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On quaternion submanifolds of codimension r (**)

1 - Preliminaries

Quaternion submanifolds of codimension 2 have been defined and studied by A. Hamoui [1] and others. Dube and Nivas [2] defined and studied the almost r-contact hyperbolic structures. In the present paper, we consider a submanifold of codimension r of a quaternion manifold and study some of its properties. We also show that the quaternion submanifold of codimension r admits an almost r-contact structure.

A *quaternion manifold* M^{4n} is a manifold admitting a set of three (1, 1) tensor fields F^*, G^*, H^* satisfying the following relations

(1.1) $$F^{*2} G^{*2} = H^{*2} = -I \qquad F^* = G^* H^* = -H^* G^*$$

(1.2) $$G^* = H^* F^* = -F^* H^* \qquad H^* = F^* G^* = -G^* F^*$$

where I denotes the unit tensor field.

If g^* is the *hermitian metric* on M^{4n} we have

(1.3) $$g^*(F^* X^*, F^* Y^*) = g^*(X^*, Y^*)$$

for arbitrary vector fields X^*, Y^* on M^{4n}.

A manifold V_n is said to possess an almost *r-contact structure* [2], if there exists a tensor field ϕ of type (1, 1), rC^∞ contravariant vector fields $\underset{1}{u}, \underset{2}{U}, \dots, \underset{r}{U}$

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and rC^∞ 1-forms $\overset{1}{u}, \overset{2}{u}, \ldots, \overset{r}{u}$ (r some finite integer) satisfying

$$\phi^2 = -I + \overset{x}{u} \otimes \underset{x}{U} \qquad \phi \underset{x}{U} = \theta_x^y \underset{y}{U}$$

(1.4)

$$\overset{x}{u}_0 \phi = \theta_y^x \overset{y}{u} \qquad \overset{x}{u}(\underset{y}{U}) = \delta_y^x + \theta_z^x \theta_y^z$$

where θ_x^y are scalar fields, δ_y^x denotes the Kronecker delta and $x, y, z = 1, \ldots, r$.

2 - Structure in M^{4n-r}

Let M^{4n-r} be *submanifold of codimension* r of the quaternion manifold M^{4n}. Let B be differential of immersion i: $M^{4n-r} \to M^{4n}$. Thus a vector field X in the tangent space of M^{4n-r} corresponds to the vector fields BX in that of M^{4n} [3]. If $\underset{x}{N}$, $x = 1, 2, \ldots, r$ be mutually orthogonal unit normals to M^{4n-r}, the transformation F^*BX and $F^*\underset{x}{N}$ can be expressed as [1], [3]

$$F^*BX = BFX + \overset{x}{u}(X)\underset{x}{N} \tag{2.1}$$

$$F^*\underset{x}{N} = B\underset{x}{U} - \theta_x^y \underset{y}{N} \tag{2.2}$$

where $\overset{x}{u}$ are rC^∞ 1-forms, F a tensorfield of type (1, 1) on M^{4n-r} and $\underset{x}{U}$ are rC^∞ vector fields on M^{4n-r} ($x = 1, \ldots, r$).

Similarly, for tensor fields G^* and H^*, we have

$$G^*BX = BGX + \overset{x}{v}(X)\underset{x}{N} \qquad G^*\underset{x}{N} = -B\underset{x}{V} - \theta_x^y \underset{y}{N} \tag{2.3}$$

$$H^*BX = BHX + \overset{x}{w}(X)\underset{x}{N} \qquad H^*\underset{x}{N} = -B\underset{x}{W} - \theta_x^y \underset{y}{N} \tag{2.4}$$

$\overset{x}{v}$, $\overset{x}{w}$, are rC^∞ 1-forms, $\underset{x}{V}$, $\underset{x}{W}$ are rC^∞ vector fields and G, H are (1, 1) tensor fields on M^{4n-r}.

We have the following

Theorem 2.1. *The quaternion submanifold M^{4n-r} of codimension r admits three almost r-contact structures, corresponding to F^*, G^*, H^*.*

Proof. Operating on (2.1) by F^* and making use of equations (1.1), (2.1) and (2.2), we obtain

$$-BX = BF^2X + \overset{y}{u}(FX)\underset{y}{N} + \overset{x}{u}(X) \qquad \{-B\underset{x}{u} - \theta_x^y \underset{y}{N}\}.$$

Comparing tangential and normal parts, we get

$$F^2 = -I + \overset{x}{u}(X)\underset{x}{U} \qquad \overset{y}{u}\circ F = \theta_x^y \overset{x}{u}. \tag{2.5}$$

Multiplying (2.2) by F^* and making use of the equation (1.1), (2.1) and (2.2), we have

$$-\underset{x}{N} = -\{BF\underset{x}{U} + \overset{z}{u}(\underset{x}{U})\underset{z}{N}\} - \theta_x^y\{-B\underset{y}{U} - \theta_y^z\underset{z}{N}\}.$$

The comparison of the tangential and normal parts, gives

$$F\underset{x}{U} = \theta_x^y\underset{y}{U} \qquad \overset{z}{u}(\underset{x}{U}) = \delta_x^z + \theta_y^z\theta_x^y \tag{2.6}$$

where $x, y, z = 1, \ldots, r$ and δ_x^z denotes the Kronecker delta.

In view of equations (2.5), (2.6), it follows that M^{4n-r} admits an almost r-contact structure.

Similarly we can prove that the quaternion submanifold M^{4n-r} of codimension r also admits almost r-contact structures with respect to the tensor fields G^* and H^*.

3 - Some other results

By virtue of equation (1.2), we have $G^*H^*BX^* = F^*BX$ which in view of (2.1), (2.3) and (2.4) takes the form

$$BGHX + \overset{y}{v}(HX)\underset{y}{N} + \overset{x}{w}(X)\{-B\underset{x}{V} - \theta_x^y\underset{y}{N}\} = BFX + \overset{y}{u}(X)\underset{y}{N}.$$

The comparison of the tangential and normal parts gives

$$GHX = FX\overset{x}{w}(X)\underset{x}{V} \qquad \overset{y}{v}(HX) = \overset{y}{u}(X) + \theta_x^y\overset{x}{w}(X) \tag{3.1}$$

where $x, y = 1, \ldots, r$.

In view of (1.2) we have $G^*H^*\underset{x}{N} = F^*\underset{x}{N}$, which by virtue of the equations

(2.2), (2.3) and (2.4) takes the form

$$-\{BG\underset{x}{W}+\overset{z}{v}(\underset{x}{W})\underset{z}{N}\}-\theta_x^y\{-B\underset{y}{V}-\theta_y^z\underset{z}{N}\}=-B\underset{x}{U}-\theta_x^z\underset{z}{N}.$$

Comparing tangential and normal parts, we get

$$G\underset{x}{W}=\underset{x}{U}+\theta_x^y\underset{y}{V} \qquad \overset{z}{v}(\underset{x}{W})=\theta_x^z+\theta_x^y\theta_y^z. \tag{3.2}$$

Similarly we can obtain sets of relations

$$HF=G+\overset{x}{u}\otimes\underset{x}{W} \qquad FG=H+\overset{x}{v}\otimes\underset{x}{U} \qquad \text{etc.}. \tag{3.3}$$

Further in view of the relation (1.2), we have $(G^*H^*+H^*G^*)BX=0$, which by virtue of equations (2.3) and (2.4) takes the form

$$BGHX+\overset{y}{v}(HX)\underset{y}{N}+\overset{x}{w}(X)\{-B\underset{x}{V}-\theta_x^y\underset{y}{N}\}$$

$$+BHGX+\overset{y}{w}(GX)\underset{y}{N}+\overset{x}{v}(X)\{-B\underset{x}{W}-\theta_x^y\underset{y}{N}\}=0.$$

Equating the tangential and normal components, we get

$$(GH+HG)X=\overset{x}{v}(X)\underset{x}{W}+\overset{x}{w}(X)\underset{x}{V} \qquad \overset{y}{v}(HX)+\overset{y}{w}(GX)=\theta_x^y\{\overset{x}{v}(X)+\overset{x}{w}(X)\}. \tag{3.4}$$

Further, we have $(G^*H^*+H^*G^*)\underset{x}{N}=0$, which in view of the equations (2.3) and (2.4), takes the form

$$BG\underset{x}{W}+\overset{z}{v}(\underset{x}{W})\underset{z}{N}+\theta_x^y(-B\underset{y}{V}-\theta_y^z\underset{z}{N})$$

$$+BH\underset{x}{V}+\overset{z}{w}(\underset{x}{V})\underset{z}{N}+\theta_x^y(-B\underset{y}{W}-\theta_y^z\underset{z}{N})=0.$$

The comparison of the tangential and normal parts gives

$$G\underset{x}{W}+H\underset{x}{V}=\theta_x^y(\underset{y}{V}+\underset{y}{W}) \qquad \overset{z}{v}(\underset{x}{W})^*+\overset{z}{w}(\underset{x}{V})=2\theta_x^y\theta_y^z, \tag{3.5}$$

where $x,\ y,\ z=1,\ \ldots,\ r$.

Results corresponding to tensor fields $(HF+FH)$ and $(FG+GF)$ can be obtained in a similar way.

References

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Abstract

We consider a submanifold of codimension r of a quaternion manifold and study some of its properties. We also show that the quaternion submanifold of codimension r admits an almost r-contact structure.

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