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Submanifolds with conformal second fundamental form and isotropic immersions (**)

1 - Introduction

Let $f\colon M \to N$ be a *Riemannian immersion* of a m-dimensional manifold M in a n-dimensional manifold N. We will denote by h the second fundamental form of the immersion.

The immersion f is said *isotropic* if there exists a differentiable function λ on M such that

$$\|h(X, X)\|^2 = \lambda^2 \|X\|^4 \qquad \forall X \in T_x M . \tag{1.1}$$

λ^2 is called the isotropy function.

The notion of *isotropic immersion* was introduced in 1965 by B. O'Neill [**9**] and his paper contains the first significant results in the case of immersions in spaces of constant curvature and for Kähler immersions.

The study of this problem was continued by many authors. Itoh and Ogiue (see for example [**6**]) studied isotropic immersions of submanifolds with constant sectional curvature (or with constant holomorphic sectional curvature). Sakamoto [**10**] considered the isotropic immersions with parallel second fundamental form of a compact simply connected manifold M in a space of constant sectional curvature (planar geodesic immersions). He proved that, if such immersions are not totally geodesic, they coincide with the standard minimal immersions of compact symmetric spaces of rank one into spheres. Such minimal immersions were introduced by Tai [**12**].

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The notion of *conformal second fundamental form* is more recent. It was introduced explicitly for the first time in 1989 by G. Jensen and M. Rigoli [7] as one of the conditions that must be fulfilled in order that the spherical Gauss map $\nu: T_1 M^{\perp} \to T_1 N$ of the unit normal bundle of M in the unit tangent bundle of N is harmonic. This condition may be expressed as follows: if $\{e_i\}$ $(i = 1, \ldots, m)$ is a local orthonormal frame of M, h is *conformal* if there exists a differentiable function ρ^2 on M, called the conformality function, such that

$$(1.2) \qquad \sum_{i,j=1}^{m} (h(e_i, e_j), v)(h(e_i, e_j), w) = \rho^2 (v, w) \qquad \forall v, w \in TM^{\perp}.$$

If the immersion is not totally geodesic, then h conformal implies that the first normal bundle $N^1 = \text{span}\{h(X, Y), X, Y \in TM\}$ of M coincides with the normal bundle $TM^{\perp}$ and that $n - m \leqslant \frac{m(m+1)}{2}$ if M is not minimal in N and $n - m \leqslant \frac{m(m+1)}{2} - 1$ if M is minimal in N.

On the other hand, the second fundamental form of any codimension one submanifold is conformal.

In a previous paper ([2]), the surfaces with conformal second fundamental form of a space form with parallel mean curvature vector field, or, in the case of minimal immersions, with the length $\|h\|$ of h constant, were characterized. It was remarked that, if a surface is minimally immersed with conformal second fundamental form in a 4-dimensional manifold, then it is isotropic.

The aim of the present paper is to point out some relationships between the isotropic immersions and the submanifolds with conformal second fundamental form (Section 2) and to give some examples of m-dimensional submanifolds $(m \geqslant 3)$ immersed in a space of constant curvature with conformal second fundamental form (Section 3).

Some of these examples can be found, in a different context, in the paper [1] of S. Chern, M. do Carmo and S. Kobayashi. For the sake of completeness, we will summarize in Example 4 (Section 3) the main results of S. Sakamoto, which give further important examples of submanifolds with the above mentioned properties, and parallel second fundamental form.

2 - Some algebraic remarks about the second fundamental form

We begin with some results of O'Neill [9] about the notion of isotropic immersion, which will be used hereafter.

The *discriminant* Δ of h is the real valued function on the Grassmannian of the two planes of T_xM defined by

$$\Delta(\pi) = (h(X, X), h(Y, Y)) - \|h(X, Y)\|^2 \tag{2.1}$$

where (X, Y) is an orthonormal basis of the plane $\pi \subset T_xM$. One says that Δ is constant if it does not depend on the plane π.

If K^M and K^N denote the sectional curvatures of M and N, respectively, then the Gauss equation relative to the immersion $f\colon M \to N$ implies

$$K^M(\pi) = K^N(\pi) + \Delta(\pi). \tag{2.2}$$

Hence, if N is of constant sectional curvature, Δ is constant if and only if M is of constant sectional curvature.

The immersion f is *isotropic* at x if and only if

$$(h(X, X), h(X, Y)) = 0 \qquad \forall X, Y \in T_xM \qquad X \perp Y. \tag{2.3}$$

If $\{e_i\}$ $(i = 1, \dots, m)$ is an orthonormal basis of T_xM, the immersion is isotropic with isotropy function λ^2 if and only if

$$(*)\qquad \begin{aligned} &\|h(e_i, e_i)\|^2 = \lambda^2 \\ &(h(e_i, e_i), h(e_i, e_j)) = 0 \\ &2\|h(e_i, e_j)\|^2 = \|h(e_i, e_i)\|^2 - (h(e_i, e_i), h(e_j, e_j)) \\ &(h(e_i, e_i), h(e_j, e_k)) + 2(h(e_i, e_j), h(e_i, e_k)) = 0 \\ &(h(e_i, e_j), h(e_h, e_k)) + (h(e_i, e_k), h(e_j, e_h)) + (h(e_i, e_h), h(e_j, e_k)) = 0 \end{aligned}$$

where i, j, h, k are all different.

Theorem 1 [9]. *Let f be an isotropic immersion with isotropy function λ^2 and constant discriminant Δ. Let*

$$p = \frac{m(m+1)}{2} \qquad q = \frac{m+2}{2(m-1)}.$$

Then

$$-q\lambda^2 \leq \Delta \leq \lambda^2. \tag{2.4}$$

Furthermore, if N_x^1 is the first normal space at x then

$$\begin{aligned} &\Delta = \lambda^2 \Leftrightarrow h \textit{ is umbilical} \Leftrightarrow \dim(N_x^1) = 1 \\ &\Delta = -q\lambda^2 \Leftrightarrow h \textit{ is minimal} \Leftrightarrow \dim(N_x^1) = p - 1 \\ &-q\lambda^2 < \Delta < \lambda^2 \Leftrightarrow \dim(N_x^1) = p. \end{aligned}$$

Here we prove the following

Theorem 2. *If $f\colon M \to N$ is an isometric immersion,* $\dim(M) > 2$ *and the first normal bundle N^1 has rank 2, then f is not isotropic.*

Proof. Set $h(e_1, e_1) = \lambda n_1$, where n_1 is a unit vector of N^1_x and $\{e_i\}$ $(i = 1, \ldots, m)$ is an orthonormal basis of $T_x M$. Two cases are possible

(1) $h(e_1, e_j) = 0 \quad \forall e_j \perp e_1$

(2) there exists at least a vector $e_j \perp e_1$ such that $h(e_1, e_j) \neq 0$.

If (1) holds, taking into account the third equation of $(*)$ one obtains $h(e_j, e_j) = \lambda n_1$ $\forall j = 1, \ldots, m$. On the other hand, as $\dim(N^1_x) = 2$, there exists a pair of vectors, say e_2 and e_3, such that $h(e_2, e_3) = \mu n_2$, $n_2 \perp n_1$. By the third equation of $(*)$ it follows that $\mu = 0$, which is a contradiction.

If (2) holds then, chosen e_2 such that $h(e_1, e_2) \neq 0$, the second of $(*)$ implies $h(e_1, e_2) = \mu n_2$ for some $n_2 \perp n_1$. As $(h(e_2, e_2), h(e_1 e_2)) = 0$, by the third equation of $(*)$, applied to the vectors e_1 and e_2, we obtain

$$h(e_2, e_2) = -\lambda n_1 = -h(e_1, e_1) \qquad \lambda^2 = \mu^2. \tag{2.5}$$

Since e_3 is normal to e_1 and e_2, we finally get

$$h(e_1, e_3) = a n_2 \qquad h(e_2, e_3) = b n_2.$$

Using the fourth $(*)$ we deduce $a = b = 0$. By the third of $(*)$, with $i = 1$, $j = 3$ and with $i = 2, j = 3$, it follows that $h(e_1, e_1) = h(e_2, e_2) = h(e_3, e_3)$, which is in contrast with (2.5).

Remark. The above result can not be improved. As a matter of fact, the Veronese immersion of $\boldsymbol{RP}^2$ in S^4 is isotropic (compare [2]). The immersion of $\boldsymbol{CP}^2$ in S^7 (rank $N^1 = 3$), which can be found in the paper [10] of Sakamoto, is isotropic.

Now, we will point out some relationships between the isotropic immersions and the immersions with conformal second fundamental form.

We will assume that *the rank of the first normal bundle coincides with the codimension of M and that it is greater that one*, as any codimension one immersion has conformal fundamental second form.

Let $\{e_\alpha\}$ be an orthonormal frame of $TM^\perp$. Consider the $(n-m)$ matrices of order m, $H_\alpha = (h^\alpha_{ij}) = (h(e_i, e_j), e_\alpha)$. Then the second fundamental form is conformal if and only if $(H_\alpha, H_\beta) = \rho^2 \delta_{\alpha\beta}$, where (H_α, H_β) is the ordinary inner product of matrices. Here the real valued function ρ^2 defined on M is the conformality function.

Theorem 3. *Let f: $M^m \to N^{m+p}$ be an isometric immersion. If the second fundamental form h is conformal and the codimension p of M is maximal, $p = \frac{1}{2}m(m+1)$, then f is isotropic. The isotropy function λ^2 coincides with the conformality function ρ^2 and the discriminant of h $\Delta = -\frac{1}{2}\lambda^2$ is constant.*

Proof. Let A be the square matrix of order p

$$A = (h^\alpha_{11}, h^\alpha_{22}, \ldots, h^\alpha_{mm}, \sqrt{2}\,h^\alpha_{12}, \ldots, \sqrt{2}\,h^\alpha_{(m-1)m}) = (h^\alpha_{ii}, \sqrt{2}\,h^\alpha_{ij})_{i<j}$$

where $\{e_i, e_\alpha\}$ is a Darboux frame relative to the immersion. As h is conformal

$$A\,{}^tA = {}^tAA = \rho^2 \mathrm{I}_p$$

which implies

$$\|h(e_i, e_i)\|^2 = \rho^2 \qquad (h(e_i, e_i), h(e_j, e_j)) = 0$$

$$2\|h(e_i, e_j)\|^2 = \rho^2 \qquad (h(e_i, e_j), h(e_i, e_k)) = 0$$

$$(h(e_i, e_j), h(e_h, e_k)) = 0$$

where i, j, h, k are all different.

Then $(*)$ hold with $\rho^2 = \lambda^2$ and it follows immediately that

$$\Delta = (h(X, X), h(Y, Y)) - \|h(X, Y)\|^2 = -\frac{1}{2}\lambda^2$$

for any pair of normal vectors X, Y.

The previous result has a partial converse

Theorem 4. *Let f: $M^m \to N^{m+p}$ be an isometric immersion. If f is isotropic with isotropy function λ^2, the codimension p is maximal, the discriminant Δ is constant and equal to $-\frac{1}{2}\lambda^2$, then the second fundamental form is conformal and the isotropy function coincides with the conformality function.*

Proof. The assumption that the codimension is maximal implies that the vectors $h(e_i, e_j)$ $(i < j)$ are not zero and orthogonal. Furthermore they are orthogonal to the subspace spanned by the vectors $h(e_i, e_i)$, which must be linear independent, because of the hypothesis on the codimension. As

$$\Delta = (h(e_i, e_i), h(e_j, e_j)) - \|h(e_i, e_j)\|^2 = -\frac{1}{2}\lambda^2$$

it follows that

$$2(h(e_i, e_i), h(e_j, e_j)) - 2\|h(e_i, e_j)\|^2 = -\|h(e_i, e_i)\|^2$$

and, by the third of ($*$)

$$(h(e_i, e_i), h(e_j, e_j)) + 2\|h(e_i, e_j)\|^2 = \|h(e_i, e_i)\|^2$$

hence $(h(e_i, e_i), h(e_j, e_j)) = 0$. It follows, therefore, by an easy argument on the matrix A (introduced in the proof of Theorem 3), that h is conformal, with conformality function λ^2.

Remark. The real projective plane $\boldsymbol{RP}^2$ immersed in $\boldsymbol{R}^5$ (with its usual metric) by the Veronese immersion followed by the natural immersion of S^4 in $\boldsymbol{R}^5$ (compare [2]) is an example of isotropic immersion with maximal codimension and not conformal second fundamental form. This immersion has constant discriminant, but it is not equal to $-\frac{1}{2}\lambda^2$. Hence the assumption that $\Delta = -\frac{1}{2}\lambda^2$ can not be omitted.

For a minimal immersion the following result holds.

Theorem 5. *Suppose $f: M^m \to N^{m+p-1}$ is a minimal immersion with conformal second fundamental form and conformality function ρ^2. Assume that the codimension $p-1$ is maximal. In other words $p-1 = \frac{1}{2}m(m+1)-1$.*

Then the immersion f is isotropic, the isotropy function is given by $\lambda^2 = \dfrac{m-1}{m}\rho^2$ and the discriminant $\Delta = -\dfrac{m+2}{2m}\rho^2$ is constant.

Proof. It suffices to prove that the immersion is isotropic and to compute λ^2 because the expression of Δ is a consequence of Theorem 1.

Let A be the $(p-1) \times p$ matrix

$$A = (h_{ii}^{\alpha}, \sqrt{2}\, h_{ij}^{\alpha}) \qquad i, j = 1, \ldots, m, \quad i < j.$$

The rank of A is $p-1$ and its m-th column is the opposite of the sum of the previous $m-1$ columns. As h is conformal, then $A\,{}^tA = \rho^2 \mathrm{I}_{p-1}$.

Let v be a vector in $\boldsymbol{R}^p$ orthogonal to every row of A and with the same norm. As the immersion is minimal, v is of the kind $(a, \ldots, a, 0, \ldots, 0)$, with m components equal to a and $ma^2 = \rho^2$. Hence the matrix $B = \binom{A}{v}$ satisfies the condition

$$B\,{}^tB = \binom{A}{v}({}^tA\ {}^tv) = \rho^2 \mathrm{I}_p .$$

Therefore $({}^tA\ {}^tv)\binom{A}{v} = \rho^2 \mathrm{I}_p$ which imply

$$\|h(e_i, e_i)\|^2 = \rho^2 - a^2 = \frac{m-1}{m}\rho^2 \qquad (h(e_i, e_i), h(e_j, e_j)) + a^2 = 0$$

$$(h(e_i, e_i), h(e_i, e_j)) = 0 \qquad (h(e_i, e_i), h(e_j, e_k)) = 0$$

$$(h(e_i, e_j), h(e_i, e_k)) = 0 \qquad 2\|h(e_i, e_j)\|^2 = \rho^2 \qquad (h(e_i, e_j), h(e_h, e_k)) = 0$$

where i, j, h, k are all different. It follows that $(*)$ hold and that $\lambda^2 = \frac{m-1}{m}\rho^2$.

Also this result has a partial converse

Theorem 6. *Let f: $M^m \to N^{m+p-1}$ be a minimal isotropic immersion, with isotropy function λ^2 and constant discriminant. Suppose that the codimension p of the immersed manifold is maximal (i.e. $p - 1 = \frac{1}{2}m(m+1) - 1$). Then*

$$(2.6) \qquad \Delta = -\frac{m+2}{2(m-1)}\lambda^2 \qquad \operatorname{rank} N^1 = p - 1 = \frac{m(m+1)}{2} - 1$$

and f has conformal second fondamental form.

Proof. The results expressed in (2.6) follow from Theorem 1 of O'Neil [9].

In the same paper it is proved that the vectors $h(e_i, e_j)$ $(i < j)$ are mutually orthogonal, that their norms are

$$\|h(e_i, e_j)\|^2 = \frac{m}{2(m-1)}\lambda^2$$

determined by the intersection of the first m rows and the first m columns of H^g_{n+1}). We remark that, by the total umbilicity of g, $f^* H^g_{n+1} = \|H^g\| \mathrm{I}_m$, where I_m is the identity matrix.

The result then follows easily examining these matrices, using the hypothesis of minimality of f (i.e. the fact that the trace of the matrices H^f_a is zero) and the relation $\|h^f\|^2 = qm\|H^g\|^2$.

Remark. The Veronese surface immersed in S^5 ([2], Theorem 3.6) is an example in which the hypotheses of this theorem hold (compare also Example 2 in number 3).

3 - Examples of submanifolds with conformal second fundamental form

The following first three examples are slight modifications of submanifolds considered in [1] in a different context.

Example 1. $S^m(R) \times S^q(R')$ in $\boldsymbol{R}^{m+q+2}$, with $\dfrac{m}{R^2} = \dfrac{q}{R'^2}$.

This is a codimension two product manifold for which the conditions of Theorem 7 hold. According to the remarks following Theorem 7 it is not an isotropic immersion.

Consider the natural immersion

$$f\colon S^m(R) \times S^q(R') \to \boldsymbol{R}^{m+q+2} \tag{3.1}$$

given by $(x, y) \mapsto (f_1(x), f_2(y))$, where f_1 and f_2 are the natural immersions of $S^m(R)$, $S^q(R')$ in $\boldsymbol{R}^{m+1}$, $\boldsymbol{R}^{q+1}$ with conformal second fundamental form, whose conformality functions are $\dfrac{m}{R^2}$ and $\dfrac{q}{R'^2}$ respectively. Hence Theorem 7 implies that f has conformal second fundamental form if and only if

$$\frac{m}{R^2} = \frac{q}{R'^2}. \tag{3.2}$$

The mean curvature vector of f in $\boldsymbol{R}^{m+q+2}$ is

$$H = \frac{1}{m+q}\left(-\frac{m}{R}n_1, \; -\frac{q}{R'}n_2\right)$$

where n_1, n_2 are the normal verctors of $S^m(R)$, $S^q(R)$ in $\boldsymbol{R}^{m+1}$, $\boldsymbol{R}^{q+1}$ respectively.

A straightforward computation shows that f is pseudoumbilical, if and only if (3.2) holds.

f induces an immersion

$$\tilde{f}\colon S^m(R) \times S^q(R') \to S^{m+q+1}(\sqrt{R^2 + R'^2}). \tag{3.3}$$

H is orthogonal to $S^{m+q+1}(\sqrt{R^2 + R'^2})$, therefore $\tilde{f}$ is a minimal map.

Example 2. This is an example of an immersion of a product of spheres in spheres, for which the second fundamental form is conformal, but, again, the immersion is not isotropic.

Let $S^m(R)$, $S^q(R')$ be represented in $\boldsymbol{R}^{m+1}$, $\boldsymbol{R}^{q+1}$ by the equations

$$\sum_{i=0}^{m} x_i^2 = R^2 \qquad \sum_{\alpha=0}^{q} y_\alpha^2 = R^2$$

and let f_1, f_2 be the canonical immersions of $S^m(R)$, $S^q(R)$ in $\boldsymbol{R}^{m+1}$, $\boldsymbol{R}^{q+1}$, respectively. Let $n_1 = \frac{1}{R}(x_i)$ and $n_2 = \frac{1}{R}(y_\alpha)$ be the unit normal vectors to $S^m(R)$ in $\boldsymbol{R}^{m+1}$ and to $S^q(R)$ in $\boldsymbol{R}^{q+1}$.

Denote by $\qquad f\colon S^m(R) \times S^q(R) \to \boldsymbol{R}^{(m+1)(q+1)}$

the immersion defined by

$$(x_i, y_\alpha) \mapsto \frac{1}{R}(x_i y_\alpha) \qquad i = 0, \dots m \quad \alpha = 0, \dots, q. \tag{3.4}$$

The image of f is contained in the sphere $S^{m+q+mq}(R)$ and (x, y) and $(-x, -y)$ are sent by f to the same point. The orthogonal unit vector at the point $f(x, y)$ to $S^{m+q+mq}(R)$ in $\boldsymbol{R}^{(m+1)(q+1)}$ is

$$N = \frac{1}{R}(x_i y_\alpha). \tag{3.5}$$

It is known [1] that the induced immersion

$$\tilde{f}\colon S^m(R) \times S^q(R) \to S^{m+q+mq}(R), \tag{3.6}$$

is minimal and when $m = q = 1$ is a two fold covering of the Clifford torus.

Now we will show that $\tilde{f}$ has conformal second fundamental form, whereas f has this property only if $m = q = 1$. Let $e_{\bar{i}}$ and $e_{\bar{\alpha}}$ ($\bar{i} = 1, \dots, m$; $\bar{\alpha} = 1, \dots, q$)

be orthonormal bases of $S^m(R)$ and $S^q(R)$ in x and y respectively, then

$$\mathrm{d}f(e_{\bar{i}}, 0) = \frac{1}{R} e_{\bar{i}} y \qquad \mathrm{d}f(0, e_{\bar{\alpha}}) = \frac{1}{R} x e_{\bar{\alpha}}, \tag{3.7}$$

where $e_{\bar{i}} y$ denotes the vector of $\boldsymbol{R}^{(m+1)(q+1)}$ with components the products of the $(m+1)$ components of $e_{\bar{i}}$ in $\boldsymbol{R}^{m+1}$ and the coordinates y_α. xe_α are likewise defined. An orthonormal basis of the normal space of the image of f in $S^{m+q+mq}(R)$ is given by $e_{\bar{i}\bar{\alpha}} = e_{\bar{i}} e_{\bar{\alpha}}$.

By (3.4), (3.5) and (3.7) one recognizes easily that the second fundamental form of the immersion f in $\boldsymbol{R}^{(m+1)(q+1)}$ is

$$\begin{aligned} h((e_{\bar{i}}, 0),(e_{\bar{j}}, 0)) &= \frac{1}{R} h^{f_1}(e_{\bar{i}}, e_{\bar{j}}) y = -\frac{1}{R^2} \delta_{\bar{i}\bar{j}} n_1 y = -\frac{1}{R} \delta_{\bar{i}\bar{j}} N \\ h((0, e_{\bar{\alpha}}),(0, e_{\bar{\beta}})) &= \frac{1}{R} x h^{f_2}(e_{\bar{\alpha}}, e_{\bar{\beta}}) = -\frac{1}{R^2} \delta_{\bar{\alpha}\bar{\beta}} x n_2 = -\frac{1}{R} \delta_{\bar{\alpha}\bar{\beta}} N \\ h((e_{\bar{i}}, 0),(0, e_{\bar{\alpha}})) &= \frac{1}{R} e_{\bar{i}} e_{\bar{\alpha}} = \frac{1}{R} e_{\bar{i}\bar{\alpha}}. \end{aligned} \tag{3.8}$$

As $h^{\tilde{f}}$ is the component of h tangent to $S^{m+q+mq}(R)$, the matrices representing $h^{\tilde{f}}$ are

$$(h^{\tilde{f}}, e_{\bar{i}\bar{\alpha}}) = \frac{1}{R}(E_{\bar{i}, \bar{\alpha}+m} + E_{\bar{\alpha}+m, \bar{i}}), \tag{3.9}$$

where $E_{p,q}$ denotes the matrix having zeroes in all entries except for the entry (p, q) where it has 1.

(3.9) shows that $\tilde{f}$ has conformal second fundamental form. In addition $(h, N) = -R^{-1} I_{m+q}$, which means that the matrices representing the second fundamental form of f with respect to the basis $(e_{\bar{i}\bar{\alpha}}, N)$ are orthogonal. They have the same norm if and only if

$$\frac{m+q}{R^2} = \frac{2}{R^2}, \tag{3.10}$$

i.e. $m = q = 1$; that is: the only case in which f has conformal second fundamental form is that of the 2-covering of the Clifford torus.

We remark that, as an application of Theorem 8, if g is a totally umbilical immersion of $S^{m+q+mq}(R)$ in $S^{(m+1)(q+1)}(R')$, with $R' = R\sqrt{\dfrac{m+q}{m+q-2}}$, then one obtains a map

$$g\tilde{f}\colon S^m(R) \times S^q(R) \to S^{(m+1)(q+1)}(R'),$$

with conformal second fundamental form.

It can be remarked that neither f nor $\tilde{f}$ are isotropic immersions. As a matter of fact, if $X = df(\lambda, \mu)$, then

$$\|h(X, X)\|^2 = \frac{1}{R^2}(\|\lambda\|^2 + \|\mu\|^2)^2 + \frac{4}{R^2}(\|\lambda\|^2\|\mu\|^2) \qquad \|h^{\tilde{f}}(X, X)\|^2 = \frac{4}{R^2}(\|\lambda\|^2\|\mu\|^2).$$

Example 3. The following example is that of the immersions of spheres into spheres, known as Veronese manifolds [6].

We define an immersion of $S^n(R)$ in $S^{n+p}(R')$ where

$$p = \frac{n(n+1)}{2} - 1 \qquad R' = R\sqrt{\frac{n}{2(n+1)}}.$$

Let E be the space of traceless symmetric matrices of order $n+1$, endowed with the usual euclidean norm, and $S^{n+p}(R')$ the sphere with centre in the origin O of E. The map $f\colon S^n(R) \to E$ defined by

$$x \mapsto \frac{1}{R\sqrt{2}}(x_i x_j - \frac{R^2}{n+1}\delta_{ij}) \qquad i, j = 1, \ldots, n+1$$

induces the map $\qquad \tilde{f}\colon S^n(R) \to S^{n+p}(R').$

The immersion $\tilde{f}$ is equivalent to the minimal immersion determined by an orthonormal basis of eigenfunctions corresponding to the second eigenvalue of the Laplacian (cf. [3]). This minimal map is called the second standard minimal immersion of $S^n(R)$ into a sphere. It is known [10] that $\tilde{f}$ is isotropic. Furthermore $\tilde{f}$ is a minimal immersion with maximal codimension and with constant discriminant.

Hence Theorem 6 implies that $\tilde{f}$ has conformal second fundamental form (this result can also be obtained by a direct computation).

Example 4 (K. Sakamoto's examples). K. Sakamoto [10] studied the isotropic immersions with parallel second fundamental form of an m dimensional manifold M in a space form $N^{m+p}(c)$. These immersions are also called *planar geodesic immersions* and they have, in particular, constant isotropy function.

Sakamoto proved that these immersions are pseudoumbilical and that they may be considered as isometric immersions into totally geodesic submanifolds $\overline{N}^{m+r}(c)$ of $N^{m+p}(c)$, where r is the constant rank of the first normal bundle $(r \leqslant \frac{m(m+1)}{2})$.

In Lemma 4.3 of Sakamoto's paper, it is proved that, if $r > 1$ and f: $M^m \to \overline{N}^{m+r}(c)$ is an isotropic immersion with parallel second fundamental form and with isotropy function λ^2, then, with respect to a suitable local orthonormal normal frame field e_α, the following two cases occur for the matrices H_α of the second fundamental form:

(1) if the immersion is minimal, all H_α are orthogonal with common length $\rho^2 = \frac{m}{2}(c - \lambda^2)$;

(2) if the immersion is not minimal and if e_{m+r} is parallel to the mean curvature vector field H, then all H_α are orthogonal with length $\frac{m}{2}(c - \lambda^2 + 2\|H\|^2)$ if $\alpha < m + r$ and $m\|H\|^2$, if $\alpha = m + r$.

In case (1), we get $0 < \lambda^2 < c$, $r \leqslant \frac{m(m+1)}{2} - 1$, and f has conformal second fundamental form.

In case (2), we have $\lambda^2 - c < 2\|H\|^2$ and M turns out to be minimally immersed in a hypersphere $\widetilde{N}^{m+q}(\widetilde{c})$ of $\overline{N}^{m+r}(c)$ with $q = r - 1$ and $\widetilde{c} = c + \|H\|^2$. f has conformal second fundamental form if and only if $c = \lambda^2$. The Veronese surface immersed in S^5 (cfr. Remark following Theorem 8) is an example of this situation. In any case $\widetilde{f}$: $M^n \to \widetilde{N}^{m+q}(\widetilde{c})$ is still isotropic with isotropy function

$$\mu^2 = \lambda^2 - \|H\|^2 \qquad \mu^2 = \frac{q\widetilde{c}}{m+q+2}$$

and it has conformal second fundamental form.

The simply connected spaces, which admit isotropic immersions in spheres with parallel second fundamental form, are the compact symmetric spaces of rank one.

Let $\widetilde{N}^{m+q}(\widetilde{c})$ be a space of constant sectional curvature $\widetilde{c}$ into which a symmetric space of rank one and maximal sectional curvature k is minimally and isotropically immersed with isotropy function μ^2 and conformal second fundamental form. Then, the only possible cases are the following:

$S^m(k)$	$k = \dfrac{m}{2(m+1)}\tilde{c}$	$q = \dfrac{m(m+1)}{2} - 1$	$\mu^2 = \dfrac{m-1}{m+1}\tilde{c}$
$\boldsymbol{CP}^m(k)$	$k = \dfrac{2m}{m+1}\tilde{c}$	$q = m^2 - 1$	$\mu^2 = \dfrac{m-1}{m+1}\tilde{c}$
$\boldsymbol{HP}^m(k)$	$k = \dfrac{2m}{m+1}\tilde{c}$	$q = (m-1)(2m+1)$	$\mu^2 = \dfrac{m-1}{m+1}\tilde{c}$
$\mathrm{Cay}\,\boldsymbol{P}^2(k)$	$k = \dfrac{4}{3}\tilde{c}$	$q = 9$	$\mu^2 = \dfrac{1}{3}\tilde{c}$.

The Veronese manifold of Example 3 coincides with the first case.

These immersions are examples of standard minimal immersions of symmetric R-spaces in spheres (see [8] for the concept of minimal immersion of R-spaces, [5] for symmetric submanifolds). In particular, Ferus and Strübing (see [**11**]), proved that an isometric immersion is symmetric if and only if it has parallel second fundamental form.

From the above remarks, the result of Sakamoto can be restated as follows: *The only minimal symmetric isotropic immersions in spheres are the standard minimal immersions as R-spaces of compact symmetric spaces of rank one.*

These minimal immersions have conformal second fundamental form.

One can remark, however, that they do not give all minimal immersions of R-spaces with conformal second fundamental form. For example the standard immersion of the Grassmannian $O(p+q)/(O(p) \times O(q))$ in the space $\boldsymbol{R}^N$ of the traceless symmetric matrices of type $p+q$ ([8]) gives rise to a minimal immersion in a hypersphere of $\boldsymbol{R}^N$. Some long, but conceptually simple, computations (the method is outlined, for instance, in [4]) shows that, if $p, q > 1$, this minimal immersion in a hypersphere has conformal second fundamental form, if and only if $p = q$ (if either p or q is equal to 1, one reduces to the case of the Veronese manifolds of Example 3).

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Summary

In this paper we point out some relationships between the isotropic immersions in spaces of constant sectional curvature and their submanifolds with conformal second fundamental form. This two notions are essentially equivalent if the dimension of the first normal space is maximal. This analogy does not hold for more general isometric immersions. Furthermore, some significant examples of submanifolds with conformal second fundamental form are given.

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