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Conformal changes of almost contact metric structures (**)

Introduction

If M^{2n+1} is a differentiable manifold endowed with an almost contact metric structure (φ, ξ, η, g), a conformal change of the metric g leads to a metric which is no more compatible with the almost contact structure (φ, ξ, η). This can be corrected by a convenient change of ξ and η which implies rather strong restrictions. Such a definition is given by I. Vaisman in [**16**]. Moreover, he characterize new types of almost contact manifolds and discuss some examples.

The aim of this paper is to continue the study of conformal changes of almost contact metric structures. In section 1 we give some results on almost contact metric manifolds. In section 2, we introduce a tensor field μ which is a conformal invariant for almost contact metric manifolds. Then, if U is a class of almost contact metric manifolds, by using the tensor field μ, we determine the class U' of all manifolds *locally conformally related* to manifolds in U (see Theorems 1 and 2). Finally, in section 3 we give some examples of locally conformal cosymplectic manifolds.

1 - Preliminaries

Let M be a $(2n+1)$-dimensional C^∞ almost contact metric manifold with metric g and almost contact structure (φ, ξ, η). Denote by $\mathfrak{X}(M)$ the Lie algebra

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of C^∞ vector fields on M. Then we have

$$\varphi^2 = -I + \eta \otimes \xi \qquad \eta(\xi) = 1 \qquad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for all $X, Y \in \mathfrak{X}(M)$, where I denotes the identity transformation. The fundamental 2-form Φ of $(M, \varphi, \xi, \eta, g)$ is defined by $\Phi(X, Y) = g(X, \varphi Y)$, for $X, Y \in \mathfrak{X}(M)$ (see [3]).

Let ∇ be the Riemannian connection of g. The covariant derivative $\nabla\Phi$ of the fundamental 2-form Φ is a covariant tensor of degree 3 which has various symmetry properties. We denote by $C(V)$ the vector space of the tensors with the same symmetries that $\nabla\Phi$, i.e., the vector space of the tensors $\alpha \in \overset{0}{\underset{3}{\otimes}} V$ satisfying

$$\alpha(x, y, z) = -\alpha(x, z, y) = -\alpha(x, \varphi y, \varphi z) + \eta(y)\alpha(x, \xi, z) + \eta(z)\alpha(x, y, \xi). \tag{1.1}$$

Here, V denotes a real vector space of dimension $2n+1$ with an almost contact structure (φ, ξ, η) and a compatible inner product $\langle , \rangle$.

In [9] it has been obtained a decomposition of $C(V)$ into twelve components $C_i(V)$ which are mutually orthogonal, irreducible and invariant subspaces under the action of $U(n) \times 1$. If $(M, \varphi, \xi, \eta, g)$ is an almost contact metric manifold and U is one of the invariant subspaces of $C(T_x M)$, where $T_x M$ is the tangent space to M at x, we say that M is of class U if $(\nabla\Phi)_x$ belongs to U, for all $x \in M$. Then, it is possible to form 2^{12} different classes of almost contact metric manifolds (where $\{0\}$ corresponds to the class C of the cosymplectic manifolds). For an extensive study of these manifolds we refer to [9].

2 - Conformal changes on almost contact metric manifolds

Let $(M, \varphi, \xi, \eta, g)$ be an almost contact metric manifold. A *conformal change* of the almost contact metric structure on M is a change of the form

$$\varphi' = \varphi \qquad \xi' = e^{-\sigma}\xi \qquad \eta' = e^{\sigma}\eta \qquad g' = e^{2\sigma} g, \tag{2.1}$$

where σ is a differentiable function of M. It is clear that $(\varphi', \xi', \eta', g')$ is again an almost contact metric structures on M. So we say that $(M, \varphi, \xi, \eta, g)$ and $(M, \varphi', \xi', \eta', g')$ are *conformally related* almost contact metric manifolds. Let U be one of these 2^{12} classes given in [9]. We denote by U' the class of all manifolds *locally conformally related to the manifolds of U*. In the words,

$(M, \varphi', \xi', \eta', g')$ belong to U' if and only if, for each m of M, there exists an open neighbourhood V of m such that $(V, \varphi', \xi', \eta', g')$ is conformally related to $(V, \varphi, \xi, \eta, g)$ of U. In such a case, for simplicity, we say that M is *l.c.* U.

If $V = M$ we say that M is *globally conformally related to a manifold of* U (or M is *g.c.* U).

Next, we introduce a *tensor field* μ, which is conformal invariant for almost contact metric manifolds and should provide information about the almost contact metric structure after a conformal change.

Let $(M, \varphi, \xi, \eta, g)$ be an almost contact metric manifold, with $\dim M \geq 5$. Then μ is the tensor field of type (1, 2) given by

$$g(\mu(X, Y), Z) = (\nabla_X \Phi)(Y, Z) - (\Pi_4(X, Y, Z) + \Pi_5(X, Y, Z) + \Pi_{12}(X, Y, Z)), \tag{2.2}$$

for $X, Y, Z \in \mathfrak{X}(M)$, where Π_4, Π_5 and Π_{12} are the tensor fields of type (0, 3) defined by

$$2(n-1)\Pi_4(X, Y, Z)$$

$$= g(\varphi X, \varphi Y)\, h\Phi(Z) - g(\varphi X, \varphi Z)\, h\Phi(Y) - \Phi(X, Y)\, h\Phi(\varphi Z) + \Phi(X, Z)\, h\Phi(\varphi Y)$$

$$\Pi_5(X, Y, Z) = \frac{\delta\eta}{2n}(\Phi(X, Z)\,\eta(Y) - \Phi(X, Y)\,\eta(Z))$$

$$\Pi_{12}(X, Y, Z) = \eta(X)\,\eta(Y)(\nabla_\xi \Phi)(\xi, Z) + \eta(X)\,\eta(Z)(\nabla_\xi \Phi)(Y, \xi)$$

where δ is the coderivative on M and

$$h\Phi(X) = \delta\Phi(\xi)\,\eta(X) - \delta\Phi(X) - (\nabla_\xi \Phi)(\xi, X)$$

for all $X \in \mathfrak{X}(M)$.

For $\dim M = 3$, we define the tensor field μ by

$$g(\mu(X, Y), Z) = (\nabla_X \Phi)(Y, Z) - \Pi_5(X, Y, Z) - \Pi_{12}(X, Y, Z).$$

We denote also by μ the tensor field of type (0, 3) given by

$$\mu(X, Y, Z) = g(\mu(X, Y), Z),$$

for $X, Y, Z \in \mathfrak{X}(M)$.

We consider a conformal change of the almost contact metric structure in M given by (2.1).

Proposition 1. *For all $X, Y, Z \in \mathcal{X}(M)$, we have*

$$\Phi'(X, Y) = e^{2\sigma}\Phi(X, Y)$$

$$\nabla'_X Y = \nabla_X Y + X(\sigma)Y + Y(\sigma)X - g(X, Y) \text{ grad } \sigma$$

$$(\nabla'_X \varphi)Y = (\nabla_X \varphi)Y + \varphi Y(\sigma)X - Y(\sigma)\varphi X - g(X, \varphi Y) \text{ grad } \sigma + g(X, Y)\varphi(\text{grad } \sigma)$$

$$(\nabla'_X \Phi')(Y, Z) = e^{2\sigma}(\nabla_X \Phi)(Y, Z)$$
$$+ e^{2\sigma}(g(X, \varphi Y)Z(\sigma) + g(X, Y)\varphi Z(\sigma) - g(X, \varphi Z)Y(\sigma) - g(X, Z)\varphi Y(\sigma))$$

$$\delta'\Phi'(X) = \delta\Phi(X) - (2n-1)\varphi X(\sigma)$$

$$h'\Phi'(X) = h\Phi(X) + 2(n-1)\varphi X(\sigma)$$

$$\delta'\eta' = e^{-\sigma}(\delta\eta - 2n\xi(\sigma))$$

$$\Pi'_4(X, Y, Z) = e^{2\sigma}(\Pi_4(X, Y, Z) + g(\varphi X, \varphi Y)\varphi Z(\sigma) - g(\varphi X, \varphi Z)\varphi Y(\sigma))$$
$$+ e^{2\sigma}(\Phi(X, Y)Z(\sigma) - \Phi(X, Y)\eta(Z)\xi(\sigma) - \Phi(X, Z)Y(\sigma) + \Phi(X, Z)\eta(Y)\xi(\sigma))$$

$$\Pi'_5(X, Y, Z) = e^{2\sigma}(\Pi_5(X, Y, Z) - \xi(\sigma)(\Phi(X, Z)\eta(Y) - \Phi(X, Y)\eta(Z)))$$

$$\Pi'_{12}(X, Y, Z) = e^{2\sigma}(\Pi_{12}(X, Y, Z) + \eta(X)\eta(Y)\varphi Z(\sigma) - \eta(X)\eta(Z)\varphi Y(\sigma))$$

where grad σ *is the vector field defined by* $g(\text{grad } \sigma, X) = X(\sigma)$.

From Proposition 1 we deduce

Proposition 2. *Let $(M, \varphi, \xi, \eta, g)$ and $(M, \varphi', \xi', \eta', g')$ be locally conformally related almost contact metric manifolds. Then the corresponding tensor fields μ and μ' coincide.*

Now, let U be a class of almost contact metric manifolds of the classification obtained in [9].

Theorem 1.

i) *For* $\dim M \geq 5$ *we have* $U' \subset U \oplus (C_4 \oplus C_5 \oplus C_{12})$. *Thus, if* $C_4 \oplus C_5 \oplus C_{12} \subset U$, *then* $U = U'$.

ii) *For* $\dim M = 3$ *we have* $U' \subset U \oplus (C_5 \oplus C_{12})$. *Thus, if* $C_5 \oplus C_{12} \subset U$, *then* $U = U'$.

Proof. Let $\dim M \geqslant 5$. From (2.2) we obtain that M is of type $U \oplus C_4 \oplus C_5 \oplus C_{12}$ if and only if $\mu \in U$.

Now, we suppose that $M \in U'$, i.e. $(M, \varphi, \xi, \eta, g)$ is locally conformally equivalent to a manifold $(M, \varphi', \xi', \eta', g')$ in U. Then $\mu' \in U$ and, by Proposition 2, we deduce that $\mu \in U$. This proves i).

If $\dim M = 3$, than $C_4 = \{0\}$, and ii) follows by a similar device.

Now, suppose that $(M, \varphi, \xi, \eta, g)$ is an almost contact metric manifold of type U and $\varphi' = \varphi$, $\xi' = e^{-\sigma}\xi$, $\eta' = e^{\sigma}\eta$, $g' = e^{2\sigma}g$, is a conformal change of the almost contact metric structure. Then, from Theorem 1, we obtain the following.

Corollary 1.

a) *If* $(d\sigma)\xi = 0$, *then* $(M, \varphi', \xi', \eta', g') \in U \oplus (C_4 \oplus C_{12})$.

b) *If* $(d\sigma)\varphi X = 0$ *for all* X *of* $\mathfrak{X}(M)$, *then* $(M, \varphi', \xi', \eta', g') \in U \oplus C_5$.

Next, suppose that $U \cap (C_4 \oplus C_5 \oplus C_{12}) = C$. We have shown that $U' \subset U \oplus (C_4 \oplus C_5 \oplus C_{12})$ (see Theorem 2.1).

The following theorem characterizes those manifolds in $U \oplus (C_4 \oplus C_5 \oplus C_{12})$ which are contained in U'. First, we define the 1-forms θ and α by

$$\theta(X) = \frac{1}{2(n-1)} h\Phi(\varphi X) \qquad \alpha(X) = (\nabla_\xi \Phi)(\xi, \varphi X)$$

for each $X \in \mathfrak{X}(M)$. Then

Theorem 2. *Let U be a class of almost contact metric manifolds and suppose that $U \cap (C_4 \oplus C_5 \oplus C_{12}) = C$. Let be $(M, \varphi, \xi, \eta, g) \in U \oplus (C_4 \oplus C_5 \oplus C_{12})$.*

a) *Suppose* $\dim M \geqslant 5$. *Then*

i) *$(M, \varphi, \xi, \eta, g) \in U'$ (that is $(M, \varphi, \xi, \eta, g)$ is locally conformally equivalent to an almost contact metric manifold in U), if and only if $\theta = \alpha$ and $\theta + \frac{\delta\eta}{2n}\eta$ is closed.*

ii) *$(M, \varphi, \xi, \eta, g)$ is globally conformally equivalent to a manifold in U, if and only if $\theta = \alpha$ and $\theta + \frac{\delta\eta}{2n}\eta$ is exact.*

b) *Suppose* $\dim M = 3$. *Then*

i) *$(M, \varphi, \xi, \eta, g) \in U'$ if and only if $\alpha + \frac{1}{2}\delta\eta\,\eta$ is closed.*

ii) *$(M, \varphi, \xi, \eta, g)$ is globally conformally equivalent to a manifold in U, if and only if $\alpha + \frac{1}{2}\delta\eta\,\eta$ is exact.*

Proof. a) We suppose that $(M, \varphi, \xi, \eta, g)$ is globally conformally equivalent to a manifold $(M, \varphi', \xi', \eta', g')$ in U. Since $U \cap (C_4 \oplus C_5 \oplus C_{12}) = C$, we have

$$h' \Phi'(\varphi X) = 0 \qquad \delta' \eta' = 0 \qquad (\nabla'_{\xi'} \Phi')(\xi', X) = 0 .$$

From these expressions, using Proposition 1, we obtain

$$h\Phi(\varphi X) = -2(n-1)\varphi^2 X(\sigma) \qquad \delta\eta = 2n\xi(\sigma) \qquad (\nabla_\xi \Phi)(\xi, \varphi X) = -\varphi^2 X(\sigma) .$$

Thus $\theta = \alpha$ and $\theta + \frac{\delta\eta}{2n}\eta$ is exact.

Conversely, if $\theta = \alpha$ and $\theta + \frac{\delta\eta}{2n}\eta$ is exact, then there exists a differentiable function σ on M such that for $X \in \mathfrak{X}(M)$

$$d\sigma(X) = \frac{h\Phi(\varphi X)}{2(n-1)} + \frac{\delta\eta}{2n}\eta(X) = (\nabla_\xi \Phi)(\xi, \varphi X) + \frac{\delta\eta}{2n}\eta(X) .$$

We put

$$\varphi' = \varphi \qquad \xi' = e^{-\sigma}\xi \qquad \eta' = e^{\sigma}\eta \qquad g' = e^{2\sigma} g .$$

Then, from Proposition 1, we have

$$h' \Phi'(\varphi X) = 0 \qquad \delta' \eta' = 0 \qquad (\nabla'_{\xi'} \Phi')(\xi', X) = 0 ,$$

and hence

$$(M, \varphi', \xi', \eta', g') \in (C_1 \oplus C_2 \oplus C_3 \oplus C_6 \oplus C_7 \oplus C_8 \oplus C_9 \oplus C_{10} \oplus C_{11})$$

$$\cap (U \oplus C_4 \oplus C_5 \oplus C_{12}) = U.$$

This proves ii).

The proof of i) is similar, except that everything is done *locally* and the Poincaré lemma is used. b) is proved in a similar way.

Also, from this theorem, if $\dim M \geqslant 5$, we obtain

Proposition 3.

i) *If* $M \in U \oplus C_4 \oplus C_{12}$, *then* M *is l.c.* U *if and only if* $\theta = \alpha$ *and* θ *is closed, and* M *is g.c.* U *if and only if, moreover,* θ *is exact.*

Thus, if $M \in U \oplus C_4$ *or* $M \in U \oplus C_{12}$, *then* M *is l.c.* U *if and only if* $M \in U$.

ii) *If* $M \in U \oplus C_4 \oplus C_5$ *or* $M \in U \oplus C_5 \oplus C_{12}$, *then* M *is l.c.U if and only if* $M \in U \oplus C_5$ *and* $\frac{\delta\eta}{2n}\eta$ *is closed, and it is g.c.U if and only if, moreover,* $\frac{\delta\eta}{2n}\eta$ *is exact. Therefore, if* $M \in C_5$ *then* M *is l.c.C.*

If $\dim M = 3$, then, we obtain

Proposition 4.

i) *If* $M \in U \oplus C_5$, *then* M *is l.c.U (g.c.U), if and only if* $\frac{\delta\eta}{2n}\eta$ *is closed (exact).*

ii) *If* $M \in U \oplus C_{12}$, *then* M *is l.c.U (g.c.U) if and only if* α *is closed (exact).*

3 - Examples

A - Let $\boldsymbol{H}^{2n+1}$ be the $(2n+1)$-dimensional hyperbolic space, i.e.,

$$\boldsymbol{H}^{2n+1} = \{(x_1, \ldots, x_{2n+1}) \in \boldsymbol{R}^{2n+1} \mid x_1 > 0\}$$

with the Riemannian metric given by $g = (x_1)^{-2} \sum_{i=1}^{2n+1} (dx_i)^2$.

$\boldsymbol{H}^{2n+1}$ is an example of Riemannian manifold of constant negative curvature $K = -1$.

The vector fields $X_i = x_1 \frac{\partial}{\partial x_i}$, $i = 1, \ldots, 2n+1$, form an orthonormal basis for this space. Let (φ, ξ, η, g) be an almost contact metric structure on $\boldsymbol{H}^{2n+1}$, and φ^i_j the components of φ with respect to the basis $\{X_1, \ldots, X_{2n+1}\}$. Let $\varphi^i_j = $ constant and $n \geqslant 2$. Then, from Corollary 1, we have

(i) *If* $\xi = x_1 \frac{\partial}{\partial x_1}$, *then the almost contact metric structures* (φ, ξ, η, g) *are of class* C_5 *and g.c.C.*

(ii) *If* $\xi = \sum_{i=2}^{2n+1} x_1 k_i \frac{\partial}{\partial x_i}$, $(k_i = \text{constant})$, *then the structures* (φ, ξ, η, g) *are of class* $(C_4 \oplus C_{12})$-$(C_4 \cup C_{12})$ *and g.c.C.*

(iii) *If* $\xi = \sum_{i=1}^{2n+1} x_1 k_i \frac{\partial}{\partial x_i}$, *where* $k_1 \neq 0$ *and* $k_i \neq 0$ *for some* $i > 1$ $(k_i = \text{constant})$, *then the structures* (φ, ξ, η, g) *are of class* $(C_4 \oplus C_5 \oplus C_{12})$-$((C_4 \oplus C_{12}) \cup C_5)$ *and g.c.C.*

B - Let $G(k)$ be the connected simply-connected 3-dimensional Lie group of real matrices of the form

$$a = \begin{pmatrix} e^{kz} & 0 & 0 & x \\ 0 & e^{-kz} & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $x, y, z \in \boldsymbol{R}$ and k is a fixed non zero real number. As easy computation shows that $\{\bar{\alpha} = e^{-kz}\,\mathrm{d}x, \bar{\beta} = e^{kz}\,\mathrm{d}y, \bar{\gamma} = \mathrm{d}z\}$ is a family of linearly independent left invariant 1-forms on $G(k)$. The corresponding dual basis of left invariant vector fields on $G(k)$ is

$$\{\bar{X} = e^{kz}\,\frac{\partial}{\partial x},\ \bar{Y} = e^{-kz}\,\frac{\partial}{\partial y},\ \bar{Z} = \frac{\partial}{\partial z}\}.$$

We have $$[\bar{X}, \bar{Z}] = -k\bar{X} \qquad [\bar{Y}, \bar{Z}] = k\bar{Y},$$

and the all the other brackets are zero. Then we easily show that $G(k)$ is a solvable non-nilpotent Lie group.

Now, let $B \in SL(2, \boldsymbol{Z})$ be an unimodular matrix, with positive real eigenvalues and differents λ and λ^{-1} and let $(a, b), (c, d) \in \boldsymbol{R}^2$ the corresponding eigenvectors. We consider the discrete subgroup $\Gamma(k)$ of $G(k)$ which consist of the matrices of the form

$$\begin{pmatrix} \lambda^p & 0 & 0 & na + mb \\ 0 & \lambda^{-p} & 0 & nc + md \\ 0 & 0 & 1 & pk^{-1}\ln\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with $n, m, p \in \boldsymbol{Z}$. We denote by $M(k) = \Gamma(k) \setminus G(k)$ the space of right cosets. Thus $M(k)$ is a compact solvmanifold of dimension 3.

If π: $G(k) \to M(k)$ is the canonical projection, then we have a basis $\{\alpha, \beta, \gamma\}$ of 1-forms on $M(k)$ verifying

$$\pi^*\alpha = \bar{\alpha} \qquad \pi^*\beta = \bar{\beta} \qquad \pi^*\gamma = \bar{\gamma}$$

$$\mathrm{d}\alpha = k\alpha \wedge \gamma \qquad \mathrm{d}\beta = -k\beta \wedge \gamma \qquad \mathrm{d}\gamma = 0\,.$$

The corresponding dual basis of vector fields is denoted by $\{X, Y, Z\}$, and we have $[X, Z] = -kX$, $[Y, Z] = kY$, being all the other brackets zero.

Alternatively, the manifold $M(k)$ may be seen as the total space of a T^2-bundle over S^1. In fact, let $T^2 = \boldsymbol{R}^2/\boldsymbol{H}^2$ the 2-dimensional tori, where $\boldsymbol{H}^2 \cong \boldsymbol{Z}^2$ is the discrete subgroup of the integral linear combinations of the basis of $\boldsymbol{R}^2$ given by $\{(a, c), (b, d)\}$ and let $\rho\colon \boldsymbol{Z} \to \mathrm{Diff}(T^2)$ be the representation defined as follows: $\rho(n)$ represents the transformation of T^2 covered by the linear transformation of $\boldsymbol{R}^2$ given by the matrix

$$\begin{pmatrix} \lambda^n & 0 \\ 0 & \lambda^{-n} \end{pmatrix}.$$

This representation determines an action A of $\boldsymbol{Z}$ on $\boldsymbol{R} \times T^2$ which is defined as follows:

$$A(n, (z, [x, y])) = (z + n, \rho(n)[x, y]).$$

Then $p\colon \boldsymbol{R} \times_Z T^2 \to S^1$ is a T^2-bundle where the projection p is given by $p[z, [x, y]] = [z]$.

Now, it is easy to see that $\psi\colon \boldsymbol{R} \times_Z T^2 \to M(k)$ given by $\psi([z, [x, y]]) = [x, y, \frac{\ln \lambda}{k} z]$ is a diffeomorphism, in such a way that $p\colon M(k) \to S^1$, $p[x, y, z] = [\frac{kz}{\ln \lambda}]$, is a T^2-bundle over S^1.

Now, let (φ, η, ξ, g) be the almost contact metric structure on $M(k)$ given by

$$\varphi = \alpha \otimes Z - \gamma \otimes X \qquad \xi = Y \qquad \eta = \beta \qquad \text{and} \qquad g = \alpha \otimes \alpha + \beta \otimes \beta + \gamma \otimes \gamma.$$

Then (φ, ξ, η, g) is an almost contact metric structure on $M(k)$ l.c.C of class C_{12} and it is not g.c.C.

Remark. It is known that $M(k)$ has no cosymplectic structures. In fact, $M(k)$ can have no normal structures, since $M(k) \times S^1$ can have no complex structures (see [**13**], [7], [6]). A generalization of $M(k)$ is given in [8].

C - Let (M, J, h) be an almost Hermitian manifold, dim $M = 2n$ $(n \geq 2)$, and θ an arbitrary 1-form on M. In $M \times \boldsymbol{R}$ we consider the almost contact metric structure (φ, ξ, η, g) given by

$$(3.1)\qquad \begin{gathered} \varphi(X, a\frac{\mathrm{d}}{\mathrm{d}t}) = (JX, -t\theta(JX)\frac{\mathrm{d}}{\mathrm{d}t}) \qquad \xi = (0, \frac{\mathrm{d}}{\mathrm{d}t}) \qquad \eta(X, a\frac{\mathrm{d}}{\mathrm{d}t}) = t\theta(X) + a \\ g((X, a\frac{\mathrm{d}}{\mathrm{d}t}), (Y, b\frac{\mathrm{d}}{\mathrm{d}t})) = h(X, Y) + ab + t^2\theta(X)\theta(Y) + (\theta(X)b + \theta(Y)a)t \end{gathered}$$

where a and b are C^∞ functions on $M \times \boldsymbol{R}$, $X, Y \in \mathfrak{X}(M)$. Then, if M is a locally conformal Kähler manifold (i.e., M is of type W_4 of the Gray-Hervella classification for almost Hermitian manifolds [10]), with Lee form w, and if we put $\theta = -\dfrac{w}{2}$, then *the above almost contact metric structure* (φ, ξ, η, g) *is of class* $C_4 \oplus C_{12}$, *and, by Proposition* 3, *it is l.c.C. Moreover, if M is not globally conformal Kähler, then $M \times \boldsymbol{R}$ is not globally conformal cosymplectic.*

In order to construct examples of locally conformal cosymplectic manifolds, let us consider the following manifolds, which are compact locally conformal Kähler manifolds, but do not admit Kähler structures.

— The Hopf manifolds (see [15]), which are defined as quotients $H = (\boldsymbol{C}^n - \{0\}) \setminus \Delta_k$, where Δ_k is the group generated by $z \to kz$ $(k \in \boldsymbol{C}, |k| \neq 0, 1)$, with the Hermitian metric

$$h = \left(\sum_{j=1}^{n} z^j \bar{z}^j\right)^{-1} \left(\sum_{i=1}^{n} \mathrm{d}z^i \otimes \mathrm{d}\bar{z}^i\right).$$

— The Inoue surfaces S_M, $S^{(+)}_{N, p, q, r; t}$ and $S^{(-)}_{N, p, q, r}$ which are the quotient manifolds $(\boldsymbol{H} \times \boldsymbol{C}) \setminus G$, where $\boldsymbol{H}$ is the upper half of the complex plane $\boldsymbol{C}$ and G is a group of analytic automorphisms of $\boldsymbol{H} \times \boldsymbol{C}$ (see [11]). The surfaces S_M are locally conformal Kähler manifolds with the metric

$$h = (w_2)^{-2} (\mathrm{d}w \otimes \mathrm{d}\bar{w}) + w_2\, \mathrm{d}z \otimes \mathrm{d}\bar{z}$$

and the surfaces $S^{(+)}_{N, p, q, r; t}$ and $S^{(-)}_{N, p, q, r}$ also are locally conformal Kähler with the metric

$$h = \frac{1 + (z_2)^2}{(w_2)^2} \mathrm{d}w \otimes \mathrm{d}\bar{w} - \frac{z_2}{w_2} (\mathrm{d}w \otimes \mathrm{d}\bar{z} + \mathrm{d}z \otimes \mathrm{d}\bar{w}) + \mathrm{d}z \otimes \mathrm{d}\bar{z}$$

where (w, z) are the coordinates in $\boldsymbol{H} \times \boldsymbol{C}$, $w_2 = \mathrm{Im}(w) > 0$ and $z_2 = \mathrm{Im}(z)$ (see [14]).

— The compact locally conformal Kähler nilmanifolds $N(r, 1) \times S^1$, where $N(r, 1) = \Gamma(r, 1) \setminus H(r, 1)$ are compact quotients of the generalized Heisenberg group $H(r, 1)$ by the subroup $\Gamma(r, 1)$ of matrices of $H(r, 1)$ with integer entries (see [5]). These nilmanifolds are examples of generalized Hopf manifolds.

— The family of compact locally conformal Kähler solvmanifolds $M^4(k, n)$, where $M^4(k, n)$ is a nontrivial circle bundle over the three-dimensional solvmanifold $M^3(k)$ considered in B (see [1]).

Then, *if M is one of the manifolds above described, we obtain examples of locally conformal cosymplectic manifolds of class* $C_4 \oplus C_{12}$, *which are not globally conformal, by considering on* $M \times \boldsymbol{R}$ *the almost contact metric structure given in* (3.1).

D - A interesting example is the $(2n+1)$-dimensional real Hopf manifold $\boldsymbol{R}H^{2n+1}$ (see [17]), which are defined as follows. We consider the transformation $\psi_\lambda : \boldsymbol{R}^{2n+1} - \{0\} \to \boldsymbol{R}^{2n+1} - \{0\}$ given by

$$\bar{x}^i = \lambda x^i \qquad \lambda \in \boldsymbol{R} \qquad \lambda > 0 \qquad \lambda \neq 1$$

and denote by Ψ_λ the infinite cyclic group generated by ψ_λ. Then

$$\boldsymbol{R}H^{2n+1} = (\boldsymbol{R}^{2n+1} - \{0\})/\Psi_\lambda .$$

Using the diffeomorphism f of $\boldsymbol{R}^{2n+1} - \{0\}$ on $S^{2n} \times \boldsymbol{R}$ given by

$$(x^i) \to \left(\frac{x^i}{\|x\|}, \frac{\ln \|x\|}{\ln \lambda}\right)$$

we obtain that $\boldsymbol{R}H^{2n+1}$ is diffeomorphic to $S^{2n} \times S^1$, which proves that $\boldsymbol{R}H^{2n+1}$ is a compact connected differentiable manifold.

Now, we consider in $\boldsymbol{R}^{2n+1} - \{0\}$ the metric

$$g = \left(\sum_{j=1}^{2n+1} (x^j)^2\right)^{-1} \left(\sum_{i=1}^{2n+1} (\mathrm{d}x^i)^2\right)$$

where $(x^1, \ldots, x^{2n+1})$ are the coordinates in $\boldsymbol{R}^{2n+1} - \{0\}$. The vector fields

$$X_i = \left(\sum_{j=1}^{2n+1} (x^j)^2\right)^{\frac{1}{2}} \frac{\partial}{\partial x^i}, \qquad i = 1, \ldots, 2n+1$$

form an orthonormal basis for the Riemann manifold $(\boldsymbol{R}^{2n+1} - \{0\})$. Let (φ, ξ, η, g) the almost contact metric structure on $\boldsymbol{R}^{2n+1} - \{0\}$ given by

$$\varphi X_i = X_{n+i} \qquad \varphi X_{n+i} = -X_i \qquad i = 1, \ldots, n$$

$$\xi = X_{2n+1} \qquad \eta = \left(\sum_{i=1}^{2n+1} (x^i)^2\right)^{-\frac{1}{2}} \mathrm{d}x^{2n+1} .$$

The structure (φ, ξ, η, g) is *g.c.C.* of type $C_4 \oplus C_5 \oplus C_{12}$, where the Lee form ω is given by

$$\omega = \left(\sum_{j=1}^{2n+1} (x^j)^2\right)^{-1} \left(\sum_{i=1}^{2n+1} x_i \mathrm{d}x^i\right).$$

Then the tensors φ, ξ, η and g on $\boldsymbol{R}^{2n+1} - \{0\}$ all descend to $\boldsymbol{RH}^{2n+1}$. We denote by $(\bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g})$ the structure induced on $\boldsymbol{RH}^{2n+1}$. Thus, $(\bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g})$ *is l.c. C. of class* $C_4 \oplus C_5 \oplus C_{12}$. Now, by the definition of the diffeomorphism f, one gets $\omega = -\ln \lambda f^* \mathrm{d}t$, and consequently, by descend to $\boldsymbol{RH}^{2n+1}$, *the structure* $(\bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g})$ *is not globally conformal cosymplectic.*

Remarks.

1 - Since $\boldsymbol{RH}^{2n+1} \approx S^{2n} \times S^1$ the Betti numbers of $\boldsymbol{RH}^{2n+1}$ are

$$b_0 = b_1 = b_{2n} = b_{2n+1} = 1 \qquad b_i = 0 \qquad 2 \leqslant i \leqslant 2n-1,$$

and thus, $\boldsymbol{RH}^{2n+1}$ can not have cosymplectic structures for $n \geqslant 2$ (see [4]).

2 - It is easy to check that $f^*(\mathrm{d}\sigma^2 + (\ln \lambda)^2 \mathrm{d}t^2) = g$, where $\mathrm{d}\sigma^2$ is the metric of S^{2n} and t is the coordinate in $\boldsymbol{R}$. Thus, $(\boldsymbol{RH}^{2n+1}, \bar{g})$ is isometric to $(S^{2n} \times S^1, \bar{h})$, being $\bar{h}$ the metric given by $\bar{h} = \mathrm{d}\sigma^2 + (\ln \lambda)^2 \theta^2$, where θ is the length element of S^1.

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Summary

In this paper conformal changes of metrics on almost contact metric manifolds are studied and some examples are given.

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