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Natural stabilization for fluids of third grade and of dipolar type (**)

1 - Introduction

In this paper we investigate two models for convection in a class of generalized fluids whose viscosity varies with temperature, namely a fluid of third grade, and a fluid of dipolar type.

The theory of a dipolar fluid was introduced by Bleustein and Green [1] and is thought capable of describing a fluid containing long molecules or a suspension of long molecular particles. These writers took account of microstructure effects by including both the gradient of velocity and the second gradient of velocity as constitutive variables, and they also found it necessary to introduce an appropriate stress tensor. Bleustein and Green [1] also solved the problem of Poiseuille flow in a pipe for a dipolar fluid and showed that a flattened velocity profile could be expected. Since then other problems have been successfully solved; Hills [9], solved the problem of slow flow past a sphere for a dipolar fluid, he established a uniqueness theorem in [10], and demonstrated continuous dependence on the data in the improperly posed backward in time problem in [11]. Straughan [16] showed new effects could be predicted in wave motion and investigated nonlinear stability for the constant viscosity Bénard problem in [17] where the micro-length associated with the theory of [1] was shown to have a strong inhibiting effect on thermal convection.

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The appropriate equations for thermal convection in a dipolar fluid are introduced in section 2.

The fluid of third grade arose as a model for incompressible, homogeneous, viscoelastic flow, when the extra stress is expressed as a function of the Rivlin-Ericksen tensors. In particular, the fluid of third grade studied here arises from the study of thermodynamics, stability and instability given by Fosdick and Rajagopal [3]. The Bénard problem for a fluid of grade three with constant viscosity was first studied by Franchi and Straughan [5]. It is appropriate to mention at this point the work of Dunn and Rajagopal [2] who develop theories for fluids of fourth and higher grade; their work reveals several novel properties which may be expected from such models.

The relevant equations for thermal convection in a layer of fluid of third grade heated from below are described in section 4.

Although the convection studies for a fluid of third grade and a dipolar fluid, [5], [17], respectively, concentrated on constant viscosity, it was recognised long ago that of fluid properties, viscosity is the one most affected by changes in temperature. Hence, we here propose to include the effect of variable viscosity. Tippelskirch [20] has shown by an experiment with liquid sulphur, that if the viscosity decreases as the temperature increases then the motion in the convection cell is one where the fluid rises in the centre and descends at the cell walls, whereas when the viscosity-temperature dependence is reserved the opposite fluid circulatory motion is observed. To account for temperature dependence in a fluid Tippelskirch [20] suggests the following relation for the viscosity, $\nu(T)$, namely

$$\nu(T) = \frac{\nu_0}{1 + \alpha T + \beta T^2} \tag{1.1}$$

for ν_0, α, β constants. We choose the linear viscosity relation of Palm et al. [14] (which contains the leading terms in a binomial expansion of (1.1)), and take

$$\nu(T) = \nu_0 (1 - \gamma(T - T_0)) \tag{1.2}$$

for γ a positive constant, or it is sometimes convenient to use the dynamic viscosity $\mu(T) = \nu(T)\rho_0$, where ρ_0 is a constant density, for which

$$\mu(T) = \mu_0 (1 - \gamma(T - T_0)) \tag{1.3}$$

with $\mu_0 = \nu_0 \rho_0$.

2 - The convection equations in a dipolar fluid

The model of Bleustein and Green [1] consists of the momentum equation

$$\rho \dot{v}_i = \rho f_i + \sigma_{ji,j} \tag{2.1}$$

the continuity equation

$$v_{i,i} = 0 \tag{2.2}$$

and the rate of work equation

$$\rho r - \rho(\dot{A} + \dot{T}S + \dot{S}T) - q_{i,i} + \tau_{ji} d_{ij} + \Sigma_{(ij)k} A_{kji} = 0 \tag{2.3}$$

where v_i, ρ, f_i, σ_{ij}, r, A, T, S, q_i are, respectively, the velocity, density, macroscopic body force, stress tensor, heat supply, Helmholtz free energy, temperature, entropy, and heat flux. Standard indicial notation is used and a superposed dot denotes the material derivative. The tensor d_{ij} and A_{ijk} are

$$d_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}) \qquad A_{ijk} = v_{i,jk} \tag{2.4}$$

and the stress tensor has form

$$\sigma_{ij} = \tau_{ji} - \Sigma_{kji,k} - \rho F_{ji} + \rho \Gamma_{ji} \tag{2.5}$$

where τ_{ij} is a symmetric stress with form

$$\tau_{ij} = -\phi \delta_{ij} + 2\mu d_{ij}. \tag{2.6}$$

ϕ is introduced since v_i is solenoidal, and μ is the dynamic viscosity (which here has form (1.3)). F_{ij} is the microscopic body force and Γ_{ij} is the dipolar inertia whose form, see Green and Naghdi [8], is

$$\Gamma_{ji} = d^2 [(\dot{v}_i)_{,j} - v_{i,k} v_{k,j} - v_{i,k} v_{j,k} + v_{k,i} v_{k,j}] \tag{2.7}$$

where $d^2 (> 0)$ is the constant inertia coefficient.

In [17] it is argued that since only the symmetric part $\Sigma_{(kj)i}$ of the dipolar stress plays any part in the equations it is reasonable to introduce only this part for any situation which the dipolar fluid will model. We adopt this premise here and then the constitutive equations of Bleustein and Green [1] yield

$$\Sigma_{(kj)i} = -\psi_k \delta_{ij} - \psi_j \delta_{ik} + h_1 \delta_{jk} A_{imm} + h_2 (A_{kji} + A_{jki}) + h_3 A_{ijk} + \gamma_d \delta_{jk} T_{,i} \tag{2.8}$$

$$q_i = \kappa T_{,i} + \bar{\alpha} A_{ikk}. \tag{2.9}$$

The function ψ_i arises because v_i is solenoidal and h_i, γ_d, κ, $\bar{\alpha}$ are constants which satisfy inequalities (15.11) of [1]: the only two of which we require here are

$$\kappa \leqslant 0 \qquad h_1 + h_3 \geqslant 0 \tag{2.10}$$

and we suppose these hold in the strict sense.

We now follow the development of [17] except we allow the viscosity to vary as in (1.3). Hence, suppose the fluid is contained in the infinite layer $z \in (0, H)$ for $H > 0$. Take $F_{ij} \equiv 0$ and use a Bousinnesq approximation so that $\rho = \rho_0$ (constant) everywhere except in the body force term in (2.1), for which

$$\rho f_i = -\rho_0 g \delta_{i3} [1 - \alpha(T - T_0)] \tag{2.11}$$

where g is gravity, T_0 is a reference temperature, and α is the coefficient of thermal expansion. By setting $k = -\dfrac{\kappa}{\rho_0} c$, with $c = T \dfrac{\partial S}{\partial T}$ (constant), it is shown in [17] that the rate of work equation may be reduced to

$$\dot{T} = k \Delta T \tag{2.12}$$

with Δ the Laplacian operator.

Hence, if we now define $\hat{\Sigma}_{(kj)i}$ as

$$\hat{\Sigma}_{(kj)i} = h_1 \delta_{jk} A_{imm} + h_2 (A_{kji} + A_{jki}) + h_3 A_{ijk} \tag{2.13}$$

then with (1.3) holding we may now show the governing equations (2.1)-(2.3) reduce to

$$\rho_0 (1 - d^2 \Delta) \dot{v}_i + \rho_0 d^2 (v_{i,k} v_{k,j} + v_{i,k} v_{j,k} - v_{k,i} v_{k,j})_{,j} \tag{2.14}$$

$$= -p_{,i} - \rho_0 g \delta_{i3} [1 - \alpha(T - T_0)] + 2\mu_0 \{[1 - \gamma(T - T_0)] d_{ij}\}_{,j} - \hat{\Sigma}_{(kj)i,jk} - \gamma_d \Delta T_{,i}$$

$$v_{i,i} = 0 \tag{2.15}$$

$$\dot{T} = k \Delta T \tag{2.16}$$

where $p = \phi - \psi_{i,i}$ acts like a pressure.

We now assume that on the boundaries

$$v_i = 0 \qquad z = 0, H \tag{2.17}$$

and

$$n_j T_{ji} = 0 \qquad z = 0, H \tag{2.18}$$

cf. [1], [9], [16], [17], and further

$$T = T_0 \quad z = 0 \quad \text{and} \quad T = T_1 \quad z = H \tag{2.19}$$

with $T_0 > T_1$.

The conduction solution to (2.14)-(2.19), whose stability we investigate is

$$\bar{v}_i \equiv 0 \qquad \bar{T} = -\beta z + T_0 \tag{2.20}$$

where $\beta = (T_0 - T_1)H^{-1}$.

To study the *nonlinear* stability of solution (2.20) we let u_i, θ, π be perturbations to $\bar{v}_i$, $\bar{T}$ and $\bar{p}$, where $\bar{p}$ is the steady pressure field found from (2.14). The resulting perturbation equations are non-dimensionalized by choosing (stars being the dimensionless variables):

$$x_i = x_i^* H \qquad u_i = u_i^* U \qquad U = \nu_0 H^{-1} \qquad \theta = T^{\#} \theta^*$$

$$T^{\#} = U\sqrt{\frac{Pr\beta}{g\alpha}} \qquad \pi = \pi^* P \qquad P = \frac{U\rho_0 \nu_0}{H}$$

$$R = \sqrt{\frac{\alpha\beta H^4 g}{\nu_0 k}} \qquad \delta = \frac{d^2}{H^2} \qquad \varepsilon = \frac{l^2}{H^2} \qquad Pr = \frac{\nu_0}{k}$$

$$t^* = t\frac{\nu_0}{H^2} \qquad \Gamma_d = \sqrt{\frac{\beta\gamma_d^2 k}{g\alpha\nu_0^3}} \qquad \Gamma = \gamma\beta H$$

and then (omitting all stars) the *non-dimensional* perturbation equations become:

$$(1 - \delta\Delta)\dot{u}_i + \delta(u_{i,k}u_{k,j} + u_{j,k}u_{i,k} - u_{k,i}u_{k,j})_{,j} = -\pi_{,i} + \delta_{i3}R\theta + 2((1 + \Gamma z)d_{ij})_{,j} - \frac{1}{\mu_0 H^2}\hat{\Sigma}_{(kj)i,kj} - \Gamma_d\Delta\theta_{,i} - 2\frac{\Gamma Pr}{R}(\theta d_{ij})_{,j} \tag{2.21}$$

$$u_{i,j} = 0 \tag{2.22}$$

$$Pr\dot{\theta} = Rw + \Delta\theta \tag{2.23}$$

where the superposed dot denotes the material derivative with respect to t and $d_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$. Equations (2.21)-(2.23) now hold on the spatial region $z \in (0, 1)$ and the boundary conditions are

$$u_i = \theta = \Sigma_{(33)i} = 0 \qquad z = 0, 1 \tag{2.24}$$

with u_i, θ, π having a «periodic shape» in (x, y).

3 - Nonlinear energy stability for solution (2.20)

We are now able to proceed in a direct manner from (2.19)-(2.21), because of the extra nonlinearities present. In the classical fluid case and in that for a micropolar fluid, a much more complicated generalized energy approach was necessary, see [6], [19]. (Further development and uses of generalized energy methods may be found in e.g. Franchi [4], Galdi [7], Mulone [12], Padula [13], Rionero and Mulone [15], or the book by Straughan [18].)

Let us now multiply (2.21) by u_i, (2.23) by θ, and integrate over a cell of solution periodicity, V. The dissipation terms may be dealt with by integration by parts, for details see [17], and we may then show

$$\frac{d}{dt}\,\frac{1}{2}(\|\boldsymbol{u}\|^2 + \delta\|\nabla\boldsymbol{u}\|^2) = R\langle\theta w\rangle - \varepsilon\|\Delta\boldsymbol{u}\|^2 - \|\nabla\boldsymbol{u}\|^2 - 2\Gamma\langle z d_{ij} d_{ij}\rangle + 2\frac{\Gamma Pr}{R}\langle\theta d_{ij} d_{ij}\rangle \tag{3.1}$$

$$\frac{d}{dt}\,\frac{1}{2}Pr\|\theta\|^2 = R\langle w\theta\rangle - \|\nabla\theta\|^2 \tag{3.2}$$

where $\|\cdot\|$ and $\langle\cdot\rangle$ denote the norm on $L^2(V)$ and integration over V, respectively.

It is sufficient to add (3.1) and (3.2), and then defining E, I and D by

$$E = \frac{1}{2}(\|\boldsymbol{u}\|^2 + Pr\|\theta\|^2 + \delta\|\nabla\boldsymbol{u}\|^2) \tag{3.3}$$

$$I = 2\langle\theta w\rangle \tag{3.4}$$

$$D = \|\nabla\theta\|^2 + \|\nabla\boldsymbol{u}\|^2 + 2\Gamma\langle z d_{ij} d_{ij}\rangle + \varepsilon\|\Delta\boldsymbol{u}\|^2 \tag{3.5}$$

we obtain

$$\frac{dE}{dt} = RI - D + 2\frac{\Gamma Pr}{R}\langle\theta d_{ij} d_{ij}\rangle . \tag{3.6}$$

It is precisely the cubic term in $\langle \theta d_{ij} d_{ij} \rangle$ which has caused problems in previous work. However, we may now argue that defining

$$\frac{1}{R_E} = \max_{\mathcal{H}} \frac{I}{D} \tag{3.7}$$

where $\mathcal{H}$ is the space of admissible solutions, then

$$\frac{dE}{dt} \leqslant -\frac{R_E - R}{R_E} D + 2\frac{\Gamma Pr}{R} \langle \theta d_{ij} d_{ij} \rangle . \tag{3.8}$$

We shall suppose

$$R < R_E \tag{3.9}$$

where R_E defines our nonlinear energy stability boundary. To manipulate the cubic term we write

$$2\langle \theta d_{ij} d_{ij} \rangle = \langle \theta u_{i,j} u_{i,j} \rangle + \langle \theta u_{i,j} u_{j,i} \rangle$$

and then integrate by parts to find

$$2\langle \theta d_{ij} d_{ij} \rangle = -\langle \theta_{,j} u_{i,j} u_i \rangle - \langle \theta u_i \Delta u_i \rangle - \langle \theta_{,i} u_{i,j} u_j \rangle . \tag{3.10}$$

It is now necessary to employ the inequality

$$\sup_V |\boldsymbol{u}| \leqslant c \|\Delta \boldsymbol{u}\| \tag{3.11}$$

in (3.10) and we may then obtain

$$2\langle \theta d_{ij} d_{ij} \rangle \leqslant 2c \|\nabla\theta\| \|\nabla \boldsymbol{u}\| \|\Delta \boldsymbol{u}\| + c \|\theta\| \|\Delta \boldsymbol{u}\|^2 \leqslant c \Big(\frac{2^{\frac{3}{2}}}{\sqrt{\delta\varepsilon}} + \frac{2^{\frac{1}{2}}}{\varepsilon\sqrt{Pr}} \Big) E^{\frac{1}{2}} D \tag{3.12}$$

where (3.3), (3.5) have been used. Thus, putting (3.12) in (3.8) we obtain

$$\frac{dE}{dt} \leqslant -\frac{R_E - R}{R_E} D + \frac{2^{\frac{1}{2}} \Gamma Pr c}{R} \Big(\frac{2}{\sqrt{\delta\varepsilon}} + \frac{1}{\varepsilon\sqrt{Pr}} \Big) E^{\frac{1}{2}} D . \tag{3.13}$$

It is now easy to show that if

$$\text{(a)} \quad R < R_E \quad \text{and} \quad \text{(b)} \quad E^{\frac{1}{2}}(0) < \frac{R\varepsilon\sqrt{\delta}(R_E - R)}{2^{\frac{1}{2}} \Gamma \sqrt{Pr}\, c R_E} \, \frac{1}{\sqrt{\delta} + 2\sqrt{\varepsilon Pr}}$$

then $E(t) \to 0$ as $t \to \infty$, cf. [18] chapter 2, and hence nonlinear stability is assured.

We wish to stress once more, that it is the natural property of the dipolar fluid equations which have allowed us to proceed to obtain nonlinear stability in such a direct manner. Thus, this mathematical analysis indicates that the dipolar fluid model has good physical properties for variable viscosity convection.

4 - The convection equations in a fluid of third grade

The paper of Fosdick and Rajagopal [3] uses the Clausius-Duhem inequality and by additionally requiring the free energy to be a minimum in equilibrium, they have shown that the stress relation for an incompressible, homogeneous fluid of third grade is

$$T = -p\boldsymbol{I} + \mu\boldsymbol{A} + \alpha_1\boldsymbol{A}_2 + \alpha_2\boldsymbol{A}^2 + \beta(\mathrm{tr}\boldsymbol{A}^2)\boldsymbol{A} \tag{4.1}$$

where $\boldsymbol{A}$ and $\boldsymbol{A}_2$ are the first two Rivlin-Ericksen tensors, defined by

$$\boldsymbol{A} = \boldsymbol{L} + \boldsymbol{L}^T \qquad \boldsymbol{A}_2 = \dot{\boldsymbol{A}} + \boldsymbol{A}\boldsymbol{L} + \boldsymbol{L}^T\boldsymbol{A} \tag{4.2}$$

$\boldsymbol{L}$ being the velocity gradient. We shall assume the normal stress coefficients α_1, α_2 and the coefficient β are constant, but we let the viscosity be a linear function of temperature of form (1.3). We observe from [3] that the coefficients satisfy the restrictions

$$\alpha_1 \geqslant 0 \qquad \beta \geqslant 0 \qquad |\alpha_1 + \alpha_2| \leqslant \sqrt{24\mu\beta}. \tag{4.3}$$

We shall assume $\alpha_1 > 0$, $\beta > 0$ and $(4.3)_3$ holds with μ replaced by the constant μ_0.

The relevant equations of [3] are then the equations of momentum, continuity, and balance of energy, which are:

$$\rho\dot{v}_i = \rho f_i + T_{ji,j} \tag{4.4}$$

$$v_{i,i} = 0 \tag{4.5}$$

$$\rho\dot{\varepsilon} = T_{ij}L_{ij} - q_{i,i} + \rho r \tag{4.6}$$

where f_i, ε, q_i and r are, respectively, body force, internal energy, heat flux, and heat supply.

It is sufficient here to set $r = 0$. Then, we may follow Franchi and Straughan

[5] and reduce the balance of energy equation, (4.6), to the form

$$\dot{T} = \kappa \Delta T \tag{4.7}$$

where κ is the thermal diffusivity. We again employ a Boussinesq approximation as in [5], and so take ρ constant everywhere except in the body force term for which

$$\rho f_i = -\rho_0 g \delta_{i3} [1 - \alpha(T - T_0)]. \tag{4.8}$$

Hence, the equations of motion are (4.4), (4.5) and (4.7), with (1.3), (4.1) and (4.8) being understood.

To investigate the problem of instability due to heating from below we suppose the fluid is contained in the layer $z \in (0, H)$. The stationary solution to (4.4), (4.5), (4.7) subject to specified constant temperatures on the boundaries,

$$T = T_0 \qquad z = 0 \qquad \text{and} \qquad T = T_1 \qquad z = H \tag{4.9}$$

$T_0 > T_1$, which also satisfies the no-slip condition

$$v_i = 0 \qquad z = 0, H \tag{4.10}$$

is

$$\overline{v}_i \equiv 0 \qquad \overline{T} = -\zeta z + T_0 \tag{4.11}$$

where $\zeta = (T_0 - T_1) H^{-1}$ and $\overline{p}$ is found from (4.4).

To investigate the nonlinear stability of (4.11) we put $v_i = \overline{v}_i + u_i$, $T = \overline{T} + \theta$, $p = \overline{p} + \pi$ and then from (4.4), (4.5) and (4.7) derive the equations for the perturbations (u_i, θ, π). The equations are non-dimensionalized according to:

$$x_i = x_i^* H \qquad u_i = u_i^* U \qquad U = \nu_0 H^{-1} \qquad \theta = \theta^* T^{\#}$$

$$T^{\#} = \sqrt{\frac{Pr \zeta}{g\alpha}}\, U \qquad \pi = \pi^* P \qquad P = \frac{U \rho_0 \nu_0}{H} \qquad R = \sqrt{\frac{\alpha \zeta H^4 g}{\kappa \nu_0}}$$

$$\Gamma_1 = \frac{\rho_0 H^2}{\alpha_1} \qquad \Gamma_2 = \frac{\rho_0 H^2}{\alpha_2} \qquad Pr = \frac{\nu_0}{\kappa}$$

$$t^* = t \frac{\nu_0}{H^2} \qquad B = \frac{\beta \nu_0}{\rho_0 H^4} \qquad \Gamma = \gamma \zeta H$$

where $Ra = R^2$, Pr are the Rayleigh and Prandtl numbers, Γ_1, Γ_2 are absorption numbers, Γ measures the viscosity variation, and B is a non-dimensional form of β. The non-dimensional equations for (u_i, θ, π) are then (omitting all stars):

$$\dot{u}_i = -\pi_{,i} + \delta_{i3} R\theta + 2((1+\Gamma z)\, d_{ij})_{,j}$$

$$+ \frac{1}{\Gamma_1}\Big(\frac{\partial}{\partial t} A_{ij} + u_k A_{ij,k} + A_{im} L_{mj} + L_{mi} A_{mj}\Big)_{,j} \tag{4.12}$$

$$+ \frac{1}{\Gamma_2}(A_{im} A_{mj})_{,j} + B[(\mathrm{tr}\,\boldsymbol{A}^2) A_{ij}]_{,j} - 2\frac{\Gamma Pr}{R}(\theta d_{ij})_{,j}$$

$$u_{i,i} = 0 \tag{4.13}$$

$$Pr\dot{\theta} = Rw + \Delta\theta \tag{4.14}$$

where now $L_{ij} = u_{i,j}$, $A_{ij} = u_{i,j} + u_{j,i}$ and $d_{ij} = \frac{1}{2} A_{ij}$.

We note that u_i, θ vanish on the boundaries $z = 0, 1$ and assume u_i, θ are periodic in x, y in the sense that the x, y planform has a repetitive plane tiling shape.

5 - Nonlinear energy analysis for the fluid of third grade

Define our energy in this case by

$$E = \frac{1}{2}(\|\boldsymbol{u}\|^2 + Pr\|\theta\|^2) + \frac{1}{4\Gamma_1}\|\boldsymbol{A}\|^2 = \frac{1}{2}(\|\boldsymbol{u}\|^2 + Pr\|\theta\|^2) + \frac{1}{2\Gamma_1}\|\nabla \boldsymbol{u}\|^2. \tag{5.1}$$

We differentiate E and substitute for the resulting time derivatives using (4.12), (4.14), and we may then show

$$\frac{dE}{dt} = 2R\langle\theta w\rangle - \|\nabla\theta\|^2 - \frac{1}{2}\|\boldsymbol{A}\|^2 - \frac{1}{2}\Big(\frac{1}{\Gamma_1} + \frac{1}{\Gamma_2}\Big)\langle \mathrm{tr}\,\boldsymbol{A}^3\rangle$$

$$- \frac{1}{2} B\langle|\boldsymbol{A}|^4\rangle - 2\Gamma\langle z d_{ij} d_{ij}\rangle + 2\frac{\Gamma Pr}{R}\langle\theta d_{ij} d_{ij}\rangle. \tag{5.2}$$

Once again, the cubic nonlinearity in (5.2) may be handled thanks to the B term. We distinguish two cases

(i) $\alpha_1 + \alpha_2 = 0$ and (ii) $0 < |\alpha_1 + \alpha_2| < \sqrt{24\beta\mu_0}$.

The third case $|\alpha_1 + \alpha_2| = \sqrt{24\beta\mu_0}$ is dismissed since it was seen in [5] not to lead to useful results.

Case (i). This case is of interest since $\alpha_1 + \alpha_2 = 0$ does hold for a fluid of second grade. Equation (5.2) now reduces to

$$\frac{dE}{dt} = 2R\langle\theta w\rangle - \|\nabla\theta\|^2 - \frac{1}{2}\|A\|^2 - \frac{1}{2}B\langle|A|^4\rangle - 2\Gamma\langle z d_{ij} d_{ij}\rangle + 2\frac{\Gamma Pr}{R}\langle\theta d_{ij} d_{ij}\rangle. \tag{5.3}$$

Define now

$$I = 2\langle\theta w\rangle \qquad D_1 = \frac{1}{2}B\langle|A|^4\rangle \qquad D = \|\nabla\theta\|^2 + \frac{1}{2}\|A\|^2 + 2\Gamma\langle z d_{ij} d_{ij}\rangle$$

and we observe using the Cauchy-Schwarz inequality that

$$2\frac{\Gamma Pr}{R}\langle\theta d_{ij} d_{ij}\rangle = \frac{\Gamma Pr}{2R}\langle\theta A_{ij} A_{ij}\rangle \leqslant \frac{\Gamma Pr}{2R}\|\theta\|\langle|A|^4\rangle^{\frac{1}{2}}. \tag{5.4}$$

Defining

$$\frac{1}{R_E} = \max_{\mathcal{H}} \frac{I}{D} \tag{5.5}$$

we assume $R < R_E$ and then using (5.3), (5.4) we find

$$\frac{dE}{dt} \leqslant -\left(\frac{R_E - R}{R_E}\right)D - D_1 + \frac{\Gamma}{R}\sqrt{\frac{Pr}{B}}E^{\frac{1}{2}}D_1 \leqslant -\mathcal{D} + \frac{\Gamma}{R}\sqrt{\frac{Pr}{B}}E^{\frac{1}{2}}\mathcal{D}, \tag{5.6}$$

where we have defined $\mathcal{D}$ by

$$\mathcal{D} = \left(\frac{R_E - R}{R_E}\right)D + D_1. \tag{5.7}$$

From (5.6) it is now easy to show $E \to 0$ as $t \to \infty$, see e.g. chapter 2 of [18], provided

$$R < R_E \qquad \text{and} \qquad E^{\frac{1}{2}}(0) < \sqrt{\frac{B}{Pr}}\,\frac{R(R_E - R)}{\Gamma R_E}.$$

Thus, we have derived sufficient conditions to ensure nonlinear stability.

Again, as in section 3, we stress that we are able to obtain nonlinear stability directly using a straightforward energy analysis only because of the extra stabilization provided by the B term in the equations for the fluid of third grade.

Case (ii). We use lemma 3 of [3] together with the arithmetic-geometric mean inequality to prove

$$\|A\|^2 + \left(\frac{1}{\Gamma_1} + \frac{1}{\Gamma_2}\right)\langle \mathrm{tr}\, A^3\rangle + B\langle |A|^4\rangle \tag{5.8}$$

$$\geqslant \left(1 - \frac{|\alpha_1 + \alpha_2|}{2\omega\rho_0 H^2\sqrt{6}}\right)\|A\|^2 + \left(\frac{\beta\mu_0}{\rho_0^2 H^4} - \frac{\omega|\alpha_1 + \alpha_2|}{2\rho_0 H^2\sqrt{6}}\right)\langle |A|^4\rangle$$

for $\omega(>0)$ at our disposal. Select now

$$\omega = \frac{\sqrt{6}\left(\dfrac{\beta\mu_0}{\rho_0 H^2} - \rho_0 H^2\right) + \sqrt{6\left(\dfrac{\beta\mu_0}{\rho_0 H^2} - \rho_0 H^2\right)^2 + (\alpha_1 + \alpha_2)^2}}{|\alpha_1 + \alpha_2|}$$

and define ε by

$$\varepsilon = \frac{1}{2} - \frac{|\alpha_1 + \alpha_2|}{4\omega\rho_0 H^2\sqrt{6}}\,. \tag{5.9}$$

Next, we employ (5.8), (5.9), together with (5.4) in (5.2), to find

$$\frac{dE}{dt} \leqslant RI - D - D_1 + \frac{\Gamma}{R}\sqrt{\frac{Pr}{B}}\,E^{\frac{1}{2}} D_1 \tag{5.10}$$

where now

$$I = 2\langle \theta w\rangle \qquad D = \|\nabla\theta\|^2 + \varepsilon\|A\|^2 + 2\Gamma\langle z d_{ij} d_{ij}\rangle \qquad D_1 = \frac{1}{2}B\langle |A|^4\rangle.$$

We again define

$$\frac{1}{R_E} = \max_{\mathscr{H}} \frac{I}{D} \qquad \mathcal{D} = \left(\frac{R_E - R}{R_E}\right)D + D_1$$

and then from (5.10) we may obtain

$$\frac{dE}{dt} \leqslant -\mathcal{D}\left(1 - \frac{\Gamma}{R}\sqrt{\frac{Pr}{B}}\,E^{\frac{1}{2}}\right). \tag{5.11}$$

Once again, nonlinear stability is, therefore, obtained provided

$$R < R_E \quad \text{and} \quad E^{\frac{1}{2}}(0) < \sqrt{\frac{B}{Pr}}\,\frac{R}{\Gamma}\,.$$

We point out that we do not concentrate on calculating R_E in this paper. However, for both cases (i) and (ii) R_E exists and is finite and non-zero, and may be found numerically using e.g. the compound matrix method, described e.g. in [18], appendix 2. The emphasis here is on showing that one may readily obtain fully nonlinear stability criteria in a direct manner, without having to resort to any sort of intricate Lyapunov function.

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Summary

The problem of convection in a dipolar fluid or a fluid of third grade is studied, when the viscosity is a linear function of temperature. In contrast with other temperature dependent viscosity theories where a generalized energy approach is necessary, see Franchi and Straughan [6], *Straughan* [19], *it is shown that fluids of third grade and of dipolar type possess just the right kinds of dissipative terms to control the extra nonlinearities which arise when the viscosity varies with temperature.*

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