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G. GANCHEV and S. IVANOV (*)

Characteristic curvatures on complex Riemannian manifolds (**)

1 - Preliminaries

In this paper we study complex Riemannian manifolds with respect to the characteristic connection and its curvature.

Let M be an n-dimensional complex manifold. We denote by (M, J) the manifold considered as a real $2n$-dimensional manifold with the induced complex structure J. The tangential space to (M, J) at $p \in M$ and its complexification are denoted by $T_p M$ and $T_p^c M$, respectively. The algebras of real differentiable vector fields, complex differentiable vector fields and vector fields of type (1, 0) on M are denoted by $\mathcal{X}M$, $\mathcal{X}^c M$ and $\mathcal{X}^{1,0} M$, respectively.

If $z^1, \ldots, z^n$ are holomorphic coordinate functions in a cordinate neighbourhood U in M and $z^\alpha = x^\alpha + iy^\alpha$, $(\alpha = 1, \ldots, n)$ then the complex vector fields

$$Z_\alpha = \frac{\partial}{\partial z^\alpha} = \frac{1}{2}\left(\frac{\partial}{\partial x^\alpha} - i\frac{\partial}{\partial y^\alpha}\right) \qquad \left(\text{resp. } Z_{\bar\alpha} = \frac{\partial}{\partial z^{\bar\alpha}} = \frac{1}{2}\left(\frac{\partial}{\partial x^\alpha} + i\frac{\partial}{\partial y^\alpha}\right)\right)$$

form a basis for $V^{1,0}$ (resp. $V^{0,1}$), where $T_p M = V$, $T_p^c M = V^c = V^{1,0} \oplus V^{0,1}$.

Definition. *A complex Riemannian metric on a complex manifold M* is a covariant symmetric 2-tensor G, which is nondegenerate at each point and

$$G(\overline{Z}, \overline{W}) = \overline{G(Z, W)} \qquad G(JZ, JW) = -G(Z, W) \qquad Z, W \in \mathcal{X}^c M.$$

The second condition is equivalent to $G(Z, \overline{W}) = 0$ for any $Z, W \in \mathcal{X}^{1,0} M$. Thus every complex Riemannian metric is completely determined by its values on $\mathcal{X}^{1,0} M$.

(*) Dept. of Geometry, Faculty of Mathematics and Informatics, Univ. Sofia, bul. Anton Ivanov 5, 1126 Sofia, Bulgaria.

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Let $z^1, \ldots, z^n$ be a local holomorphic coordinate system in M. Unless otherwise stated, Greek indices $\alpha, \beta, \gamma, \ldots$ run from 1 to n, while Latin capitals $A, B, C \ldots$ run throug $1, \ldots, n, \bar{1}, \ldots, \bar{n}$.

In terms of local coordinates we set $G_{AB} = G(Z_A, Z_B)$. Then, the defining conditions for a complex Riemannian metric are

$$G_{\bar{\alpha}\bar{\beta}} = \overline{G_{\alpha\beta}} \qquad G_{\bar{\alpha}\beta} = G_{\alpha\bar{\beta}} = 0 . \tag{1}$$

Given a complex Riemannian metric G on M, we define the *tensor field* $\tilde{G}$ on M by the equality

$$\tilde{G}(Z, W) = G(JZ, W) \qquad Z, W \in \mathcal{X}^c M .$$

It is clear that $\tilde{G}$ is *a complex Riemannian metric on M*. This metric is said to be *associated with* G. In local coordinates

$$\tilde{G}_{\alpha\beta} = iG_{\alpha\beta} \qquad \tilde{G}_{\bar{\alpha}\bar{\beta}} = -iG_{\bar{\alpha}\bar{\beta}} . \tag{2}$$

We will refer to the pair (M, G), where M is a complex manifold and G is a complex Riemannian metric on M, as a *complex Riemannian manifold.*

Complex Riemannian manifolds with holomorphic metric (i.e. $\bar{\partial} G_{\alpha\beta} = 0$ in local coordinates) have been studied in [4], [5], [6], [7], [8]. We call such manifolds *holomorphic* (*complex analytic*) Riemannian manifolds.

Given a complex Riemannian manifold (M, G), the complex Riemannian metric G induces a real pseudo-Riemannian metric g on the manifold (M, J). Thus, every n-dimensional complex Riemannian manifold (M, G) can be considered as a real $2n$-dimensional manifold (M, J, g) with a complex structure J and a metric g of signature (n, n) such that

$$g(JX, JY) = -g(X, Y) \qquad X, Y \in \mathcal{X} M .$$

Such an approach has been used in [1], [2].

We call the triple (M, J, g) the *realization* of (M, G).

2- The characteristic connection

Let (M, G) be a complex Riemannian manifold and ∇ be the Levi-Civita connection of G. The *fundamental tensor* Φ on M is defined by

$$\Phi(Z, W) = \tilde{\nabla}_Z W - \nabla_Z W \qquad Z, W \in \mathcal{X}^c M \tag{3}$$

where $\tilde{\nabla}$ is the Levi-Civita connection of the associated metric $\tilde{G}$.

We denote the fundamental tensor of type (0, 3) by the same letter

$$\Phi(X, Y, Z) = G(\Phi(X, Y), Z) \qquad X, Y, Z \in \mathfrak{X}^c M.$$

In local coordinates we put $\Phi_{AB,C} = \Phi^S_{AB} G_{SC}$. The essential components (which may not be zero) of Φ are

$$\Phi_{\alpha\beta,\bar{\gamma}} = \partial_{\bar{\gamma}} G_{\alpha\beta} \qquad \Phi_{\overline{\alpha\beta},\gamma} = \overline{\Phi_{\alpha\beta,\bar{\gamma}}}. \tag{4}$$

Theorem 1. *On a complex Riemannian manifold* (M, G) *there exists a unique linear connection* D *with components* D^A_{BC} *such that*

$$D^C_{AB} = D^C_{BA} \qquad \textit{i.e. } D \textit{ is symmetric}$$
$$D^{\gamma}_{\alpha\bar{\beta}} = D^{\bar{\gamma}}_{\alpha\beta} = 0 \qquad \textit{i.e. } D \textit{ is almost complex}$$
$$D_\alpha G_{\beta\gamma} = 0.$$

Proof. Existence. We define

$$D^C_{AB} = \Gamma^C_{AB} + \frac{1}{2}\Phi^C_{AB} - \frac{1}{2} G^{CS}(\Phi_{SA,B} + \Phi_{SB,A}) \tag{5}$$

where Γ^C_{AB} are the components of the Levi-Civita connection ∇ of G. By direct computations we check that D satisfies the conditions of the theorem.

Uniqueness. Let D' be another connection satisfying the conditions of Theorem 1. Put $S^C_{AB} = D^C_{AB} - D'^C_{AB}$. Then

$$S^C_{AB} = S^C_{BA} \qquad S^{\bar{\gamma}}_{\alpha\beta} = S^{\gamma}_{\alpha\bar{\beta}} = 0; \qquad S^{\sigma}_{\alpha\beta} G_{\sigma\gamma} + S^{\sigma}_{\alpha\gamma} G_{\sigma\beta} = 0. \tag{6}$$

From these equalities it follows immediately that $S^C_{AB} = 0$. i.e. $D' = D$.

Further we call the linear connection D of Theorem 1 the *characteristic connection* of the complex Riemannian manifold (M, G).

Taking into account the defining equality (5) of the characteristic connection and the properties of the fundamental tensor, we obtain.

Corollary 2. *On a complex Riemannian manifold* (M, G) *there exists a unique linear connection* D *such that*

D *is symmetric*

D *is almost complex*

$D_A G_{BC} = \Phi_{BC,A}$, *i.e. the covariant derivative of the metric* G *is equal to the fundamental tensor* Φ.

Let (M, G) be a complex Riemannian manifold with *characteristic connection* D *and characteristic curvature tensor* K.

From the definition of D we have

$$K(X, Y, Z, U) = -K(Y, X, Z, U)$$

$$K(X, Y)Z + K(Y, Z)X + K(Z, X)Y = 0$$

$$K(X, Y, Z, U) = -K(X, Y, JZ, JU)$$

$$K(\overline{X}, \overline{Y}, \overline{Z}, \overline{U}) = \overline{K(X, Y, Z, U)}$$

for all $X, Y, Z, U \in \mathfrak{X}^c M$.

Using the properties of D we obtain

Proposition 3. *Let* K_{ABCD} *be the components of the characteristic curvature tensor with respect to a holomorphic coordinate system. Then*

$$K_{\alpha\beta\gamma\delta} = G_{\sigma\delta}\{\partial_\alpha \Gamma^\sigma_{\beta\gamma} - \partial_\beta \Gamma^\sigma_{\alpha\gamma} + \Gamma^\lambda_{\beta\gamma}\Gamma^\sigma_{\alpha\lambda} - \Gamma^\lambda_{\alpha\gamma}\Gamma^\sigma_{\beta\lambda}\}$$

$$K_{\bar{\alpha}\beta\gamma\delta} = \partial_{\bar{\alpha}}\Gamma^\sigma_{\beta\gamma}G_{\sigma\delta},$$

$$K_{\bar{\alpha}\bar{\beta}\gamma\delta} = 0.$$

As usual we consider $r_{\alpha\beta} = K_{\lambda\alpha\beta\mu}G^{\lambda\mu}$ and $\tau = G^{\alpha\beta}r_{\alpha\beta}$.

Definition. The manifold (M, G) is said to be a K_1-manifold [3], if the tensor K satisfies the condition

$$K(JX, JY, Z, U) = -K(X, Y, Z, U) \qquad X, Y, Z, U \in \mathfrak{X}^c M$$

Proposition 4. *A complex Riemannian manifold* (M, G) *is a* K_1*-manifold iff the characteristic connection is holomorphic.*

Proof. From Proposition 3 it follows that (M, G) is a K_1-manifold iff $K^\lambda_{\bar{\alpha}\beta\gamma} = \partial_{\bar{\alpha}}D^\lambda_{\beta\gamma} = 0$ i.e. the characteristic connection D is holomorphic.

In order to characterize complex Riemannian manifolds with flat characteristic connection we need another definition

Definition. A complex Riemannian manifold is said to be *anti-holomorphic* if every point has an open neighbourhood, parametrized by holomorphic coordinates such that the components $G_{\alpha\beta}$ of the metric G are anti-holomorphic functions.

Theorem 5. *Let (M, G) be a complex Riemannian manifold. The following conditions are equivalent*

(M, G) is an anti-holomorphic manifold

The characteristic connection D is flat.

Proof. Let (M, G) be an anti holomorphic manifold with local holomorphic parametrization such that $\partial_\alpha G_{\beta\gamma} = 0$. Then the components of D are zero and hence $K = 0$.

To prove the inverse, we need the following

Lemma. *Let (M, G) be a complex Riemannian manifold with flat characteristic connection D. Then there exist local holomorphic coordinates such that the components of D are zero.*

Proof of the Lemma. Let $D^\lambda_{\beta\gamma}$ be the components of D in a holomorphic coordinate system $(w^1, \ldots, w^n)$. To find a holomorphic coordinate system $(z^1, \ldots, z^n)$ in which the components of D are zero, we have to solve the system

$$\frac{\partial^2 z^\sigma}{\partial w^\lambda \partial w^\mu} + D^\sigma_{\beta\gamma} \frac{\partial z^\beta}{\partial w^\lambda} \frac{\partial z^\gamma}{\partial w^\mu} = 0 . \tag{7}$$

The condition $K^\lambda_{\bar{\alpha}\beta\gamma} = 0$ implies that $D^\lambda_{\beta\gamma}$ are holomorphic functions. The integrability condition for the system (7) is $K^\lambda_{\alpha\beta\gamma} = 0$, which proves the assertion.

To complete the proof of the theorem, let $(z^1, \ldots, z^n)$ be a local holomorphic coordinate system as in the Lemma. Then the condition $D_\alpha G_{\beta\gamma} = 0$ reduces to $\partial_\alpha G_{\beta\gamma} = 0$.

3 - Holomorphic characteristic curvature

Let E^c be a nondegenerate complex 2-plane in $V^c = T^c_p M$, i.e. the restriction of G onto E^c has maximal rank. If $\{X, Y\}$ is an arbitrary basis for E^c, the *complex sectional curvature* of E^c with respect to R is defined in the usual manner

$$K(E^c) = \frac{R(X, Y, Y, X)}{\pi_1(X, Y, Y, X)}$$

where π_1 is the tensor of type (0,4) defined by

$$\pi_1(X, Y, Z, U) = G(Z, Y)G(X, U) - G(Z, X)G(Y, U) \qquad X, Y, Z, U \in V^c .$$

Any complex 2-plane in $V^{1,0}$ is said to be a *holomorphic 2-plane.*

Next manifolds we shall consider are described by the following

Definition. A complex Riemannian manifold (M, G) is said to be of *pointwise constant holomorphic characteristic curvature* if $K(E^c, p) = c(p)$ for all non-degenerate holomorphic 2-planes E^c in $V^{1,0} = T_p^{1,0}(M)$.

In this definition sectional curvatures $K(E^c, p)$ are taken with regard to the characteristic curvature tensor K of the manifold.

We have

Proposition A. [3]. *A complex Riemannian manifold (M, G) is of constant holomorphic characteristic curvature iff in local holomorphic coordinates*

$$K_{\alpha\beta\gamma\delta} = c(p)(G_{\beta\gamma}G_{\alpha\delta} - G_{\alpha\gamma}G_{\beta\delta})$$

where $c(p) = \dfrac{\tau}{n(n-1)}$.

For an arbitrary complex Riemannian manifold (M, G) the equality $D_\alpha G_{\beta\gamma} = 0$ implies

Lemma 6. *The local components of the characteristic curvature tensor K satisfy the following identities*

$$D_\alpha K^{\mu}_{\beta\gamma\lambda} + D_\beta K^{\mu}_{\gamma\alpha\lambda} + D_\gamma K^{\mu}_{\alpha\beta\lambda} = 0 \qquad \textit{(Bianchi identity)}$$

$$D_\alpha K^{\alpha}_{\beta\gamma\lambda} = D_\beta r_{\gamma\lambda} - D_\gamma r_{\beta\lambda}$$

$$\partial_\alpha \tau = 2D_\sigma r^{\sigma}_{\alpha} .$$

For a K_1-manifold it follows that $\partial_{\bar\gamma} r_{\alpha\beta} = 0$ i.e. $r_{\alpha\beta}$ *are holomorphic functions.*

Proposition 7. (Schur's type theorem). *Let (M, G) be of pointwise constant holomorphic characteristic curvature $c(p)$. If* $\dim M = n \geqslant 3$, *then $c(p)$ is an anti-holomorphic function.*

Proof. Applying Proposition A and Lemma 6 we obtain $(n-2)\partial_\alpha c = 0$. Hence, by $n \geqslant 3$, $\partial_\alpha c = 0$.

Proposition 8. *Let* (M, G) *be a* K_1*-manifold of pointwise constant characteristic curvature* $c(p)$ *and* $\dim M = n \geqslant 2$. *Then in local holomorphic coordinates*

$$G_{\alpha\beta}\partial_{\bar\gamma}c + c\,\partial_{\bar\gamma}G_{\alpha\beta} = 0\,.$$

Proof. Applying Proposition A and Lemma 6, we obtain

$$r_{\alpha\beta} = (n-1)\,c(p)\,G_{\alpha\beta}\,. \tag{8}$$

By the conditions of the proposition the functions $r_{\alpha\beta}$ are analytic, i.e. $\partial_{\bar\gamma}r_{\alpha\beta} = 0$. Then (8) implies the assertion.

Now we shall prove the main result of this paper.

Theorem 9. (Classification theorem). *Let* (M, G) $(\dim M = n \geqslant 3)$ *be a* K_1*-manifold of pointwise constant holomorphic characteristic curvature* $c(p)$ *which is not identically zero. Then*

i) *If c is constant, then* (M, G) *is a holomorphic Riemannian manifold locally holomorphical isometric to a complex sphere*

ii) *If c is not constant, then* (M, G) *is locally conformal equivalent to the unit complex sphere.*

Proof. i) Applying Proposition 8, we obtain $\partial_{\bar\gamma}G_{\alpha\beta} = 0$ i.e. (M, G) is a holomorphic Riemannian manifold of constant holomorphic sectional curvature. The assertion follows from the following Theorem B [3].

Theorem B. [3]. *Any complex analytic Riemannian manifold of constant holomorphic sectional curvature k is locally holomorphical isometric to a complex sphere if* $k \neq 0$ *or to* $\boldsymbol{C}^n$ *if* $k = 0$.

ii) Since $c(p) \neq 0$ we can consider the metric G' determined by $G'_{\alpha\beta} = c\,G_{\alpha\beta}$. We claim that G' is a holomorphic metric of constant curvature 1. Using the defining equality for G' we compute

$$\partial_{\bar\gamma}G'_{\alpha\beta} = \partial_{\bar\gamma}c \qquad G_{\alpha\beta} + c\,\partial_{\bar\gamma}G_{\alpha\beta}\,.$$

Applying Proposition 8, we obtain that G' is a holomorphic metric. If $\Gamma^\gamma_{\alpha\beta}$ are the local components of the Levi-Civita connection of G' we find $\Gamma^\gamma_{\alpha\beta} = D^\gamma_{\alpha\beta}$. Hence,

$R^{\lambda}_{\alpha\beta\gamma} = K^{\lambda}_{\alpha\beta\gamma}$, where $R^{\lambda}_{\alpha\beta\gamma}$ are the local components of the curvature tensor of G'. From Proposition A it follows that

$$R^{\lambda}_{\alpha\beta\gamma} = G'_{\beta\gamma}\delta^{\lambda}_{\alpha} - G'_{\alpha\gamma}\delta^{\lambda}_{\beta} .$$

Hence, (M, G) is of constant holomorphic sectional curvature 1. Now the statement follows by applying Theorem B.

The above theorem does not include the case $c(p) = 0$, i.e. the flat case $K = 0$.

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Summary

We introduce the characteristic connection on a complex Riemannian manifold. In the class of complex Riemannian manifolds with holomorphic characteristic connection we study manifolds of pointwise constant characteristic holomorphic sectional curvatures and prove a classification theorem.

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