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An example of an almost hermitian flat manifold which is not hermitian (**)

1 - An almost hermitian manifold which is not hermitian

We consider the space $\mathbb{R}^4$ as a *differentiable manifold.* Let $x = (x^1, x^2, x^3, x^4)$ be a point of $\mathbb{R}^4$ and let T_x be the tangent space of $\mathbb{R}^4$ at the point x.

Let $\{e_j\}$ be the natural basis defined by $e_j = \partial/\partial x^j$. We consider the canonical metric g on $\mathbb{R}^4$ given by $g_{ij} = g(e_i, e_j) = \delta_{ij}$. The manifold $\mathbb{R}^4$ with the Riemannian metric g becomes a *riemannian manifold.*

In the vector space T_x we consider the basis $\{E_j\}$ defined by

$$\begin{aligned} E_1 &= e_1 \qquad & E_2 &= e_2 \cos x^1 + e_3 \sin x^1 , \\ E_4 &= e_4 \qquad & E_3 &= - e_2 \sin x^1 + e_3 \cos x^1 . \end{aligned} \tag{1}$$

We note that the basis, which we have constructed, is orthonormal i.e. $g(E_i, E_j) = \delta_{ij}$.

Then we define an almost complex structure. In fact, the following identities

$$J(E_1) = E_2 \qquad J(E_2) = - E_1 \qquad J(E_3) = E_4 \qquad J(E_4) = - E_3 \tag{2}$$

define in each T_x *a homomorphism* J. It is immediate that J is an isomorphism and that $J^2 = -1$. Hence in this way $\mathbb{R}^4$ becomes *an almost complex manifold.*

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We also observe that for each point x of $\mathbb{R}^4$ we have

$$g(E_i, E_j) = g(JE_i, JE_j).$$

This implies that the almost complex structure J is compatible with the metric structure g, i.e. for any vectors X, Y on $\mathbb{R}^4$ we have

$$g(X, Y) = g(JX, JY). \tag{3}$$

It is also easy to observe that both structures J and g are smooth as x varies on $\mathbb{R}^4$. Hence it follows that $\mathbb{R}^4$ with the introduced structures is *an almost hermitian manifold.*

Since with respect to the coordinates x^k $(k = 1, 2, 3, 4)$ the coefficients of g_{ij} of the metric g are constant, then the curvature tensor R vanishes at each point x of $\mathbb{R}^4$. It means that $\mathbb{R}^4$ with the introduced structure is an *almost hermitian flat manifold.*

As the point x varies in $\mathbb{R}^4$, equation (1) defines four vector fields also denote by E_j. In such a way get four basic C^∞-vector fields on $\mathbb{R}^4$. It is an easy computation to get the following relations

$$\begin{aligned} &[E_1, E_2] = E_3 \qquad [E_1, E_3] = -E_2 \\ &[E_1, E_4] = [E_2, E_3] = [E_2, E_4] = [E_3, E_4] = 0. \end{aligned} \tag{4}$$

At this point it is convenient to recall *the Nijenhuis tensor* N which expresses the torsion of an alomst complex structure J, defined by

$$\frac{1}{2}N(X, Y) = [JX, JY] - [X, Y] - J[JX, Y] - J[X, JX]$$

where X, Y are arbitrary C^∞-vector fields on the manifold (cf. [2], p. 123).

In our case, applying (2) and (4), we get

$$\frac{1}{2}N(E_1, E_4) = [JE_1, JE_4] - [E_1, E_4] - J[JE_1, E_4] - J[E_1, JE_1]$$

$$= -[E_2, E_3] - [E_1, E_4] - J[E_2, E_4] + J[E_1, E_3] = -JE_2 = E_1 \neq 0.$$

It follows that $N \neq 0$. So we conclude that *the almost complex structure J is not integrable. Therefore $\mathbb{R}^4$ with the given structure is not a hermitian manifold.*

2 - Remarks

The example, we constructed in the first part of the paper, permits us to make some remarks.

First of all we recall that a manifold M is called *para-kähler* (cf. G. B. Rizza, [3], p. 51) iff for each point $x \in \mathbb{R}^k$ and for each X, Y, Z, W vectors of the tangent space T_x we have

$$R(X, Y, Z, W) = R(X, Y, JZ, JW). \tag{5}$$

This identity is known as the *Kähler identity.*

We recall also that, if M is a hermitian manifold, then for each point $x \in M$ and for each vectors X, Y, Z, W of the tangent space $T_x(M)$ we have

$$\begin{aligned} R(X, Y, Z, W) - R(JX, JY, Z, W) - R(X, Y, JZ, JW) \\ + R(JX, JY, JZ, JW) - R(X, JY, Z, JW) - R(JX, Y, Z, JW) \\ - R(X, JY, JZ, W) - R(JX, Y, JZ, W) = 0 \end{aligned} \tag{6}$$

(cf. [1], Corollary 3.2, p. 603). Relation (6) is called the *A. Gray identity.*

We are now able to prove the propositions:

P_1. *There exist para-kähler manifolds which are not Kähler.*

P_2. *There exist almost hermitian manifolds, statisfying the A. Gray identity, which are not hermitian.*

In fact, the almost hermitian manifold $(\mathbb{R}^4, g, J)$, we constructed in **1** provides an example for both P_1 and P_2. Since the metric structure is flat, i.e. $R = 0$, then identities (5) and (6) obviously hold. On the other hand since $N \neq 0$ then the manifold $(\mathbb{R}^4, g, J)$ is not hermitian. Therefore it cannot be a Kähler manifold neither.

We would like to underline that proposition P_1 was already known (cf. e.g. [4], p. 249). However the example in the first part of this paper appears much more simpler than the one in [4].

As a final remark we would like to note that proposition P_2 suggests the study of the almost hermitian manifolds the which satisfy Gray's identity.

References

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Abstract

In the present paper we construct an almost hermitian flat manifold which is not hermitian. This example demonstrates that there exist para-kähler manifolds which are not Kähler, and that there exist almost hermitian manifolds satisfying the A. Gray identity, which are not hermitian.

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