho(X,\ JY) + \frac{1}{4}\tau\omega(X) = 0\,. \tag{4.14}$$

Moreover, (4.14) leads to (4.13), since $B = \omega(X)X + \omega(JX)JX + \omega(Y)Y + \omega(JY)JY$.

The formula (4.12) is a consequence of (4.6), (4.13).

For a fixed $x \in M$, such that $\omega_x \neq 0$, let $\{Y,\ JY,\ \frac{B}{\|B\|},\ \frac{JB}{\|B\|}\}$ be an orthonormal basis of T_xM. Then, using (4.13), one has

$$\|\omega\|_x^2\rho_x(Y,\ Y) = \frac{\tau}{2}\|\omega\|_x^2 - \rho_x(B,\ B) = \frac{\tau}{4}\|\omega\|_x^2\,.$$

This relation, together with (4.13) and the J-invariance of $\rho - \frac{\tau}{4}g$ implies $\omega_x \otimes \rho_x = \frac{\tau}{4}\omega_x \otimes g_x$. Therefore, one has $\omega \otimes \rho = \frac{\tau}{4}\omega \otimes g$.

Finally, the condition (a) is an easy consequence of (3.8), (4.11), (4.12), (4.13).

Theorem 4.3. *Let $(M,\ g,\ J)$ be a 4-dimensional conneted l.c.K. manifold, with $\nabla\rho = 0$ and pointwise constant antiholomorphic sectional curvature ν. Then $(M,\ g,\ J)$ is either a conformally flat Kähler manifold or is an Einstein manifold. In the last case, $(M,\ g)$ has constant sectional curvature, iff it is conformal-*

ly flat, otherwise (M, g, J) is a g.c.K. manifold, whose Weyl tensor never vanishes.

Since $\nabla\rho = 0$, (M, g) is an Einstein manifold or is locally isometric to the product of Einstein manifolds.

In the first case, the formula (2.1) reduces to $R = \nu\pi_1 + \dfrac{\tau - 12v}{12}\pi_2$. Since $C(R) = W_+ = -\dfrac{\tau - 12\nu}{12}(\pi_1 - \pi_2)$, (M, g) is conformally flat, iff $R = \nu\pi_1$. Moreover, (4.12) implies that (M, g, J) is a g.c.K. manifold when $C(R)$ never vanishes.

If (M, g) is not an Einstein manifold, let U be an open neighbourhood of a fixed $x \in M$, isometric to the product $M_1 \times M_2 \times \ldots \times M_k$ of Einstein manifolds, such that $\rho|_U = \mu_1 g_1 + \ldots + \mu_k g_k$.

Then, $k = 2$, since any M_i turns out to be an hermitian manifold (see the proof of the Theorem 4.1). Thus, given $X \in TM_1$, $Y \in TM_2$, with $\|X\| = \|Y\| = 1$, one has

$$0 = R(X, Y, X, Y) = \nu \qquad 0 = R(X, JX, Y, JY) = \frac{1}{3}(\mu_1 + \mu_2),$$

that is $\nu|_U = 0$ and $\tau = 2(\mu_1 + \mu_2) = 0$. This implies $C(R)|_U = 0$. Moreover, since $\rho_x \neq 0$ for any $x \in U$, the formula (4.11) implies $\omega|_U = 0$.

This proves that in this case (M, g, J) is a conformally flat Kähler manifold.

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Sunto

Si studiano le varietà localmente conformi a una varietà di Kähler, di curvatura sezionale antiolomorfa puntualmente costante. Si caratterizzano le suddette varietà e si prova un teorema di classificazione. Inoltre, si esamina il caso in cui il tensore di Ricci è parallelo.

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