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On the triconfluent Heun equation $[0, 0, 1_5]$ (**)

1 - Introduction

The normal form of the triconfluent Heun equation with Ince symbol $[0, 0, 1_6]$ is

$$y'' + (A_0 + A_1 x + A_2 x^2 + A_3 x^3 + A_4 x^4)\, y = 0 \tag{1.1}$$

see, for example, [2].

If $A_4 = 0$, the corresponding form of the differential equation with one singularity, irregular and of the fifth type, results. This latter case, however, has various properties which cannot readily be deduced from (1.1), so that a separate treatment is worthwhile. For a discussion of Ince's classification of linear differential, see [5], Chapter 20.

The canonical form of the differential equation $[0, 0, 1_5]$ is taken to be

$$y'' - 2xy' - 4(a + kx^3)\, y = 0 \tag{1.2}$$

and is of some theoretical interest in its own right as well as being associated with the general cubic anharmonic oscillator. Explicit solutions of the biconfluent Heun equation have recently been obtained by Exton [3], and a similar approach is made in this study.

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2 - An associated differential equation

If we put $c = \frac{1}{2}$ and replace z by x^2 in the equation

$$zy'' + (c - z)\,y' - ay = kz^{\frac{3}{2}}\,y \tag{2.1}$$

we recover (1.2). Equation (2.1) is seen to be a confluent hypergeometric equation with an extra term on the right. Let

$$y = \sum_{r=0}^{\infty} k^r y_r(z) \tag{2.2}$$

in (2.1), when on equating coefficients of powers of the parameter k, it follows that

$$zy''_0 + (c - z)\,y'_0 - ay_0 = 0 \tag{2.3}$$

$$zy''_r + (c - z)\,y'_r - ay_r = z^{\frac{3}{2}}\,y_{r-1}\,. \tag{2.4}$$

A solution of (2.3) is the confluent hypergeometric function $y_0 = {}_1F_1(a;\ c;\ z)$. We can write

$$zy''_1 + (c - z)\,y'_1 - ay_1 = \sum_{m_0=0}^{\infty} \frac{(a,\ m_0)}{(c,\ m_0)\,(1,\ m_0)} z^{m_0 + \frac{3}{2}} \tag{2.5}$$

where the Pochhammer symbol is given by

$$(a,\ n) = a(a + 1)\ldots(a + n - 1) = \Gamma(a + n)/\Gamma(a) \qquad (a,\ 0) = 0\,.$$

Utilise the inhomogeneous confluent hypergeometric function $\theta_\sigma(a;\ c;\ z)$ given by

$$\theta_\sigma(a;\ c;\ z) = \frac{z^\sigma}{\sigma(c - 1 + \sigma)}\,{}_2F_2(a + \sigma + 1,\ 1;\ c + \sigma + 1,\ \sigma + 2;\ z)\,.$$

See [1], p. 121. It follows that

$$y_1 = \sum_{m_0=0}^{\infty} \frac{(a,\ m_0)}{(c,\ m_0)\,(1,\ m_0)}\,\theta_{m_0 + \frac{5}{2}}(a;\ c;\ z) = \frac{z^{\frac{5}{2}}}{(\frac{5}{2})\,(c + \frac{3}{2})}$$

(2.6)

$$\cdot \sum_{m_0,\ m_1=0}^{\infty} \frac{(a,\ m_0)\,(c + \frac{3}{2},\ m_0)\,(\frac{5}{2},\ m_0)\,(a + \frac{5}{2},\ m_0 + m_1)\,z^{m_0 + m_1}}{(a + \frac{5}{2},\ m_0)\,(c,\ m_0)\,(1,\ m_0)\,(c + \frac{5}{2},\ m_0 + m_1)\,(\frac{7}{2},\ m_0 + m_1)}\,.$$

On progressively applying the above process, we find that

$$\text{(2.7)}\qquad y_r = \frac{z^{\frac{5r}{2}}}{(\frac{5}{2})^{2r} r!(\frac{2c}{5}+\frac{3}{5}, r)} \sum_{m_0, \ldots, m_r = 0}^{\infty} \frac{(a, m_0)(c+\frac{3}{2}, m_0)(\frac{5}{2}, m_0)}{(a+\frac{5}{2}, m_0)(c, m_0)(1, m_0)}$$

$$\cdot \frac{(a+\frac{5}{2}, m_0+m_1)(c+4, m_0+m_1)(5, m_0+m_1)}{(a+5, m_0+m_1)(c+\frac{5}{2}, m_0+m_1)(\frac{7}{2}, m_0+m_1)} \cdots$$

$$\cdot \frac{(a+\frac{5r}{2}-\frac{5}{2}, m_0+\ldots+m_{r-1})(c+\frac{5r}{2}-1, m_0+\ldots+m_{r-1})(\frac{5r}{2}, m_0+\ldots+m_{r-1})}{(a+\frac{5r}{2}, m_0+\ldots+m_{r-1})(c+\frac{5r}{2}-\frac{5}{2}, m_0+\ldots+m_{r-1})(\frac{5r}{2}-\frac{3}{2}, m_0+\ldots+m_{r-1})}$$

$$\cdot \frac{(a+\frac{5r}{2}, m_0+\ldots+m_r) z^{m_0+\ldots m_r}}{(c+\frac{5r}{2}, m_0+\ldots+m_r)(1+\frac{5r}{2}, m_0+\ldots+m_r)}.$$

The convergence of the series solution (2.2) may be established in a similar way to that used in connection with the solution of the biconfluent Heun equation by Exton [3]. This holds for all values of the variable and the parameters, real or complex, provided that $c \neq 0, -1, -2, \ldots$. Hence, (2.2) with (2.7) denotes a solution of (2.1) near the origin with zero exponent which is taken as standard and denoted by the symbol $y(a; c; k; z)$.

On the replacing y by $z^{1-c}y$ in the subsidiary equation (2.1), we obtain

$$\text{(2.8)}\qquad zy'' + (2-c-z)y' - (a+1-c)y = kz^{\frac{3}{2}} y$$

which is of the same form as (2.1). Thus, if $Z(a; c; k; z)$ is a solution of this equation, then so also is $z^{1-c}Z(a+1-c; 2-c; k; z)$. Corresponding solutions of the differential equation $[0, 0, 1_5]$ then follow by putting $c = \frac{1}{2}$ and $z = x^2$ in the above results.

3 - The behaviour at the point at infinity

The solutions of the equation (1.2) at infinity are essentially different from those of the triconfluent Heun equation $[0, 0, 1_6]$ at the same point. With regard to the equation $[0, 0, 1_5]$, we have an aggregate of subnormal solutions instead of a regular solution and a normal solution. This would be expected on account of the odd species of the irregular singularity at infinity.

In order to investigate this matter, we first of all replace x by ξ^2 in (1.2) and obtain

$$\xi y'' - (1+\xi^4)y' - 4\xi^3(a+k\xi^6)y = 0. \tag{3.1}$$

Then let $y = \exp(\alpha\xi^5 + \beta\xi^4 + \gamma\xi^3 + \delta\xi^2 + \varepsilon\xi)Y$, and $\xi = \dfrac{1}{\zeta}$ so that (3.1) becomes

$$\zeta^3 Y'' - [10\alpha\zeta^{-3} + (8\beta-1)\zeta^2 + 6\gamma\zeta^{-1} + 4\delta + 2\varepsilon\zeta - 3\zeta^2]Y' \tag{3.2}$$

$$+[(25\alpha^2 - 4k)\zeta^{-9} + 5\alpha(8\beta-1)\zeta^{-8} + 2(8\beta^2 + 15\alpha\gamma - 2\beta)\zeta^{-7} + (24\beta\gamma + 20\alpha\delta - 3\gamma)\zeta^{-6}]Y$$

$$+[(9\gamma^2 + 16\beta\delta + 10\alpha\varepsilon - 2\delta)\zeta^{-5} + (8\beta\varepsilon + 12\gamma\delta + 15\alpha - \varepsilon)\zeta^{-4} + (4\delta^2 + 6\gamma\varepsilon + 8\beta - 4a)\zeta^{-3}]Y$$

$$+[(4\delta\varepsilon + 3\gamma)\zeta^{-2} + \varepsilon(4\varepsilon+1)\zeta^{-1} - \varepsilon]Y = 0.$$

If the parameters α, β, γ, δ and ε are selected so that the terms in ζ^{-9}, ζ^{-8}, ζ^{-7}, ζ^{-6} and ζ^{-5} are removed from the coefficient of Y in the previous equation, it is found that

$$\alpha = \pm\frac{2}{5}k^{\frac{1}{2}} \qquad \beta = \frac{1}{8} \qquad \gamma = \pm\frac{1}{48}k^{-\frac{1}{2}} \qquad \delta = 0 \quad \text{and} \quad \varepsilon = \pm\frac{1}{1024}k^{-\frac{3}{2}}.$$

Hence, the required aggregate of subnormal solutions of (1.3) may be written

$$\exp[\pm\frac{2}{5}k^{\frac{1}{2}}x^{\frac{5}{2}} + \frac{1}{8}x^2 \pm \frac{1}{48}k^{-\frac{1}{2}}x^{\frac{3}{2}} \pm \frac{1}{1024}k^{-\frac{3}{2}}x^{\frac{1}{2}}] \quad [1 + O(x^{-\frac{1}{2}})] \tag{3.3}$$

since the associated indicial function is of the first degree.

An explicit form of these solutions is obtainable by expanding the solution $y(a; \frac{1}{2}; k; x^2)$ as a Neumann series of modified Bessel functions of argument $\pm\frac{2}{5}k^{\frac{1}{2}}x^{\frac{5}{2}}$ and order $\frac{3}{2} + 2N$, $N = 0, 1, 2, \ldots$ in a manner similar to that used by Exton [4] by means of the asymptotic representation of the Bessel function. The reader is referred to this paper for the rather lengthy details.

References

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Summary

The linear differential equation of the second order with only one singularity, irregular of the fifth type cannot conveniently be regarded as a special case of equation with Ince symbol $[0, 0, 1_6]$. *A separate treatment is necessary and explicit solutions are obtained as power series of a parameter relative to the ordinary point at the origin. An aggregate of subnormal solutions applicable to the irregular singularity at infinity is sketched and a means of deducing an explicit representation is indicated.*

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