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SIMON SALAMON (*)

Special structures on four-manifolds (**)

Introduction

In this note the formalism of spinors is used to analyse properties of almost complex structures on manifolds in dimension four. The integrability property of an almost complex structure is, in the presence of a compatible metric, exactly complementary to the condition that the almost complex structure give rise to a symplectic form. This fact is particularly evident when one studies the situation of a 4-manifold which possesses two anti-commuting almost complex structures. The most striking instance of this is when the manifold has a hyperkähler metric, and our remarks serve to place these metrics in a more general setting.

Consequences of the existence of a complex structure for the Riemann curvature tensor are pursued in the last section. A 4-manifold has an abundance of orthogonal complex structures if and only if it is self-dual, which means that half of its conformal Weyl tensor vanishes. Thus the material below is closely connected with the more general theory of self-duality.

1 - Two-forms and spinors

Let M denote a 4-dimensional oriented manifold with a Riemannian metric g, which in general we do not assume is complete. Indeed, much of the initial di-

(*) Mathematical Institute, 24-29 St. Giles', Oxford, OX1 3LB, United Kingdom.

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scussion is algebraic in nature, and for this purpose we shall work at a fixed point m of M.

An almost complex structure on M is a smooth endomorphism J of TM satisfying $J^2 = -1$. The value of such a structure at m is completely determined by the i-eigenspace $T^{1,0}$ for the action of J on the complexified tangent space $(T_m M)_c$, and there are decompositions

$$(1.1) \qquad (T_m M)_c = T^{1,0} \oplus T^{0,1} \qquad (T_m^* M)_c = \Lambda^{1,0} \oplus \Lambda^{0,1}$$

where $T^{0,1} = \overline{T^{1,0}}$ and $\Lambda^{0,1} = \overline{\Lambda^{1,0}}$ is the annihilator of $T^{1,0}$. More generally, the complexified space $(\bigwedge^k T_m^* M)_c$ of k-forms contains a distinguished subspace $\Lambda^{p,q}$ isomorphic to $\bigwedge^p \Lambda^{1,0} \otimes \bigwedge^q \Lambda^{0,1}$, elements of which are called *forms of type* (p, q).

An almost complex structure J is said to be *orthogonal* if

$$(1.2) \qquad g(JX, JY) = g(X, Y) \qquad X, Y \in T_m M.$$

This condition can be rewritten as $g(X - iJX, Y - iJY) = 0$, and is equivalent to the assertion that the subspace $T^{1,0}$ is totally isotropic relative to g. We shall consider only those almost complex structures which are orthogonal and *compatible with the orientation*, in the sense that at each point of m there is an orthonormal basis $\{e^1, e^2, e^3, e^4\}$ of $T_m^* M$ such that

$$(1.3) \qquad \Lambda^{1,0} = \operatorname{span}\{e^1 - ie^2, e^3 - ie^4\}.$$

This is consistent with a dual action of J on 1-forms given by $Je^1 = e^2$, $Je^3 = e^4$.

In terms of the above basis, the *fundamental* 2-*form* ω defined by

$$(1.4) \qquad \omega(X, Y) = g(X, JY)$$

equals

$$e^1 \wedge e^2 + e^3 \wedge e^4 = \frac{1}{2i}((e^1 - ie^2) \wedge (e^1 + ie^2) + (e^3 - ie^4) \wedge (e^3 + ie^4))$$

and is of type (1, 1) relative to J. It follows that

$$(1.5) \qquad (\bigwedge^2 T^* M)_c = \Lambda^{2,0} \oplus \Lambda^{0,2} \oplus \langle\omega\rangle \oplus \Lambda_0^{1,1}$$

where $\Lambda_0^{1,1}$ denotes the space of (1,1)-forms orthogonal to the span $\langle\omega\rangle$ of ω.

The decomposition (1.5) exhibits the reduction of structure group to $U(2)$ that results from the choice of the orthogonal almost complex structure J. However, part of this decomposition is already detected by the metric and orientation without reference to J. For

$$\Lambda^2_+ = [\Lambda^{2,0} \oplus \Lambda^{0,2}] \oplus \langle \omega \rangle \qquad \Lambda^2_- = [\Lambda_0^{1,1}] \tag{1.6}$$

are the invariant eigenspaces of the Hodge $*$-operator. Square brackets here indicate the *inverse to complexification*: in general if U is a complex vector space endowed with complex conjugation (i.e., an antilinear map $\sigma\colon U \to U$ with $\sigma^2 = 1$) then we use $[U]$ to denote the real vector space of fixed points (i.e., $\{u \in U\colon \sigma u = u\}$). The resulting decomposition

$$\bigwedge^2 T_m^* M = \Lambda^2_+ \oplus \Lambda^2_- \tag{1.7}$$

is $SO(4)$-invariant.

The origin of the summands $\Lambda^2_\pm$ is rendered clearer using the isomorphism

$$\mathrm{Spin}(4) \cong SU(2) \times SU(2)$$

where Spin (4) is by definition the *simply-connected double-covering group of* $SO(4)$. This allows one to introduce complex 2-dimensional vector spaces V_+, V_- such that

$$T_m^* M \cong [V_+ \otimes V_-] \tag{1.8}$$

and in this way *spinors*, i.e. elements of $V_\pm$, appear more fundamental than 1-forms or tangent vectors. The use of spinors to prove statements below is by no means essential, but their great convenience was made apparent for example in the extensive paper by Atiyah, Hitchin and Singer [3].

Complex conjugation on $T_m^* M$ is induced from quaternionic structures on V_+ and V_- (so that $\sigma = j_+ \otimes j_-$ where $j_\pm\colon V_\pm \to V_\pm$ is antilinear and $j_\pm^2 = -1$). Each $V_\pm$ is also equipped with a real symplectic form $\eta_\pm$ which trivializes $\bigwedge^2 V_\pm$, and these forms satisfy

$$g(v_1 \otimes w_1, v_2 \otimes w_2) = \eta_+(v_1, v_2)\,\eta_-(w_1, w_2) \tag{1.9}$$

for $v_1, v_2 \in V_+$ and $w_1, w_2 \in V_-$. (Here the metric g is extended to be symmetric and complex linear.) It follows that a 1-form is *isotropic* relative to g if and only if it is a simple or indecomposable tensor product in (1.8). Moreover

$$\bigwedge^2 T^*M \cong \bigwedge^2[V_+ \otimes V_-] \cong [S^2V_+ \otimes \bigwedge^2 V_-] \oplus [\bigwedge^2 V_+ \otimes S^2 V_-] \cong [S^2V_+] \oplus [S^2V_-]$$

and we may identify $[S^2 V_\pm]$ with the space $\Lambda^2_\pm$ of (1.7).

The set $\mathscr{J}$ of orthogonal almost complex structures on $T_m M$ compatible with the orientation is parametrized, by means of (1.6), by the 2-sphere consisting of elements ω of Λ^2_+ of a fixed norm. Given such an almost complex structure J, the totally isotropic space $\Lambda^{1,0}$ necessarily has the form $\langle v\rangle \otimes V_-$ for some $v \in V_+$ determined up to multiplication by a non-zero complex scalar. In this way we see that $\mathscr{J}$ is also parametrized by the complex projective line $\boldsymbol{P}(V_+)$.

In the sequel it will be convenient to suppose that v is unitary in the sense that

$$\eta_+(v, \tilde{v}) = 1 \tag{1.10}$$

where $\tilde{v}$ is an abbreviation for jv. In this case, $\{v, \tilde{v}\}$ constitutes a special unitary basis of V_+, and the only ambiguity in the definition of v involves multiplication by $e^{i\theta}$ for some $\theta \in [0, 2\pi)$. In terms of the identification

$$S^2 V_+ = \Lambda^{2,0} \oplus \Lambda^{0,2} \oplus \langle\omega\rangle$$

we may now take

$$\Lambda^{2,0} = \langle v \otimes v\rangle \qquad \Lambda^{0,2} = \langle \tilde{v} \otimes \tilde{v}\rangle \qquad \omega = iv \vee \tilde{v} \tag{1.11}$$

where $v \vee \tilde{v}$ denotes the symmetric product $v \otimes \tilde{v} + \tilde{v} \otimes v$.

2 - Complex and symplectic structures

Let us suppose now that M is *spin*, so that it possesses a principal Spin(4)-bundle P with the property that $P/\boldsymbol{Z}_2$ is isomorphic to the bundle of oriented orthonormal frames. This will always be valid if we replace M by a suitable open set, and such a restriction will suffice for the predominantly local theory below. Then there exist rank 2 vector bundles over M with fibres V_+, V_-, and these vector bundles have connections compatible with the Levi Civita one induced on T^*M via (1.8), so that

$$\nabla_X(v \otimes w) = \nabla_X v \otimes w + \otimes \nabla_X w,$$

where v, w are sections of the bundles with fibres V_+, V_- respectively.

Suppose that J is an orthogonal almost complex structure on M corresponding to a 2-form ω and a section v of the spin bundle with fibre V_+, as above.

The covariant derivative ∇v is a section of

$$V_+ \otimes (T_m^* M)_c \cong V_+ \otimes (V_+ \otimes V_-),$$

and we may write

$$\nabla v = v(vw_1 + \tilde{v}w_2) + \tilde{v}(vw_3 + \tilde{v}w_4)$$

where w_i is a section of the bundle with fibre V_-. (Tensor product signs have been omitted and the bracketing mimics that of the preceding line.)

Assuming that v is normalized as in (1.10), we have

$$0 = \eta_+(\nabla v, \tilde{v}) + \eta_+(v, \nabla\tilde{v}) = vw_1 + \tilde{v}w_2 + \tilde{v}\tilde{w}_1 - v\tilde{w}_2 = 2\,\mathrm{Re}\,(v(w_1 - \tilde{w}_2)),$$

so $w_1 = \tilde{w}_2$, and $\phi = vw_1 + \tilde{v}w_2$ satisfies $\phi + \bar{\phi} = 0$. Let $\alpha = vw_3$ and $\bar{\beta} = \tilde{v}w_4$, so that α, β are both (1, 0)-forms relative to the almost complex structure J. We may now write

$$\nabla v = v\phi + \tilde{v}(\alpha + \bar{\beta}). \tag{2.1}$$

The almost complex structure J is really determined by the line bundle $\langle v\rangle$ generated by the section v, rather than by v itself. The purely imaginary 1-form ϕ represents a canonical Hermitian connection on this line bundle, and if v is replaced by $e^{i\theta}v$ for some real-valued function θ then ϕ is replaced by $\phi + i\,\mathrm{d}\theta$. On the other hand, $\alpha + \bar{\beta}$ represents the *second fundamental form* of $\langle v\rangle$, and it follows that the derivative of J is completely determined by the pair (α, β).

An almost complex structure is said to be *integrable* or a *complex structure* if there exist complex coordinates z^1, z^2 in a neighbourhood of each point such that $\mathrm{d}z^1$, $\mathrm{d}z^2$ are (1, 0)-forms. An obvious necessary condition for integrability is that the exterior derivative of any form θ of type (1, 0) has no component of type (0, 2), or equivalently that

$$g(\mathrm{d}\theta, \gamma) = 0 \quad \text{for all } \gamma \in \Lambda^{2,0}. \tag{2.2}$$

The sufficiency of this condition is the Newlander-Nirenberg theorem.

Since we are concerned exclusively with *orthogonal* almost complex structures, we shall say that one of these, J, is *symplectic* if its fundamental 2-form ω defined by (1.4) is closed.

Lemma 1. *In the notation of* (2.1), *J is symplectic if and only if $\alpha = 0$; J is integrable if and only if $\beta = 0$.*

Proof. Because the Levi Civita connection has no torsion, $d\omega$ equals minus the skew-symmetric part of $\nabla\omega$. Using (1.11) and (2.1), it follows that $-\frac{1}{2}d\omega$ equals

$$\operatorname{Re}(d(iv\tilde{v})) = \operatorname{Re}(iv\tilde{v}\wedge\phi + i\tilde{v}\tilde{v}\wedge(\alpha+\bar{\beta}) + iv\tilde{v}\wedge\bar{\phi} - ivv\wedge(\bar{\alpha}+\beta))$$

$$= 2\operatorname{Re}(i\tilde{v}\tilde{v}\wedge(\alpha+\bar{\beta})) = 2\operatorname{Re}(i\tilde{v}\tilde{v}\wedge\alpha)$$

the last equality because $\tilde{v}\tilde{v}$ spans $\Lambda^{0,2}$. Thus $d\omega = 0$ if and only if $\alpha = 0$.

Using (1.11) and (2.2), it follows that J is integrable if and only if $g(\nabla v, vv) = 0$, where the induced metric on $V_+ \otimes V_+$ is calculated in a similar way to (1.9). The equality amounts to $\beta = 0$.

The condition that J is complex can now be expressed in terms of the line bundle $\langle v\rangle$; it is equivalent to saying that $\langle v\rangle$ is stable by the operator ∇_A whenever A is a vector field of type (1, 0). Equivalently, $\nabla_A B$ has type (1, 0) whenever A, B are themselves vector fields of type (1, 0).

A Riemannian metric g is called *Kähler* if it admits an orthogonal complex structure such that the corresponding 2-form ω is closed. This is equivalent to asserting the existence of an *almost* complex structure J for which $\nabla J = 0$ (or equivalently $\nabla\omega = 0$), since the covariant derivative of J encapsulates both the obstruction to integrability and $d\omega$. Moreover, $\langle v\rangle$ and the bundle with fibre $T^{1,0}$ are then stable by ∇_A for *all* vector fields A.

If the spinor section v represents a Kähler metric, then $2\,d\phi$ is a (1, 1)-form that equals the curvature of the *canonical bundle* $\langle v\rangle \otimes \langle v\rangle$ (see (1.11)) which is well known to represent the *Ricci form* of the Kähler manifold. Its trace (given by $*(2\,d\phi\wedge\omega)$) will be proportional to the scalar curvature s of M. In the Kähler case, $d\phi$ completely determines the curvature of the spin bundle $\langle v\rangle \oplus \langle\tilde{v}\rangle$, and s determines the semi Weyl tensor W_+ defined in section **5** below.

3 - Anti-commuting almost complex structures

Let $\{e^1, e^2, e^3, e^4\}$ be a basis of $T^*_m M$ whose elements are orthogonal and of equal norm. The associated basis

$$(3.1)\qquad \omega = e^1\wedge e^2 + e^3\wedge e^4 \qquad \omega' = e^1\wedge e^3 + e^4\wedge e^2 \qquad \omega'' = e^1\wedge e^4 + e^2\wedge e^3$$

of Λ^2_+ determines a triple J, J', J'' of almost complex structures whose dual action on forms is given by

$$
\begin{aligned}
&Je^1 = e^2 && Je^3 = e^4 \\
&J'e^1 = e^3 && J'e^4 = e^2 \\
&J''e^1 = e^4 && J''e^2 = e^3 .
\end{aligned}
\tag{3.2}
$$

Observe that J, J', J'' are uniquely specified by the property that

$$
e^1 - ie^2 \qquad e^1 - ie^3 \qquad e^1 - ie^4 \tag{3.3}
$$

are forms of type (1, 0) relative to each of them in turn. For the action of (for example) J on the real 2-dimensional orthogonal complement of e^1 and e^2 is specified by the orientation of M.

The equations (3.2) show that

$$JJ' = J'' = -J'J,$$

reflecting the Lie algebra structure of $\mathfrak{su}(2)$. Indeed, it is easy to see that if two orthogonal almost complex structures J, J' anti-commute then their associated 2-forms ω, ω' are orthogonal. Conversely, if J, J' are any two anti-commuting almost complex structures on a 4-manifold M then there exists a Riemannian metric and an orientation on M such that J, J' can be expressed as in (3.2).

Suppose now that J and J' are two complex structures on M corresponding to the respective sections $\langle v\rangle$, $\langle v'\rangle$ of the bundle with fibre $\boldsymbol{P}(V_+)$, where

$$v' = v + \lambda\tilde{v}$$

and λ is a complex valued function. We may furthermore assume that v satisfies (1.10), although it is convenient to retain v' unnormalized. From (1.11), ω, ω' are orthogonal if and only if

$$0 = g(v \vee \tilde{v}, (v + \lambda\tilde{v}) \vee (\tilde{v} - \bar{\lambda}v)) = |\lambda|^2 - 1,$$

that is, $|\lambda| = 1$. If $JJ' + J'J = 0$, we may always choose v such that J corresponds to $\lambda = 1$. In this case it is easy to check, using (1.11), that the almost complex structure JJ' corresponds to $\lambda = -i$. (The value $-i$ arises rather than i because of earlier conventions such as JJ' being the composition of J' followed by J.)

The next result unifies some previously-known lemmas [2], [15]; it shows that anti-commuting complex and symplectic structures curiously behave as if they are assigned the parities even and odd respectively.

Proposition 1. *Let J, J' be anti-commuting almost complex structures on M. If J, J' are integrable, then JJ' is integrable. If J, J' are symplectic, then JJ' is integrable. If J is integrable and J' is symplectic, then JJ' is symplectic.*

Proof. We start with the following general calculation.

$$\nabla v' = v\phi + \tilde{v}(\alpha + \bar{\beta}) + \lambda(\tilde{v}\bar{\phi} - v(\bar{\alpha} + \beta)) = v'\phi' + \tilde{v}'(\alpha' + \overline{\beta'}),$$

where α', β' are assumed to have type (1, 0) relative to the almost complex structure determine by v', and

$$(3.4)\qquad \begin{aligned} (1 + |\lambda|^2)\,\phi' &= (1 - |\lambda|^2)\,\phi + \bar{\lambda}(\alpha + \bar{\beta}) - \lambda(\bar{\alpha} + \beta) \\ (1 + |\lambda|^2)(\alpha' + \overline{\beta'}) &= \alpha + \bar{\beta} + \lambda^2(\bar{\alpha} + \beta) - 2\lambda\phi. \end{aligned}$$

Taking $\lambda = 1$ we obtain

$$\mathrm{Re}\,\alpha' + \mathrm{Re}\beta' = \mathrm{Re}\,\alpha + \mathrm{Re}\beta \qquad \mathrm{Im}\,\alpha' - \mathrm{Im}\beta' = -\phi.$$

Taking $\lambda = -i$ and indicating the quantities associated with this value of λ by double primes, we have

$$\alpha'' + \overline{\beta''} = \mathrm{Im}\,\alpha - \mathrm{Im}\beta + i\phi = \mathrm{Im}\,\alpha - \mathrm{Im}\beta - i(\mathrm{Im}\,\alpha' - \mathrm{Im}\beta').$$

The conclusion of the proof of all the results rests on Lemma 1 and the following observation. Two forms θ, θ' that have type (1, 0) relative to J, J' respectively and sharing the same real part can be expressed in the form $e^1 - ie^2$, $e^1 - ie^3$ of (3.3) with respect to a suitable basis. Then

$$\mathrm{Im}\,\theta \mp i\,\mathrm{Im}\,\theta' = -i(e^2 \mp ie^3)$$

has type (1, 0) or (0, 1) relative to JJ' according to sign.

In terms of fundamental 2-forms, the condition $JJ' + J'J = 0$ translates into $\omega \wedge \omega' = 0$. If ω, ω' are both closed then $\omega - i\omega'$ is a *holomorphic symplectic form* for the complex structure JJ'. Indeed, such a complex symplectic structure arises whenever one has two real closed 2-forms ω, ω' satisfying $\omega \wedge \omega = \omega' \wedge \omega' \neq 0$ and $\omega \wedge \omega' = 0$ everywhere. Although spinors provide a unified approach to Proposition 1, simple proofs of each of the three implications can also be given using differential forms and the reader is invited to supply these.

4 - Examples

Proposition 1 gives rise to three particular classes of 4-manifolds, as follows:

a - *Hyperkähler manifolds*, those which admit anti-commuting almost complex structures J, J' which are both complex and symplectic

b - *Hypercomplex (non-hyperkähler) manifolds*, those which admit anti-commuting almost complex structures J, J' which are both complex (but neither is symplectic)

c - *Complex symplectic (non-hyperkähler) manifolds*, those which admit anti-commuting almost complex structures J, J' which are both symplectic (but neither is complex).

Each of these classes has some elementary compact representatives, formed by taking the quotient of a flat space by a discrete group preserving the appropriate complex or symplectic structures. We describe these basic examples and comment on the classification of the other members of each class in turn.

a - Euclidean space $\boldsymbol{R}^4$ with its standard metric is certainly hyperkähler since there is a triple of closed 2-forms as in (3.1). These forms are translation-invariant, and therefore induce a hyperkähler structure on any torus $T^4 = \boldsymbol{R}^4/\boldsymbol{Z}^4$. Setting $J'' = JJ'$, one obtains a whole 2-sphere

$$\{aJ + a'J' + a''J'' \colon a^2 + (a')^2 + (a'')^2 = 1\} \tag{4.1}$$

of parallel complex structures relative to each of which g is therefore a Kähler metric, and indeed one with zero Riemann tensor. On the other hand, any compact complex surface admitting a non-flat hyperkähler metric is necessarily a $K3$ surface, which is a complex surface characterized by $b_1 = 0$ and $c_1 = 0$. Conversely any such surface carries a hyperkähler metric whose existence was proved by Yau. These facts were explained by Calabi [7], who first coined the name *hyper-Kähler*, and discovered (independently with Eguchi and Hanson [8]) such a matric on the cotangent bundle $T^*\boldsymbol{C}P^2$.

Describing hyperkähler metrics on compact 4-manifolds explicitly remains a major problem. Various *approximations* to the $K3$ metric were described by Hitchin [**10**]; his ideas were developed by Kronheimer [**13**] who used what is now a well-established quotient construction to describe complete hyperkähler metrics on the minimal resolution of $\boldsymbol{C}^2/K$, where K is a finite subgroup of $SU(2)$. These metrics approach the Euclidean metric on $\boldsymbol{C}^2/K$ to order $O(1/r^4)$.

Other examples of complete hyperkähler metrics in four dimensions are described in [2] which includes a general account of such metrics with a 3-dimensional group of isometries permuting the elements of (4.1).

b - Let λ be a complex number with $|\lambda| \neq 1$, and let Γ be the infinite cyclic group generated by $\lambda \in \boldsymbol{C}$. Then

$$X = \frac{\boldsymbol{R}^4 \setminus \{0\}}{\Gamma}$$

is diffeomorphic to $S^1 \times S^3$, and having odd first Betti number $b_1 = 1$ cannot admit a Kähler metric. In contrast to the case of a torus, the discrete group Γ does not act isometrically, but it does preserve the 2-sphere of complex structures determined by (3.2) and (4.1). Relative to each of these, X is a *primary Hopf surface*, that is a complex surfaces covered by $\boldsymbol{C}^2 \setminus \{0\}$ with fundamental group $\boldsymbol{Z}$.

In the terminology of G-structures, a hypercomplex structure as defined above is the same thing as a $GL(1, \boldsymbol{H})$-structure admitting a torsion-free connection; such connections were considered by Obata [15]. If the $GL(1, \boldsymbol{H})$-structure has vanishing curvature tensor then it is integrable in the sense that it admits compatible affine coordinate systems. Compact complex surfaces admitting such integrable hypercomplex structures have been classified by Kato [11], and are all quotients of $\boldsymbol{C}^2 \setminus \{0\}$ by a subgroup Γ of $GL(1, \boldsymbol{H})$. Finite subgroups of $SU(2)$ again crop up, as Γ contains such a subgroup K for which Γ/K is infinite cyclic, and the resulting Hopf surface is diffeomorphic to a S^3/K-bundle over S^1.

The above results were extended by Boyer who proved that Hopf surfaces exhaust the class of compact hypercomplex manifolds which are not hyperkähler [5]. A key ingredient is a theorem of Vaisman [20] that asserts that only Hopf surfaces admit metrics that are conformally flat and locally conformally Kähler. Hypercomplex structures also play a key role in the work of Ashketar et al. [1].

c - The fundamental example in this case is the nilmanifold cited by Thurston [18] as an example of a symplectic non-Kähler manifold. This is based on the *Heisenberg group*

$$H = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \boldsymbol{R} \right\}.$$

Let
$$Y = (H_Z \backslash H) \times S^1,$$

where H_Z denotes the subgroup of H defined by restricting x, y, z to be integers, and $H_Z \backslash H$ is the set of right cosets. The 1-forms

$$e^1 = \mathrm{d}z - x\,\mathrm{d}y \qquad e^2 = \mathrm{d}x \qquad e^3 = \mathrm{d}y$$

are left-invariant, and are therefore globally defined on Y. They satisfy

$$\mathrm{d}e^1 = -e^2 \wedge e^3 \qquad \mathrm{d}e^2 = 0 \qquad \mathrm{d}e^3 = 0.$$

Let e^4 be a closed 1-form on S^1, and consider the structures defined by (3.1) and (3.2). Then ω, ω' are closed, and $J'' = JJ'$ is integrable.

The function $\mu = x - iy$ defines a holomorphic mapping from Y to a torus $T^2 = \boldsymbol{C}/\boldsymbol{Z}^2$ realizing Y as a holomorphic bundle over T^2 with tori as fibres. Indeed, μ may be regarded as a *moment mapping* for the action of a group $\boldsymbol{C}^*$ preserving the holomorphic symplectic form

$$\omega - i\omega' = (e^1 - ie^4) \wedge (e^2 - ie^3).$$

This description of Y as a holomorphic bundle with base and fibres elliptic curves exhibits it as one of a family of non-Kähler surfaces with $b_1 = 3$. The holomorphic symplectic form trivializes the canonical bundle, and the Hodge number $h^{2,0}$ is non-zero. A moduli space of such complex surfaces on the underlying smooth structure of Y may be obtained by choosing a holomorphic connection for the fibration μ in order to *add* complex structures on the base and fibres. Moreover, Kodaira's Theorem 19 in [**12**] asserts that any manifold of type **c** above is of the form $\boldsymbol{C}^2/L$ where L is a group of affine transformations leaving invariant the standard symplectic form $\mathrm{d}z^1 \wedge \mathrm{d}z^2$.

The manifold Y, and for that matter $H_Z \backslash H$, is a basic example in the theory of differential algebras in rational homotopy theory [**9**]. Indeed, the differential algebra generated by the invariant forms e^1, e^2, e^3, e^4 provides a *minimal model* for the de Rham algebra of Y; it faithfully represents the cohomology of Y and is built up in a simple way because of its nilpotency. This minimal model is not *formal* as it possesses a non-vanishing Massey triple product $\langle [e^2], [e^3], [e^2] \rangle$ represented by the cohomology class $-2[e^1 \wedge e^2]$ which does not belong to the ideal in $H^2(Y, \boldsymbol{R})$ generated by $[e^2]$. The existence of such a non-vanishing class is not possible on a manifold admitting a Kähler metric.

5 - Curvature considerations

The Riemann curvature tensor R of M is defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z \tag{5.1}$$

where ∇ denotes covariant differentiation on TM. It may be viewed as an endomorphism of the bundle $\bigwedge^2 T^*M$ of 2-forms by means of the formula

$$g(R(X \wedge Y), Z \wedge W) = g(R(X, Y)Z, W). \tag{5.2}$$

Well-known symmetries of R guarantee that the definition makes sense. The following result is well known.

Proposition 2. *Relative to* (1.7), *R may be written in block form*

$$\left(\begin{array}{c|c} W_+ + \frac{1}{12} sI & B^T \\ \hline B & W_- + \frac{1}{12} sI \end{array}\right)$$

where $W_\pm$ is a traceless self-adjoint endomorphism of $\Lambda^2_\pm$, B is a linear mapping from Λ^2_+ to Λ^2_-, and s is a smooth function.

The four quantities W_+, W_-, B, s represent the four components of R relative to the irreducible $SO(4)$-modules constituting the space of curvature tensors, first described by Singer and Thorpe [17]. The trace of R as an endomorphism of $\bigwedge^2 T^*M$ equals one half the scalar curvature $s = g^{jl} R^i_{jil}$; the latter is also the trace of the *Ricci tensor* $R_{jl} = R^i_{jil}$ which at each point is an element of the 10-dimensional space $S^2 T^*_m M$ of symmetric 2-tensors. In general, the traceless part of the Ricci tensor can be identified with B by means of the isomorphism

$$S^2 T^*_m M \cong S^2 [V_+ \otimes V_-] \cong [S^2 V_+] \otimes [S^2 V_-] \oplus \mathbf{R}$$

which implies that

$$\mathrm{Hom}(\Lambda^2_+, \Lambda^2_-) \cong \Lambda^2_+ \otimes \Lambda^2_- \tag{5.3}$$

is isomorphic to the space $S^2_0 T^*_m M$ of symmetric tensors orthogonal to g.

The remaining components W_+, W_- of R constitute what is effectively the *Weyl tensor*, and is conformally invariant in the following sense. If the metric g is replaced by $\tilde{g} = e^f g$, where f is some smooth function, then the modified metric $\tilde{g}$ has $\tilde{W}_\pm = e^{-f} W_\pm$. Moreover, there exist local coordinates $x^1, \ldots, x^4$ relative to which $g = e^{-f} \sum dx^i \otimes dx^i$ if and only if $W_+ = W_- = 0$. By analogy to (5.3) we have

$$\mathrm{Hom}(\Lambda_+^2, \Lambda_+^2) \cong [S^2 V_+] \otimes [S^2 V_+] \cong [S^4 V_+] \oplus [S^2 V_+] \oplus \boldsymbol{R}.$$

The traceless self-adjoint endomorphisms of Λ_+^2 generate the space $[S^4 V_+]$, and it is this that contains the *semi Weyl tensor* W_+. The spinor description of this tensor was exploited in **[16]**.

The following result appears in **[19]**, albeit in a different guise.

Lemma 2. *If J is an orthogonal complex structure compatible with the orientation, then W_+ has no component in the space* $\mathrm{Hom}(\Lambda^{0,2}, \Lambda^{2,0}) \cong \cong \Lambda^{2,0} \otimes \Lambda^{2,0}$.

Proof. This follows immediately from (5.1) and the remarks in the paragraph after the proof of Lemma 1. To shed more light on it, consider the curvature

$$R_+ \in \mathrm{Hom}(V_+, V_+) \otimes (\Lambda^2 T_m^* M)_c$$

of the spin bundle with fibre V_+, which is associated to R by means of the natural inclusion $(\Lambda_+^2)_c \hookrightarrow \mathrm{Hom}(V_+, V_+)$ and may be identified with the first column of the 2×2 block matrix in Proposition 2. Formula (2.1) becomes $\nabla v = v\phi + \tilde{v}\alpha$, and $R_+(v)$ equals the skew-symmetric part of $\nabla^2(v)$ which is

$$\nabla v \wedge \phi + v\, d\phi + \nabla \tilde{v} \wedge \alpha + \tilde{v}\, d\alpha = v(d\phi - \bar{\alpha} \wedge \alpha) + \tilde{v}(d\alpha + 2\alpha \wedge \phi).$$

The image of the (0, 2)-form $\widetilde{vv} \in (\Lambda_+^2)_c$ under R is proportional to $d\alpha + 2\alpha \wedge \phi$, where α has type (1, 0) relative to J. Thus the component of type (2, 0) of $R(\widetilde{vv})$, or equivalently of $W_+(\widetilde{vv})$, equals zero.

The curvature condition of Lemma 2 imposes two real conditions on W_+. If M admits at least three independent complex structures in a neighbourhood of a point $m \in M$ then $W_+(m) = 0$. An oriented 4-manifold is called *anti-self-dual* or *half conformally flat* if W_+ is identically zero. The powerful integrability theorem of [3] implies that, conversely, if M is anti-self-dual then there exist infinitely many orthogonal complex structures compatible with the orientation in a neighbourhood of any point of M. This characterization of anti-self-duality emphasizes that it is a conformally-invariant notion, as a complex structure remains orthogonal when the metric is multiplied by a positive function.

The manifolds of type **a** and **b** above are very special examples with W_+ everywhere zero. The remarks at the end of section 2 furnish another important class, namely complex surfaces admitting a Kähler metric whose scalar curvature s vanishes. Separately the conditions $\nabla J = 0$ and $s = 0$ are not conformally invariant although their combination is. The theory of these *scalar-flat Kähler surfaces* has been recently developed extensively by LeBrun and Singer [**14**].

Finally, suppose that M is an oriented Riemannian manifold for which (at least on some open set) W_+ is nowhere zero. As a real element of $S^4 V_+$ the tensor W_+ may be viewed at each point as a quartic polynomial and its roots determine four sections $\langle v \rangle$, $\langle \tilde{v} \rangle$, $\langle v' \rangle$, $\langle \tilde{v}' \rangle$ of the bundle with fibre $\boldsymbol{P}(V_+)$ associated to almost complex structures J, $-J$, J', $-J'$ relative to each of which R satisfies the condition in Lemma 2. It is an interesting problem to classify classes of metrics in terms of properties of these almost complex structures. In particular, let us say that W_+ is *degenerate* if the pairs $\pm J$, $\pm J'$ coincide; this occurs if and only if the self-adjoint endomorphism of Λ^2_+ determined by W_+ has two coincident eigenvalues at each point. Such manifolds include the conformally Kähler surfaces studied by Derdziński [**6**], and also the example Y above.

References

[1] A. ASHTEKAR, T. JACOBSON and L. SMOLIN, *A new characterization of half-flat solutions to Einstein's equations*, Comm. Math. Phys. **115** (1988), 631-648.

[2] M. F. ATIYAH and N. J. HITCHIN, *The Geometry and Dynamics of Magnetic Monopoles*, Princeton Univ. Press, Princeton, USA 1988.

[3] M. F. ATIYAH, N. J. HITCHIN and I. M. SINGER, *Self-duality in four-dimensional Riemannian geometry,* Proc. Roy. Soc. London A **362** (1978), 425-461.

[4] A. L. BESSE, *Einstein Manifolds*, Springer, Berlin 1987.

[5] C. P. BOYER, *A note on hyperhermitian four-manifolds*, Proc. Amer. Mat. Soc. **102** (1988), 157-164.

[6] A. DERDZIŃSKI, *Self-dual Kähler manifolds and Einstein manifolds of dimension four*, Compositio Math. **49** (1983), 405-433.

[7] E. CALABI, *Isometric families of Kähler structures*, in The Chern Symposium 1979, ed. W. Y. Hsiang et al., Springer, Berlin 1980.

[8] T. EGUCHI and A. J. HANSON, *Asymptotically flat self-dual solutions to Euclidean gravity*, Phys. Lett. B **74** (1978), 249-251.

[9] P. A. GRIFFITHS and J. W. MORGAN, *Rational homotopy theory and differential forms*, Progress in Mathematics, **16**, Birkhäuser, Basel, Schweiz 1981.

[10] N. J. HITCHIN, *Twistor construction of Einstein metrics,* in Global Riemannian Geometry, ed. T. J. Willmore et al., Ellis Horwood, Chichester 1984.

[11] M. KATO, *Compact differentiable 4-folds with quaternionic structures*, Math. Ann. **248** (1980), 79-96.

[12] K. KODAIRA, *On the structure of complex analytic surfaces, I*, Amer. J. Math. **86** (1964), 751-798.

[13] P. B. KRONHEIMER, *The construction of ALE spaces as hyper-Kähler quotients*, J. Differential Geom. **29** (1989), 665-683.

[14] C. LEBRUN and M. SINGER, *Existence and deformation theory for scalar-flat Kähler metrics on compact complex surfaces*, Invent. Math. (to appear).

[15] M. OBATA, *Affine connections on manifolds with almost complex, quaternionic or Hermitian structure*, Japan J. Math. **26** (1956), 43-79.

[16] S. SALAMON, *Harmonic 4-spaces*, Math. Ann. **269** (1984), 169-178.

[17] I. M. SINGER and J. A. THORPE, *The curvature of 4-dimensional Einstein spaces*, in Global Analysis in honor of Kodaira, ed. D. C. Spencer et. al., Princeton Univ. Press, Princeton, USA 1969.

[18] W. P. THURSTON, *Some examples of symplectic manifolds*, Proc. Amer. Math. Soc. **55** (1976), 467-468.

[19] F. TRICERRI and L. VANHECKE, *Curvature tensors on almost Hermitian manifolds*, Trans. Amer. Math. Soc. **267** (1981), 365-398.

[20] I. VAISMAN, *Generalized Hopf surfaces*, Geom. Dedicata **13** (1982), 231-255.

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