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On commutativity of right s-unital rings (**)

1 - Introduction

Throughout, R will represent an associative ring. As usual, we write $[x, y] = xy - yx$ for x, y in R. A ring R is called *right s-unital* (resp. *left s-unital*) if for each x in R, $x \in xR$ (resp. $x \in Rx$); R is called *s-unital* if R is both right and left s-unital.

Recently Streb [8] gave a classification for non-commutative rings, which has been used by several authors in obtaining a number of commutativity theorems (cf. [4], [5], [6] and [7]). Further, it can be observed from the proof of Corollary 1 of [8] that if R is a non-commutative right s-unital ring, then there exists a factor subring S of R, which is of type (a), (b), (c) or (d):

(a) $M(p) = \begin{pmatrix} 0 & GF(p) \\ 0 & GF(p) \end{pmatrix}$ $\quad p$ a prime.

(b) $M_\sigma(K) = \left\{ \begin{pmatrix} a & b \\ 0 & \sigma(a) \end{pmatrix} \mid a, b \in K \right\}$

where K is a finite field with a non-trivial automorphism σ.

(c) A non-commutative ring without divisors of zero.

(d) $S = \langle 1 \rangle + T$, T is a non-commutative subring of S such that $T[T, T] = [T, T]T = 0$.

This gives us the following lemma, which will be useful in the subsequent discussion.

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Lemma 1. *Let P be a ring property, which is inherited by factor subrings. If no rings of type* (a), (b), (c) *or* (d) *satisfy P, then every right s-unital ring satisfying P is commutative.*

We remark that the above lemma may be looked as dual of the metatheorem of [5].

In the present paper, we consider the following *ring properties*

P_1 For every y in R there exists $f(X) \in X^2\mathbf{Z}[X]$, such that $[yx^m - x^n f(y), x] = 0$ for all x in R, where m, n are fixed non-negative integers.

P_2 For every x, y in R there exist non-negative integers m, n and $f(X) \in X^2\mathbf{Z}[X]$ such that $[yx^m - x^n f(y), x] = 0$.

2 - Main results

The following theorems give two commutativity conditions for right s-unital rings.

Theorem 1. *A right s-unital ring R is commutative if* (*and only if*) *it satisfies the property* P_1.

Theorem 2. *A right s-unital ring R is commutative if* (*and only if*) *it satisfies the property* P_2 *and for each x, y in R there exist* $g(X)$, $h(X) \in X^2\mathbf{Z}[X]$ *such that* $[x - g(x), y - h(y)] = 0$.

In preparation for the proof of the above theorems, we start with the following lemmas. Proofs of Lemma 2, Lemma 3 and Lemma 4 can be seen in [2], [3] and [4] respectively; whereas Lemma 5 can be proved dualizing the proof of Lemma 1 of [6].

Lemma 2. *If for every x, y in R there exists* $f(x) \in X^2\mathbf{Z}(X)$ *such that* $[x - f(x), y] = 0$, *then R is commutative.*

Lemma 3. *Let f be a polynomial in non-commuting indeterminates* $x_1, x_2, \ldots, x_n$ *with integer coefficients. Then the following statements are equivalent:*

(i) *For any ring R satisfying $f = 0$, the commutator ideal of R is a nil ideal.*

(ii) *Every semi-prime ring satisfying $f = 0$ is commutative.*

(iii) *For every prime p, the ring $(GF(p))_2$ of 2×2 matrices over $GF(p)$ fails to satisfy $f = 0$.*

Lemma 4. *Let R be a non-commutative ring with the property that for each x, y in R there exist $f(X)$, $g(X) \in X^2\mathbf{Z}[X]$ such that $[x - f(x), y - g(y)] = 0$. Then there exists a factor subring of R, which is of type* (a) *or* (b).

Lemma 5. *If R is right s-unital and not left s-unital, then R has a factor subring of type* (a).

Proof of Theorem 1. Suppose that R satisfies the property P_1. First consider the ring of the type (a). Then in $M(p)$, we see that $[e_{22}, e_{12}e_{22}^m - e_{22}^n f(e_{12})] = -e_{12} \neq 0$ for all integers $m \geqslant 0$, $n \geqslant 0$ and $f(X) \in X^2\mathbf{Z}[X]$. Thus, no rings of type (a) satisfy P_1 and by Lemma 5, R is s-unital. Hence, in view of Proposition 1 of [3], we may assume that R has unity 1.

Next, consider the ring $M_\sigma(K)$, a ring of type (b).

Let $x = \begin{pmatrix} a & 0 \\ 0 & \sigma(a) \end{pmatrix}$ $(a \neq \sigma(a))$ and $y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

then $[yx^m - x^n f(y), x] = -[x, y]x^m = -(a - \sigma(a))(\sigma(a))^m y \neq 0$ for all integers $m \geqslant 0$, $n \geqslant 0$ and $f(X) \in X^2\mathbf{Z}[X]$.

If $m = n = 0$, then $[y - f(y), x] = 0$ and R is commutative by Lemma 2. Henceforth, we may assume that $m > 0$ or $n > 0$. Property P_1 may be written as

$$[x, y]x^m = x^n[x, f(y)]. \tag{1}$$

Multiply (1) by $(x+1)^n$, obtaining

$$(x+1)^n[x, y]x^m = (x+1)^n x^n[x, f(y)] \tag{2}$$

then replace x by $x + 1$ in (1) and multiply the result by x^n, thereby obtaining

$$x^n[x, y](x+1)^m = (x+1)^n x^n[x, f(y)]. \tag{3}$$

Comparing of (2) and (3) now yields

$$x^n[x, y](x+1)^m = (x+1)^n[x, y]x^m. \tag{4}$$

But $x = e_{12} - e_{22}$, $y = e_{12}$ fail to satisfy the polynomial identity (4) in $M(p)$, p a prime. Hence in view of Lemma 3, the commutator ideal of R is nil and therefore, R has no factor subrings of type (c).

Finally, consider $S = \langle 1 \rangle + T$, where T is a non-commutative subring of S such that $T[T, T] = [T, T]T = 0$. Suppose that R has a factor subring S. Choose $s, t \in T$ such that $[s, t] \neq 0$. A simple computation gives that $[s, t] = [s, t](s+1)^m$ for any non negative integer m. Further, by hypothesis there exists $f(x) \in X^2\boldsymbol{Z}[X]$ such that

$$[s, t] = [s, t](s+1)^m = (s+1)^n[s, f(t)] = [s, f(t)].$$

Now since $f(X) \in X^2\boldsymbol{Z}[X]$ and $T[T, T] = [T, T]T = 0$, we get $[s, f(t)] = 0$. Hence $[s, t] = 0$, a contradiction.

Thus we have seen that no rings of type (a), (b), (c) or (d) satisfy P_1. Hence by Lemma 1, R is commutative.

Proof of Theorem 2. As above, it is easy to see that no rings of type (a) or (b) satisfy P_2. Combining this fact with the Lemma 4, we get the required result.

By looking at the special case where $f(X)$ is a power of X, we obtain the following corollaries. In very recent past, many results of these types have been obtained by several authors using different techniques (for some references, see [**1**]).

Corollary 1. *A right s-unital ring R is commutative if (and only if) for each y in R there exists an integer $p > 1$ such that $[yx^m - x^n y^p, x] = 0$ for all x in R, where m, n are fixed non-negative integers.*

Corollary 2. *A right s-unital ring R is commutative if (and only if) for each x, y in R there exist integers $m \geqslant 0$, $n \geqslant 0$, $p > 1$ and $g(X), h(X) \in X^2\boldsymbol{Z}[X]$ such that $[yx^m - x^n y^p, x] = 0$ and $[x - g(x), y - h(y)] = 0$.*

Remark. The justification of the additional condition

$$[x - g(x), y - h(y)] = 0$$

in the hypothesis of Theorem 2 remains open.

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Summary

It is shown that a right s-unital ring R is commutative if and only if for each non-negative integers m, n and for each y in R there exists $f(X) \in X^2\mathbf{Z}[X]$ such that $[yx^m - x^n f(y), x] = 0$ for all x in R. Further, the result has been extended to the case when the exponents m and n depend on the choice of x and y.

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