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On (λ, $\sigma\mu$)-bases (**)

Introduction

This article is devoted to the study of *λ-bases*, wherein λ is equipped with the more general topology, called *$\sigma\mu$-topology*, introduced by Ruckle [**10**], μ being an arbitrary sequence space.

We characterize analytically *fully-λ-bases* and *fully-λ^{μ}-bases*.

Efforts have been made to identify topologically a sequentially complete space, having a fully-λ-base (or a fully-λ^{μ}-base), with a Köthe space, thereby providing a far reaching generalization of the famous classical theorem, which tells that a sequentially complete space with an absolute base is nothing but a Köthe space (cf. [**9**]).

Most of the results are motivated by their corresponding analogues in the case of traditional normal topology.

1 - Fundamentals

For various terms definitions and notations on nuclearity and sequence spaces we refer, respectively, to [**9**], [**6**] and [**11**]. We know that the *normal topology* $\eta(\lambda, \lambda^{\times})$ on a sequence space λ is generated by the family $\{p_y : y \in \lambda^{\times}\}$

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of semi-norms where

$$p_y(x) = \sum_{i \geq 1} |x_i y_i| \qquad x \in \lambda .$$

$\lambda^\times$ being the *Köthe dual* of λ.

Towards the generalization of this topology Ruckle [10] introduced the concept of *$\sigma\mu$-topology* associated with a sequence space μ on an arbitrary sequence space λ. Indeed, the *μ-dual* of λ is the subspace of ω defined by

$$\lambda^\mu = \{b \in \omega : ab \in \mu, \ \forall a \in \lambda\} .$$

Similarly, we can define another subspace of ω, namely, the *μ-dual* $\lambda^{\mu\mu}$ of λ^μ, where

$$\lambda^{\mu\mu} = (\lambda^\mu)^\mu = \{c \in \omega : bc \in \mu, \ \forall b \in \lambda^\mu\} .$$

λ is said to be *μ-perfect* if $\lambda = \lambda^{\mu\mu}$. To topologize the spaces λ and λ^μ, let us assume that D_μ is the family of semi-norms, generating the topology on μ. For $b \in \lambda^\mu$ and $p \in D_\mu$, we define

$$p_b(a) = p(\{a_n b_n\}) \qquad a \in \lambda .$$

Then the topology generated by the family $\{p_b : p \in D_\mu, \ b \in \lambda^\mu\}$ of semi-norms on λ is called the *$\sigma\mu$-topology* and is denoted by $T_{\sigma\mu}$. Similarly the $\sigma\mu$-topology $T^*_{\sigma\mu}$ on λ^μ is generated by the collection $\{p_a : p \in D_\mu, \ a \in \lambda\}$ of semi-norms where

$$p_a(b) = p(\{a_n b_n\}) \qquad b \in \lambda^\mu .$$

Remarks 1.1.

(i) For $\mu = l^1$. λ^μ is the *Köthe dual* (or *α-dual*) $\lambda^\times$ (cf. [8]); $(\lambda, T_{\sigma\mu}) = (\lambda, \eta(\lambda, \lambda^\times))$ and $(\lambda^\mu, T^*_{\sigma\mu}) = (\lambda^\times, \eta(\lambda^\times, \lambda))$.

(ii) For $\mu = cs$ (*convergent series*), λ^μ is the *β-dual* λ^β (cf. [3]).

(iii) For $\mu = bs$ (*bounded partial sum*), λ^μ is the *γ-dual* λ^γ (cf. [3]).

For further informations regarding the topological aspects of $(\lambda, T_{\sigma\mu})$ and $(\lambda^\mu, T^*_{\sigma\mu})$ we refer to [4].

To begin with, we have

Example 1.2. Let μ be l^{∞}, i.e. the set of all bounded sequences equipped with the usual supnorm topology and λ be φ, the space spanned by the unit vectors $\{e^n: n \geqslant 1\}$, $e^n = \{0, \ldots, 0, 1, 0, \ldots\}$. Then

$$\varphi l^{\infty} = \omega \qquad \varphi l^{\infty} l^{\infty} = \varphi .$$

Thus φ is l^{∞}-perfect. Observe that the topology $T_{\sigma l^{\infty}}$ is generated by the family $\{p_b: b \in \omega\}$, where

$$p_b(a) = \sup_{n \geqslant 1} |a_n b_n| \qquad a \in \varphi .$$

Since $(\varphi, \eta(\varphi, \omega))$ is nuclear (cf. [6]), using the Grothendieck-Pietsch Criterion we find that for each $b \in \omega$, $b_n > 0$, there exists, $c \in \omega$, $c_n > 0$ such that $\{\frac{b_n}{c_n}\} \in l^1$. Therefore, for $a \in \varphi$

$$p_b(a) = \sum_{n \geqslant 1} |a_n b_n| \leqslant (\sum_{n \geqslant 1} \frac{b_n}{c_n}) p_c(a) .$$

Consequently $$(\varphi, T_{\sigma l^{\infty}}) = (\varphi, \eta(\varphi, \omega)) .$$

Remarks 1.3. The above example reveals that *the nuclearity of* $(\lambda, T_{\sigma\mu})$ *does not necessarily imply the nuclearity of* (μ, T_{μ}). Equivalently, (μ, T_{μ}) may not be nuclear but still $(\lambda, T_{\sigma\mu})$ may turn out to be nuclear. However, if T_{μ} is the normal topology $\eta(\mu, \mu^{\times})$, then nuclearity of $(\mu, \eta(\mu, \mu^{\times}))$ always imply the nuclearity of $(\lambda, T_{\sigma\mu})$ whatever may be λ; which is a consequence of the following result contained in [12].

Proposition 1.4. *The space* $(\lambda, T_{\sigma\mu})$ *is nuclear iff* $\lambda^{\mu}\mu^{\times} = l^1 \lambda^{\mu}\mu^{\times}$.

Following the proof of the above result one can establish

Proposition 1.5. *The space* $(\lambda^{\mu}, T^{*}_{\sigma\mu})$ *is nuclear iff* $\lambda\mu^{\times} = l^1 \lambda\mu^{\times}$.

Note that for a nuclear sequence space $(\mu, \eta(\mu, \mu^{\times}))$, λ^{μ} is always nuclear.

Remarks 1.6. The example 1.2 also underlines that even the nuclearity of $(\lambda, T_{\sigma\mu})$ and of its μ-dual $(\lambda^{\mu}, T^{*}_{\sigma\mu})$ does not necessarily yield the nuclearity of (μ, T_{μ}), for in this case $(\lambda^{\mu}, T^{*}_{\sigma\mu}) = (\omega, \eta(\omega, \varphi))$ which is nuclear.

Let E be an *l.c. TVS* and λ a sequence space carrying the $\sigma\mu$-topology.

Definition 1.7. A Schauder base $\{x_i, f_i\}$ for E is said to be a *semi-λ-base*, if for each $p \in D_E$

$$\{f_i(x)\, p(x_i)\} \in \lambda \qquad x \in E$$

and it is called a *Q-fully-λ-base* if there exists a permutation π such that for each $p \in D_E$ the map $\psi_p^\pi: E \to \lambda$ is continuous where

$$\psi_p^\pi(x) = \{f_{\pi(i)}(x)\, p(x_{\pi(i)})\} \qquad x \in E.$$

If π is the identity permutation, one gets what is called a *fully-λ-base*. Thus a fully-λ-base is a Q-fully-λ-base. However, the converse is not true. For an example we refer to [2].

The impact of nuclearity of $(\mu, \eta(\mu, \mu^\times))$ on λ and λ^μ is reflected in the following two results

Proposition 1.8. *Let $(\mu, \eta(\mu, \mu^\times))$ be a perfect nuclear sequence space. If λ is μ-perfect, then $\lambda = S$, where*

$$S = \{x \in \omega: \{x_n y_n z_n\} \in l^\infty,\ \forall y \in \lambda^\mu,\ \forall z \in \mu^\times\}.$$

Proof. Let $x \in S$ and $y \in \lambda^\mu$, $z \in \mu^\times$ be choosen arbitrarily. In view of Remark 1.3 $(\lambda, T_{\eta\mu})$ is nuclear. Thus by Proposition 1.4 we get $u \in \lambda^\mu$ and $v \in \mu^\times$ such that $\{ \frac{y_n z_n}{u_n v_n} \} \in l^1$. Now there exists $M = M(u, v)$ such that $|x_n u_n v_n| \leqslant M$ $\forall n$.

From the inequality

$$\sum_{n \geqslant 1} |x_n y_n z_n| \leqslant M \sum_{n \geqslant 1} \left| \frac{y_n z_n}{u_n v_n} \right| < \infty$$

it follows that $xyz \in l^1$ for all $y \in \lambda^\mu$ and for all $z \in \mu^\times$. Thus

$$xy \in \mu^{\times\times} = \mu \qquad \forall y \in \lambda^\mu$$

which implies $x \in \lambda^{\mu\mu} = \lambda$, and this completes the proof.

Proposition 1.9. *Let $(\mu, \eta(\mu, \mu^{\times}))$ be a perfect nuclear sequence space. Then*

$$\lambda^{\mu} = \{z \in \omega : \sup_{n \geq 1} |z_n y_n x_n| < \infty, \ \forall x \in \lambda, \ \forall y \in \mu^{\times}\}.$$

Proof. The proof is similar to that of the above result.

Remarks 1.10.

(i) If $(\mu, \eta(\mu, \mu^{\times}))$ is perfect nuclear space and λ is μ-perfect, then $T_{\sigma\mu}$ on λ can be generated by the family $\{p_{y,z} : y \in \lambda^{\mu}, z \in \mu^{\times}\}$ of semi-norms where

$$p_{y,z}(x) = \sup_{n \geq 1} \{|x_n y_n z_n|\} \qquad x \in \lambda.$$

(ii) For a perfect nuclear space $(\mu, \eta(\mu, \mu^{\times}))$, $T^{*}_{\sigma\mu}$ on λ^{μ} can be generated by the family $\{P_{x,y} : x \in \lambda, y \in \mu^{\times}\}$ where

$$P_{x,y}(z) = \sup_{n \geq 1} \{|x_n y_n z_n|\} \qquad z \in \lambda^{\mu}.$$

2 - Analytic characterizations

This section starts with a characterization of fully-λ-bases.

Proposition 2.1. *Let $(\mu, \eta(\mu, \mu^{\times}))$ be a perfect sequence space. Suppose λ is a μ-perfect sequence space, such that $(\lambda, T_{\sigma\mu})$ is nuclear. Then a Schauder base $\{x_n, f_n\}$ in an l.c. TVS E is a fully-λ-base, iff to each $p \in D_E$, $a \in \lambda^{\mu}_{+}$ and $b \in \mu^{\times}_{+}$ there corresponds $q \in D_E$, such that*

$$\sup_{n \geq 1} \{|f_n(x)|\, p(x_n)\, a_n b_n\} \leq q(x), \qquad \forall x \in E. \tag{1}$$

Proof. The *only if* part being obvius, we prove the *if* part. Let $p \in D_E$, $a \in \lambda^{\mu}_{+}$ and $b \in \mu^{\times}_{+}$ be choosen arbitrarily. The nuclearity of $(\lambda, T_{\sigma\mu})$ yields $c \in \lambda^{\mu}_{+}$ and $g \in \mu^{\times}_{+}$, such that

$$M = \sum_{n \geq 1} \frac{a_n b_n}{c_n g_n} < \infty .$$

By the hypothesis we can find $q \in D_E$ such that

$$\sup_{n \geq 1} \{ |f_n(x)| \, p(x_n) \, c_n g_n \} \leqslant q(x) \qquad x \in E .$$

Since the inequality

$$\sum_{n \geq 1} |f_n(x)| \, p(x_n) \, a_n b_n \leqslant M q(x) \tag{2}$$

is valid for each $x \in E$, the continuity of ψ_p follows. Also (2) means that

$$\{ f_n(x) \, p(x_n) \, a_n \} \in \mu \qquad \forall a \in \lambda^{\mu}$$

and this implies $\{f_n(x) \, p(x_n)\} \in \lambda$ as λ is μ-perfect. Thus $\{x_n, f_n\}$ is a fully-λ-base for E.

This result in particular, gives

Corollary 2.2. *Let $(\lambda, \eta(\lambda, \lambda^{\times}))$ be a perfect nuclear sequence space. Then a Schauder base $\{x_n, f_n\}$ in an l.c. TVS E is a fully-λ-base, iff to each $p \in D_E$ and $y \in \lambda_{+}^{\times}$ there exists $g \in D_E$, such that*

$$\sup_{n \geq 1} \{ |f_n(x)| \, p(x_n) \, y_n \} \leqslant g(x) \qquad x \in E .$$

(*compare with Proposition* 3.7 [7]).

Example 2.3. Let $\lambda(P)$ be a nuclear G_{∞}-space and S be a sequence space, such that S is $\lambda(P)$-perfect and $S^{\lambda(P)} \subseteq l^{\infty}$. The G_{∞}-character of P yields a $c \in P$ for each pair of a and b in P, such that $a_i b_i \leqslant c_i$. Now take any $x \in \lambda(P)$, $y \in S^{\lambda(P)}$ and $a \in P$. Then for any $b \in P$ we have the inequality

$$\sum_{i \geq 1} |\langle e_i, x \rangle| \, p_a(e_i) |y_i| \, b_i \leqslant \sup_{i \geq 1} |y_i| \sum_{i \geq 1} |x_i c_i|$$

which implies that $\{e_i, e_i\}$ is a fully-S-base for $\lambda(P)$ as S is $\lambda(P)$-perfect.

We also have an analytic characterization of fully-λ^{μ}-base.

Proposition 2.4. *Let $(\mu, \eta(\mu, \mu^{\times}))$ be a perfect nuclear sequence space. Then a Schauder base $\{x_n, f_n\}$ in an l.c. TVS E is a fully-λ^{μ}-base, iff to each*

$p \in D_E$, $z \in \lambda$ and $y \in \mu_+^\times$ there corrisponds $q \in D_E$ such that

$$\sup_{n \geq 1} \{ | f_n(x) | \, p(x_n) \, | z_n | \, y_n \} \leq q(x) \qquad x \in E .$$

Proof. Proof is left out, being similar to that of Proposition 2.1.

In particular, this gives

Corollary 2.5. *Let $(\lambda^\times, \eta(\lambda^\times, \lambda))$ be nuclear. Then a Schauder base $\{x_n, f_n\}$ in an l.c. TVS E is a fully-$\lambda^\times$-base, iff to each $p \in D_E$ and $y \in \lambda_+$ there exists $q \in D_E$, such that*

$$\sup_{n \geq 1} \{ | f_n(x) | \, p(x_n) \, y_n \} \leq q(x) \qquad x \in E .$$

This section ends with

Proposition 2.6. *Let $(\mu, \eta(\mu, \mu^\times))$ be a perfect sequence space. Suppose λ is μ-perfect and $\{x_n, f_n\}$ is an equicontinuous base for a nuclear space E. Then $\{x_n, f_n\}$ is a fully-λ-base, iff to each $p \in D_E$, $a \in \lambda^\mu$ and $b \in \mu^\times$, there exists $g \in D_E$, such that*

$$\sup_{n \geq 1} \{ | f_n(x) | \, p(x_n) \, | a_n b_n | \} \leq g(x) \qquad x \in E . \tag{3}$$

Proof. The proof follows, mutatis mutandis, on lines similar to that of Proposition 2.1, as nuclearity of E is equivalent to the fact that given $p \in D_E$ one can always choose $q \in D_E$ with $\{ \frac{p(x_i)}{q(x_i)} \} \in l^1$ (cf. [6]).

Remarks 2.7. It becomes clear from the above result that, if λ and μ are arbitrary sequence spaces, then an equicontinuous semi-λ-base $\{x_n, f_n\}$ in a nuclear space E is a fully-λ-base, iff for each $p \in D_E$, $a \in \lambda^\mu$ and $b \in \mu^\times$, there exists $q \in D_E$, such that

$$\sup_{n \geq 1} \{ | f_n(x) \, a_n b_n | \, p(x_n) \} \leq q(x) \qquad x \in E .$$

3 - Basis Theorem and Application

The result to follow, establishes that sequentially complete spaces with fully-λ-bases are nothing but a particular type of Köthe spaces.

Theorem 3.1. *Suppose X is a sequentially complete space having a fully-λ-base $\{x_n, f_n\}$. Let $y \in \lambda^{\mu}$ and $z \in \mu^{\times}$ be such that $y_n \geq \varepsilon > 0$, $z_n \geq l > 0$, $\forall n$, for some ε and l. Then X can be topologically identified with a Köthe space $\lambda(P_0)$ where*

$$P_0 = \{p(x_n)\, a_n\, b_n : p \in D_X,\ a \in \lambda_+^{\mu},\ b \in \mu_+^{\times}\}.$$

Proof. In view of the existence of mapping Ψ_p the function $\Psi: X \to \lambda(P_0)$, where $\Psi(x) = \{f_n(x)\}$, $x \in X$, defines a linear mapping. It is injective as $\{x_n, f_n\}$ is a Schauder base. To prove the surjectivity, let $\alpha \in \lambda(P_0)$. Then for $p \in D_x$, we have

$$\sum_{n \geq 1} |\alpha_n|\, p(x_n) \leq \frac{1}{\varepsilon l} \sum_{n \geq l} |\alpha_n|\, p(x_n)\, y_n z_n < \infty .$$

Since X is sequentially complete, there exists $x \in X$ such that $x = \sum_{n \geq 1} \alpha_n x_n$ giving $\Psi(x) = \alpha_n$. For proving the continuity of Ψ, let $\alpha \in P_0$. Then $\alpha_n = p(x_n)\, a_n\, b_n$ for some $p \in D_X$, $a \in \lambda_+^{\mu}$ and $b \in \mu_+^{\times}$. Thus we get

$$\hat{p}_{\alpha}(\Psi(x)) = \sum_{n \geq 1} p(x_n)\, |f_n(x)|\, a_n\, b_n \leq q(x) \qquad x \in X.$$

The continuity of Ψ^{-1} now follows from the inequality

$$p(\Psi^{-1}(\alpha)) \leq \frac{1}{\varepsilon l} \sum_{n \geq 1} |\alpha_n|\, p(x_n)\, y_n z_n = \frac{1}{\varepsilon l} \hat{p}_{y,z}(\alpha)$$

for $\alpha \in \lambda(P_0)$, where $p \in D_X$ and $\hat{p}_{y,z}$ denotes the seminorm on $\lambda(P_0)$ resulting from the sequence $\{p(x_n)\, y_n z_n\}$.

Remarks 3.2. The above result, which is a far reaching generalization of the classical theorem that a sequentially complete space with an absolute base is a Köthe space (cf. [9]), provides us in particular Proposition 3.8 [7].

The proof of the following result is analogous to that of Proposition 3.1.

Proposition 3.3. *Let $y \in \lambda$ and $z \in \mu^{\times}$ be such that $y_n \geqslant \varepsilon > 0$ and $z_n \geqslant l > 0$ for all n, for some ε and l. If a sequentially complete space X possesses a fully-λ^{μ}-base, then it can be identified topologically with a Köthe space $\lambda(P_1)$ where*

$$P_1 = \{p(x_n)\, a_n\, b_n : p \in D_X,\ a \in \lambda_+,\ b \in \mu_+^{\times}\}.$$

Nuclearity of $(\lambda, T_{\sigma\mu})$ is strong enough to ensure the nuclearity of a sequentially complete space having a fully-λ-base. This is virtually the subject matter of the following

Proposition 3.4. *Let $(\lambda, T_{\sigma\mu})$ be nuclear. Suppose X is a quasi-complete space having a fully-λ-base $\{x_n, f_n\}$. If there exist $y \in \lambda^{\mu}$ and $z \in \mu^{\times}$ with $y_n \geqslant \varepsilon > 0$ and $z_n \geqslant l > 0$, for all n, for some ε and l; then X is semi-Montel.*

Proof. In view of Proposition 3.1, X can be topologically identified with $\lambda(P_0)$ where

$$P_0 = \{\{p(x_n)\, a_n\, b_n\} : p \in D_X,\ a \in \lambda_+^{\mu},\ b \in \mu_+^{\times}\}.$$

Thus X is nuclear, iff $\lambda(P_0)$-nuclear. Since $(\lambda, T_{\sigma\mu})$ is assumed to be nuclear, by Proposition 1.4 for each $a \in \lambda_+^{\mu}$ and $b \in \mu_+^{\times}$ there exist $c \in \lambda_+^{\mu}$ and $g \in \mu_+^{\times}$ with $\{\frac{a_n b_n}{c_n g_n}\} \in l^1$. Consider arbitrary $p \in D_X$, $a \in \lambda_+^{\mu}$ and $b \in \mu_+^{\times}$. Then $\lambda(P_0)$ is nuclear because

$$\{\frac{p(x_n)\, a_n\, b_n}{p(x_n)\, c_n\, g_n}\} \in l^1.$$

Consequently E is a quasi-complete nuclear space and hence it is semi-Montel by Corollary 3.15.4 [5] (compare with Corollary 4.3 [1]).

Note. This result in particular includes Corollary 4.3 [1] and Proposition 4.3.4 [4].

The following Corollary, which results from the above Proposition in view of Remark 1.3, marks the end of the present article.

Corollary 3.5. *Let X be an infrabarrelled quasi-complete space with a fully-λ-base $\{x_n, f_n\}$. Suppose that $(\mu, \eta(\mu, \mu^{\times}))$ is nuclear and that there exist*

$a \in \lambda^{\mu}$ and $b \in \mu^{\times}$ such that $a_n \geqslant \varepsilon > 0$, $b_n \geqslant l > 0$ for all n, for some ε and l. Then X is Montel.

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Summary

See Introduction.

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