Riv. Mat. Univ. Parma (5) **1** (1992), 51-56

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A note on link digraphs (**)

1 - Preliminaries

In this note we consider the class of *asymmetric digraphs.* By an asymmetric digraph we mean a digraph $G = (V(G), U(G))$ with the vertex set $V(G)$ and the arc set $U(G)$ satisfying the following condition $(x, y) \in U(G) \Rightarrow (y, x) \notin U(G)$ for all $x, y \in V(G)$.

From now on we think of a digraph as being asymmetric.

Definition 1. The *outlink* (*inlink*) *of a vertex v of a digraph H* is the subdigraph of H induced by all vertices $w \in V(H)$, such that $(v, w) \in U(H)$ ($(w, v) \in U(H)$, respectively).

Definition 2. A digraph G is called an *outlink* (*inlink*) *digraph,* if there exists a digraph H, such that oulinks (inlinks) of all vertices in H are isomorphic to G. Then we say that H *has constant outlink* (*inlink*).

Now let $G = (V(G), E(G))$ be a simple graph with the vertex set $V(G)$ and the edge set $E(G)$.

Definition 3. By an *orientation of G* we mean a digraph $\vec{G}$ with the vertex set $V(G)$, such that for every $\{x, y\} \in E(G)$, either (x, y) or (y, x) belongs to $U(\vec{G})$ and no other arcs belong to $U(\vec{G})$.

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(**) Received October 22, 1991. AMS classification 05 C 20.

We introduce the class of starlike graphs. They are a special type of the so-called treelike graphs, see [5].

Definition 4. The graph G is said to be *starlike*, if every two cliques of G have exactly one common vertex.

If $m \geqslant 1$ is the number of cliques of a starlike graph, then we denote it by $S_{1,m}$. Further notations and definitions not given here can be found in [3], [4].

2 - Some results

In [2] we posed the following problem. *Let G be a simple graph. Is there an orientation of G, which is an outlink digraph?* In this note we answer to this question in the case when G is a starlike graph.

First we give some results about outlink (inlink) digraphs.

Lemma 1. *If G_1, G_2 are outlink (inlink) digraphs, then $G_1 \cup G_2$ is an outlink (inlink) digraph.*

Proof. Let H_i be a digraph with the constant outlink G_i, $i = 1, 2$. Then $H = H_1 \times H_2$, defined as follows

$$V(H_1 \times H_2) = V(H_1) \times V(H_2) \qquad ((v_1, v_2), (v_3, v_4)) \in U(H_1 \times H_2)$$

$$\text{iff } (v_1, v_3) \in U(H_1) \wedge (v_2 = v_4) \qquad \text{or} \qquad (v_1 = v_3) \wedge (v_2, v_4) \in U(H_2)$$

has the constant outlink isomorphic to $G_1 \cup G_2$. The same reasoning applies to the case of inlink digraphs.

Let G be a digraph. By the symbol $G \overrightarrow{+} v$ ($G \overleftarrow{+} v$) we denote a digraph, which arises from G by addition of a new vertex v and of the arcs from every vertex of G to v (from v to every vertex of G). Then we obtain

Lemma 2. *Let G be an outlink (inlink) digraph. Then the digraphs $G \overrightarrow{+} v$, $G \overleftarrow{+} v$ are also outlink (inlink) digraphs.*

Proof. Let H_i be a digraph with an outlink of every vertex isomorphic to G, for $i = 1, 2, 3$, and v_1, v_2, v_3 be the arbitrarily chosen vertices in H_2, H_3, H_1, respectively. Then the digraph $F = (V(F), U(F))$, presented in Fig. 1 and de-

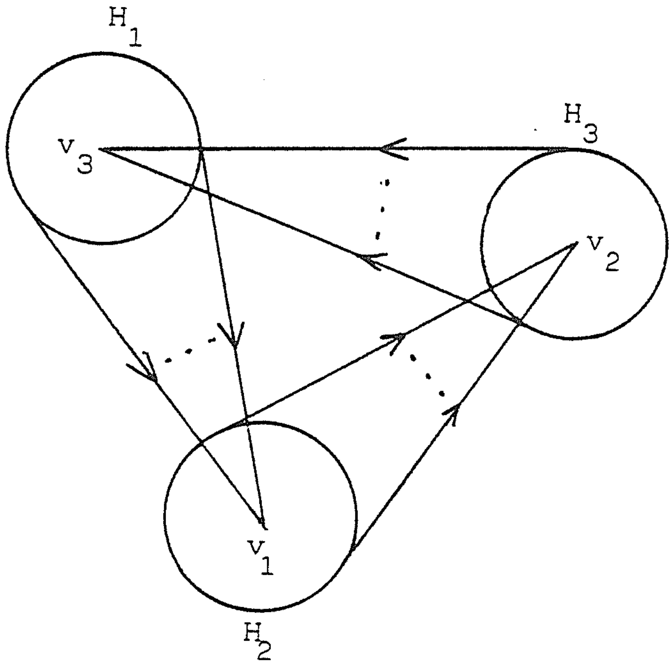

Figure 1.

fined as follows

$$V(F) = \bigcup_{i=1}^{3} V(H_i) \qquad U(F) = \bigcup_{i=1}^{3} U(H_i) \cup \bigcup_{i=1}^{3} \{(w, v_i) \colon \; w \in V(H_i)\},$$

has the constant outlink isomorphic to $\overrightarrow{G+v}$.

Now let H be a digraph with the outlink of every vertex isomorphic to G. We proceed as follows: for every vertex v, $v \in V(H)$, we add two new vertices u_1, u_2 and the arcs from v to u_1, from u_1 to u_2, from u_2 to v and from u_i to every vertex of the outlink of v in H, for $i = 1, 2$ (see Fig. 2).

The digraph $E = (V(E), U(E))$, such that

$$V(E) = V(H) \cup \{u_1^i, u_2^i; \; i = 1, 2, \; \ldots, \; |V(H)|\}$$

$$U(E) = U(H) \cup \bigcup_{i=1}^{|V(H)|} \{(v_i, u_1^i), (u_2^i, v_i), (u_1^i, u_2^i)\}$$

$$\cup \bigcup_{i=1}^{|V(H)|} \{(u_1^i, x), (u_2^i, x) \colon \; x \in V(L^o(v_i, H))\},$$

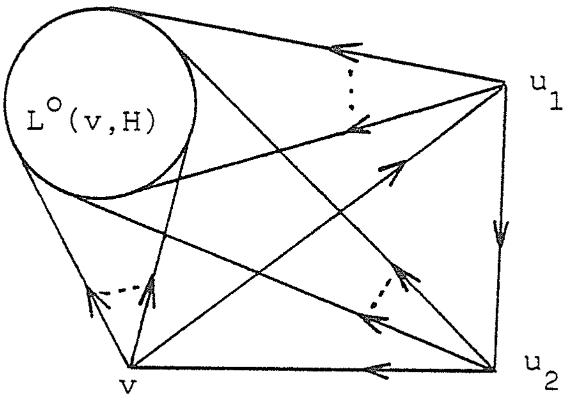

Figure 2.

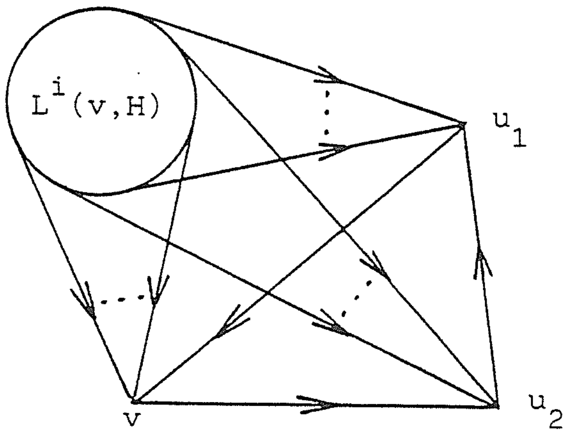

Figure 3.

where $L^o(\underleftarrow{v_i}, H)$ denotes the outlink of v_i in H, has the constant outlink isomorphic to $G + v$.

The second part of out thesis, concerning inlink digraphs, follows by the same method as above, using the construction presented in Fig. 3 (for $G \overrightarrow{+} v$) and Fig. 4 (for $G \overleftarrow{+} v$). This completes the proof.

From Lemmas 1, 2 we get

Theorem 1. *Let G_i be outlink (inlink) digraphs, $i = 1, 2, 3, \ldots, k$, $k \in N$. Then $\bigcup_{i=1}^{k} G_i \overrightarrow{+} v$, $\bigcup_{i=1}^{k} G_i \overleftarrow{+} v$ are outlink (inlink) digraphs.*

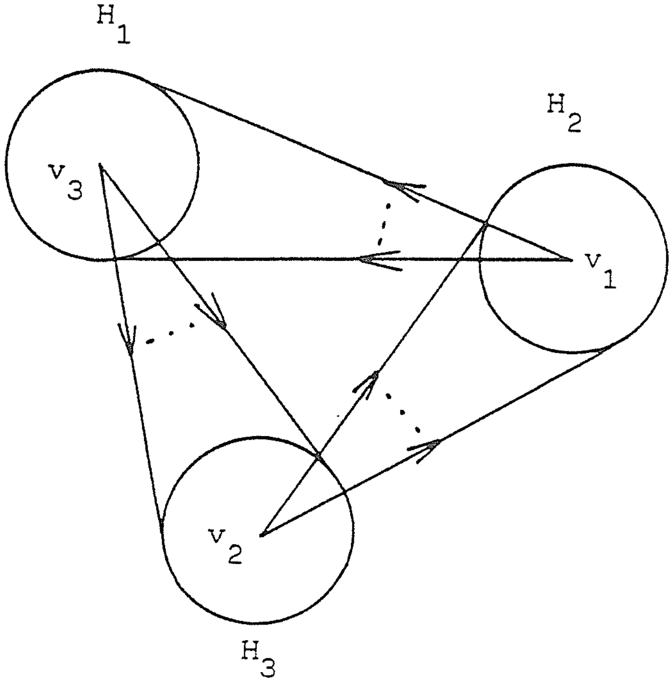

Figure 4.

Corollary 1. *Let $S_{1,m}$ be a starlike graph, $m \geq 1$. There exist two orientations of $S_{1,m}$, which are oulink (inlink) digraphs.*

Proof. The result follows from the above theorem and from the fact that for every n, the n-vertex transitive tournament is an outlink (inlink) digraph [**1**].

In [**1**] Harary suggested the problem, wheather instars and outstars are outlink (inlink) digraphs. Assuming that G_i is a single vertex digraph, for $i \geq 1$, the above theorem gives an affirmative answer to this question.

References

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Summary

In this note we deal with the outlink (inlink) digraphs. The outlink (inlink) of a vertex of a digraph is the subdigraph induced by its outneighbourhood (inneighbourhood). In a digraph with constant outlink (inlink) all outlinks (inlinks) are isomorphic. Then an outlink (inlink) digraph is the constant outlink (inlink) of some digraph. We prove that a starlike graph has at least two orientations, which are outlink (inlink) digraphs. As a corollary from the above result, we obtain the affirmative answer to Harary's question [**1**], *if instars and outstars are oulink (inlink) digraphs.*

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