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S. M. KHURSHEED HAIDER, V. A. KHAN and S. I. HUSAIN (*)

Reduction in codimension of proper mixed foliated semi-invariant submanifold of a Sasakian space form $\overline{M}(-3)$ (**)

1 - Introduction

Semi-invariant submanifolds of a Sasakian manifold have been introduced and extensively studied by A. Bejancu and N. Papaghiuc [3], [4] etc. We call M a *mixed foliated semi-invariant submanifold* if $D \oplus \{\xi\}$ is integrable and $h(X + \xi, Z) = 0$ for each $X \in D$ and $Z \in D^{\perp}$. It is easy to see that, given a Sasakian space form $\overline{M}(c)$ of constant ϕ-holomorphic sectional curvature c, in order that it may admit a mixed foliated proper semi-invariant submanifold, it is necessary that $c \leqslant 1$. This and some other considerations motivate us to study proper mixed foliated semi-invariant submanifods of a Sasakian space form of constant curvature -3. The present paper is mainly concerned with the reduction in the dimension of such an ambient space in which a proper mixed foliated semi-invariant submanifold is immersed.

2 - Preliminaries

Let $\overline{M}$ be a $(2m + 1)$-dimensional *almost contact metric manifod* with structure tensors (ϕ, ξ, η, g) where ϕ is a tensor field of type (1, 1), ξ is a vector field, η

(*) Dept. of Math., Aligarh Muslim Univ., Aligarh 202002, India.
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is a 1-form and g is the Riemannian metric on $\overline{M}$. These tensors satisfy [6]

$$\begin{aligned} &\phi^2 X = -X + \eta(X)\xi \qquad \phi(\xi) = 0 \qquad \eta(\xi) = 1 \qquad \eta(\phi X) = 0 \\ &g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \qquad \eta(X) = g(X, \xi) \end{aligned} \tag{2.1}$$

for any vector fields X, Y tangent to $\overline{M}$. We denote by $\overline{\nabla}$ the Riemannian connection defined by the metric g on $\overline{M}$. It is known that $\overline{M}$ is a *Sasakian manifold* if and only if

$$(\overline{\nabla}_X \phi) Y = g(X, Y)\xi - \eta(Y)X, \qquad \overline{\nabla}_X \xi = -\phi X.$$

Let M be an m-dimensional Riemannian manifold with induced metric g isometrically immersed in $\overline{M}$. We assume that the structure vector field ξ of $\overline{M}$ is *tangent* to M and denote by $\{\xi\}$, the distribution spanned by ξ. Also we denote by TM and $T^{\perp}M$ the tangent and the normal bundles to M respectively.

The submanifold M of the Sasakian manifold $\overline{M}$ is called *semi-invariant* if it is endowed with the pair of distributions $(D, D^{\perp})$ satisfying the following conditions [3]

$TM = D \oplus D^{\perp} \oplus \{\xi\}$, and D, $D^{\perp}$, $\{\xi\}$ are mutually orthogonal,

the distribution D is invariant by ϕ, i.e. $\phi D_x = D_x$ for each $x \in M$,

the distribution $D^{\perp}$ is anti-invariant by ϕ, i.e. $\phi D_x^{\perp} \subset T_x^{\perp} M$ for each $x \in M$.

The semi-invariant submanifold M is called *anti-invariant* submanifold (resp. *invariant* submanifold) if $D = 0$ (resp. $D^{\perp} = 0$). M is called *proper* if neither $D = 0$ nor $D^{\perp} = 0$. It follows that the normal bundle $T^{\perp}M$ splits as $T^{\perp}M = \phi D^{\perp} \oplus u$, where u is the orthogonal complement of $\phi D^{\perp}$ and is a subbundle of $T^{\perp}M$, invariant under ϕ. Assume $\dim D = 2p$ and $\dim D^{\perp} = q \geqslant 2$.

Let ∇ be the Riemannian connection on M, then the *Gauss and Weingarten formulas* are given respectively by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.3}$$

$$\overline{\nabla}_X N = -A_N X + \nabla_X^{\perp} N \tag{2.4}$$

for each N normal to M. h is the second fundamental form and A is related to h by

$$g(A_N X, Y) = g((h(X, Y), N) \tag{2.5}$$

and $\nabla^{\perp}$ denotes the connection in the normal bundle $T^{\perp}M$ of M.

The *equations of Codazzi and Ricci* are respectively by

$$(2.6)\qquad \begin{aligned}[\overline{R}(X, Y)Z]^{\perp} = \nabla_X^{\perp} h(Y, Z) - \nabla_Y^{\perp} h(X, Z)\\ -h(\nabla_X Y, Z) - h(Y, \nabla_X Z) + h(\nabla_Y X, Z) + h(X, \nabla_Y Z)\end{aligned}$$

$$(2.7)\qquad \overline{R}(X, Y, N, N_1) = R^{\perp}(X, Y, N, N_1) - g([A_N, A_{N_1}](X), Y)$$

where $[\;]^{\perp}$ denotes the normal component, $\overline{R}$ and $R^{\perp}$ are the *curvature tensors* associated with $\overline{\nabla}$ and $\nabla^{\perp}$ respectively.

For a submanifold M, the *first normal space* N_x^1 and the *first osculating space* O_x^1 at $x \in M$ are defined by

$$N_x^1 = \{h_x(X_x, Y_x): X_x, Y_x \in T_x M\} \qquad O_x^1 = T_x M \oplus N_x^1$$

where $T_x M$ is the tangent space of M at x. A subspace $\overline{U}$ of $T_x \overline{M}$ is said to define a *Lie-triple system* if $\overline{R}_x(X_x, Y_x) Z_x \in \overline{U}$ for $X_x, Y_x, Z_x \in \overline{U}$. For a Lie triple system $\overline{U}$ in a symmetric space $\overline{M}$, there exists a unique complete totally geodesic submanifold M' of $\overline{M}$ such that $T_x M' = \overline{U}$, [8].

The *curvature tensor* $\overline{R}$ of $\overline{M}(-3)$ is given by [6]

$$(2.8)\qquad \begin{aligned}\overline{R}(X, Y)Z = \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y - g(X, Z)\eta(Y)\xi + g(Y, Z)\eta(X)\xi\\ -g(\phi Y, Z)\phi X + g(\phi X, Z)\phi Y - 2g(X, \phi Y)\phi Z.\end{aligned}$$

The 2-form Ω on M is defined by $\Omega(X, Y) = g(X, \phi Y)$. Ω is skew-symmetric [3], that is

$$(2.9)\qquad g(X, \phi Y) = -g(\phi X, Y)$$

and the covariant derivative of ϕ is defined by

$$(2.10)\qquad (\overline{\nabla}_X \phi) Y = \overline{\nabla}_X \phi Y - \phi(\overline{\nabla}_X Y).$$

The projection morphism of TM to D and $D^{\perp}$ are denoted respectively by P and Q. Using this notation we have

$$(2.11)\qquad X = PX + QX + \eta(X)\xi \qquad \phi N = BN + CN.$$

where $BN \in D^{\perp}$ and $CN \in u$ [4]. The semi-invariant submanifold M is called *D-totally geodesic* if $h(X, Y) = 0$ for each $X, Y \in D$. It is known that the simply connected manifolds of constant curvature are symmetric [9].

It is worth completing this section with the following result.

Proposition 1. *If M is a proper mixed foliated semi-invariant submanifold of a Sasakian space form $\overline{M}(c)$, then $c \leq 1$.*

Proof. We take $X, Y \in D$, $Z \in D^{\perp}$ such that $Z = \phi N$ for $N \in \phi D^{\perp}$. Subtracting (1.14) from (1.15) of [4] remarking that $CN = 0$, we have $\nabla_Y Z = B\nabla_Y^{\perp} N - \phi P A_N Y$. Using this together with $h([X, Y], Z) = 0$ in Codazzi equation, we obtain $[\overline{R}(X, Y)Z]^{\perp} = h(Y, \phi P A_N X) - h(X, \phi P A_N Y)$. Replacing X by ϕY and using $h(Y, A_N Y) = h(\phi Y, A_N \phi Y)$ we finally have

$$[\overline{R}(\phi Y, Y)Z]^{\perp} = 2h(\phi Y, A_N \phi Y).$$

Furthermore, from the curvature equation of the Sasakian space form $\overline{M}(c)$, $[\overline{R}(\phi Y, Y)Z]^{\perp} = \dfrac{1-c}{2} g(X, \phi Y)N$. Hence $h(\phi Y, A_N \phi Y) = \dfrac{1-c}{4} g(X, \phi Y)N$. Taking inner product with N and using the fact that $g(A_N \phi Y, A_N \phi Y) \geq 0$, the assertion immediately follows.

This and some other considerations motivate us to take $c = -3$.

3 - Reduction in codimension

In the present section we study proper mixed foliated semi-invariant submanifolds of a Sasakian space form $\overline{M}(-3)$. First we give some basic lemmas

Lemma 1. *Let M be a mixed foliated semi-invariant submanifold of a Sasakian space form $\overline{M}(-3)$. Then*

$$g(\nabla_U Z, X) = g(\phi A_{\phi Z} U, X) \tag{3.1}$$

for each $X \in D$, $Z \in D^{\perp}$ and U tangent to M.

Proof. $g(\nabla_U Z, X) = -g(Z, \nabla_U X)$. Putting $X = \phi Y$ for $Y \in D$ and using (2.10), (2.2), (2.3) and (2.1) we get

$$g(\nabla_U Z, \phi Y) = g(\phi Z, h(U, Y)) \tag{3.2}$$

which proves our assertion.

Proposition 2. *Let M be a proper mixed foliated semi-invariant submanifold of a Sasakain space from $\overline{M}(-3)$. Then $h(X, Y) \in \phi D^{\perp}$ for each $X, Y \in D$.*

Proof. The equation (2.8) gives

$$[\bar{R}(X, Y)Z]^{\perp} = -2g(X, \phi Y)\phi Z \tag{3.3}$$

for each $X, Y \in D$ and $Z \in D^{\perp}$. Using this in (2.6), we get

$$-2g(X, \phi Y)\phi Z = h(X, \nabla_Y Z) - h(Y, \nabla_X Z) \tag{3.4}$$

where we have used $h([X, Y], Z) = 0$. Taking the inner product with ϕW, where $W \in D^{\perp}$, we get

$$-2g(X, \phi Y)g(\phi Z, \phi W) = g(h(X, \nabla_Y Z), \phi W) - g(h(Y, \nabla_X Z), \phi W).$$

Replacing X by ϕX and using (2.1), (2.5), (3.1) and (3.4) we finally have

$$2g(X, Y)g(Z, W) = g(A_{\phi W}Y, A_{\phi Z}X) + g(A_{\phi W}X, A_{\phi Z}Y). \tag{3.5}$$

Moreover, $R^{\perp}(X, Y)\phi Z \in \phi D^{\perp}$ because $\nabla^{\perp}_X \phi Z \in \phi D^{\perp}$. Therefore for $N \in u$, equation (2.8) together with (2.7) implies that

$$g([A_{\phi Z}, A_N](X), Y) = 0. \tag{3.6}$$

Furthermore, we replace X by ϕX and take the inner product with $N \in u$ in (3.4). Then, using similar techniques as in (3.5), we get

$$g(A_N X, A_{\phi Z}Y) + g(A_N Y, A_{\phi Z}X) = 0. \tag{3.7}$$

Adding (3.6) and (3.7), we have $A_{\phi Z}(A_N X) = 0$, $X \in D$. Clearly for each $X \in D$, $A_N X \in D$, replacing X by $A_N X$ in (3.5) and using the fact that $A_{\phi Z}(A_N X) = A_{\phi W}(A_N X) = 0$, we obtain

$$2g(A_N X, Y)g(Z, W) = 0. \tag{3.8}$$

Since M is proper, therefore $(A_N X, Y) = 0$, from which our assertion follows.

Note. We also observe that $h(\xi, X)$, $h(\xi, \xi)$ and $h(X, Z + \xi)$ belong to $\phi D^{\perp}$ for each $X \in D$ and $Z \in D^{\perp}$.

From equation (3.5) it follows

Corollary 1. *Let M be a proper mixed foliated semi-invariant submanifold of a Sasakin space from $\overline{M}(-3)$. Then*

$$A_{\phi Z}^2 X = X, \qquad X \in D \quad \text{and } Z \text{ is a unit vector in } D^{\perp}$$

$$A_{\phi Z} A_{\phi W} X = -A_{\phi W} A_{\phi Z} X, \qquad X \in D \qquad Z, W \in D^{\perp} \qquad \text{and } Z \perp W.$$

Corollary 2. *There exist no D-totally geodesic proper mixed foliated semi-invariant submanifold of a Sasakian space form $\overline{M}(-3)$.*

Proof. We take $X, Y \in D$ and $Z \in D^{\perp}$. Then putting $W = Z$ in (3.5) we get

$$g(X, Y)\, g(\phi Z, \phi Z) = g(h(X, A_{\phi Z} Y), \phi Z)$$

where we have used (2.1) and (2.5). On the contrary, suppose M is D-totally geodesic, then $g(X, Y)\, g(\phi Z, \phi Z) = 0$, which ensures that either $D = 0$ or $D^{\perp} = 0$, i.e., M is not proper, which is a contradiction. Thus M can not be D-totally geodesic.

Lemma 2. *Let M be a proper mixed foliated semi-invariant submanifold of $\overline{M}(-3)$, satisfying $h(Z, W) \in \phi D^{\perp}$ for $Z, W \in D^{\perp}$. Then $T_x M \oplus \phi D_x^{\perp}$ is the first osculating space at $x \in M$.*

Proof. In order to show that $T_x M \oplus \phi D_x^{\perp}$ is the first osculating space at $x \in M$ it is sufficient to show that

$$\phi D^{\perp} = \{h(X, Y) \colon X, Y \in \mathfrak{X}(M)\}$$

where $\mathfrak{X}(M)$ denotes the set of all vector fields. By the use of Proposition 2 we get $\{h(X, Y) \colon X, Y \in \mathfrak{X}(M)\} \subset \phi D^{\perp}$. In fact equality holds. For otherwise taking a unit vector $\phi Z \in \phi D^{\perp}$ such that $g(h(X, Y), \phi Z) = 0$ for all $X, Y \in \mathfrak{X}(M)$ we get $g(A_{\phi Z} X, Y) = 0$ for all Y. In particular if we take X in D and $Y = A_{\phi Z} X$ then $g(A_{\phi Z} X, A_{\phi Z} X) = 0$ which is impossible by corollary (3.1) unless M is anti-invariant. This completes the proof of the lemma.

We are now in a position to state the main result.

Theorem 1. *Let M be a $(2p + q + 1)$-dimensional proper mixed foliated semi-invariant submanifold of a simply connected Sasakian space form $\overline{M}(-3)$ of dimension n $(n \geqslant 2p + 2q + 1)$ satisfying $h(Z, W) \in \phi D^{\perp}$ for $Z, W \in D^{\perp}$. Then* *there exists a complete totally geodesic invariant submanifold M' of dimension $2p + 2q + 1$ of $\overline{M}$ such that M is a proper mixed foliated semi-invariant submanifold of M'.*

Proof. Using (2.8) and lemma (3.2) it is obvious that the osculating space O_x^1 is a Lie-triple system. Hence by [8] there exists a complete totally geodesic submanifold M' of $\overline{M}(-3)$ of dimension $2p + 2q + 1$. Now, the second fundamental form of M' satisfies the classical equation of Codazzi, and hence, by theorem 3.1 of [4], M' is either an anti-invariant or an invariant submanifold. In our case M' is obviously invariant, In fact M' itself is a Sasakian space form of constant curvature -3, and M is its proper mixed foliated semi-invariant submanifold.

This gives the required reduction.

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Summary

A theorem on the reduction in codimension of a proper mixed foliated semi-invariant submanifold of a Sasakian space form $\overline{M}(-3)$ is established.

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