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On thermodynamics and stability problems in linear viscoelasticity (**)

1 - Introduction

In the last decade much research has been performed in the framework of linear viscoelasticity (cf. [4], [6]). In particular, thermodynamic restrictions on the mathematical modelling and well posedness of the quasi-static and dynamic problem have been the subject of a number of papers. As shown repeatedly (cf. [2], ch. 4), the two topics are closely related, though compatibility with thermodynamics does not guarantee, per se, well posedness of the pertinent problems.

Recent papers on the stability for the dynamic problem of linear viscoelasticity seem to indicate that a further analysis of the connection between thermodynamic restrictions, mathematical modelling and stability is in order. While a general connection, if possible, is much too far, this note has three specific purposes. First, to review the statement of the second law of thermodynamics and the relation with a property that is usually called dissipativity. Second, to show that some inequalities, sometimes involved in stability problems, are not a consequence of thermodynamics. Third, to prove that some models are overly restrictive in that make the viscoelastic material be elastic.

To establish the necessary notation (cf. [2]), we consider a viscoelastic solid occupying a region $\mathscr{R} \subset \mathscr{E}^3$ in a reference, stress-free configuration. The motion

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of the solid is described by the displacement $\boldsymbol{u}(\boldsymbol{x}, t)$ of any point $\boldsymbol{x} \in \mathscr{R}$ as a function of the time t. The behaviour of the solid is described by letting the Cauchy stress $\boldsymbol{T}$, at any point $\boldsymbol{x}$, be a linear functional of the (infinitesimal) strain tensor $\boldsymbol{E} = \operatorname{sym} \partial \boldsymbol{u}/\partial \boldsymbol{x}$,

$$\boldsymbol{T}(t) = \boldsymbol{G}_0 \boldsymbol{E}(t) + \int_0^\infty \boldsymbol{G}'(s) \boldsymbol{E}(t-s)\, \mathrm{d}s \tag{1.1}$$

where $\boldsymbol{G}' \in L^1(\boldsymbol{R}^+)$ and the values of $\boldsymbol{G}_0$, $\boldsymbol{G}'$ are fourth-order tensors; $\boldsymbol{G}_0$ is called *the instantaneous elastic modulus*. We denote by $\boldsymbol{E}^t$, on $\boldsymbol{R}^+$, the history of $\boldsymbol{E}$ up to time t. To save writing we denote by $\mathscr{T}(\boldsymbol{E}^t)$ the constitutive functional for $\boldsymbol{T}$. The function

$$\boldsymbol{G}(s) = \boldsymbol{G}_0 + \int_0^s \boldsymbol{G}'(\xi)\, \mathrm{d}\xi$$

is called the *relaxation function*. The possible dependence of $\boldsymbol{G}$ on $\boldsymbol{x}$ is understood and not written. The limit $\boldsymbol{G}_\infty = \lim \boldsymbol{G}(s)$ as $s \to \infty$ exists and is called the *equilibrium elastic modulus*. The fact that the material is a solid is reflected in the inequality

$$\boldsymbol{G}_\infty > 0 . \tag{1.2}$$

Of course, by (1.2) we may understand that $\boldsymbol{G}_\infty$ is symmetric. As shown in a moment, the symmetry is implied by thermodynamics.

2 - Second law and dissipativity

A cycle (for an isothermal, viscoelastic solid) in the time interval $[0, d)$ is a one-parameter family of histories $\boldsymbol{E}^\tau$ such that $\boldsymbol{E}^d = \boldsymbol{E}^0$. The initial history (state) $\boldsymbol{E}^0$ is regarded as known. We let the process $P(\tau)$, $\tau \in [0, d)$, such that

$$\boldsymbol{E}^\tau(s) = \begin{cases} \boldsymbol{E}(0) + \int_0^{\tau - s} P(\xi)\, \mathrm{d}\xi & s \in [0, \tau) \\ \boldsymbol{E}^0(s - \tau) & s \in [\tau, \infty) \end{cases}$$

be a piecewise continuous function on $\boldsymbol{R}^+$. Letting a superposed dot denote (partial) time differentiation, we state the second law by saying that, for any cycle in $[0, d)$, the inequality

$$\int_0^d \mathscr{T}(\boldsymbol{E}^t)\cdot\dot{\boldsymbol{E}}(t)\,\mathrm{d}t \geqslant 0 \tag{2.1}$$

holds. The functional $\mathscr{T}(\boldsymbol{E}^t)$ is said to be *compatible with the second law of thermodynamics* if the inequality (2.1) is identically true.

The functional $\mathscr{T}(\boldsymbol{E}^t)$ is compatible with the second law of thermodynamics if and only if ([2], ch. 3)

$$\boldsymbol{G}_0 = \boldsymbol{G}_0^T \qquad \boldsymbol{G}_\infty = \boldsymbol{G}_\infty^T \tag{2.2}$$

the superscript T denoting transpose, and

$$\int_0^\infty [\boldsymbol{E}_1\cdot\boldsymbol{G}'(s)\boldsymbol{E}_1+\boldsymbol{E}_2\cdot\boldsymbol{G}'(s)\boldsymbol{E}_2]\sin\omega s\,\mathrm{d}s+\int_0^\infty \boldsymbol{E}_1\cdot[\boldsymbol{G}'(s)-\boldsymbol{G}'^T(s)]\boldsymbol{E}_2\cos\omega s\,\mathrm{d}s<0 \tag{2.3}$$

for every ω in the set of strictly positive reals $\boldsymbol{R}^{++}$ and every pair of symmetric tensors $\boldsymbol{E}_1$, $\boldsymbol{E}_2$. In particular (2.3) implies that

$$\boldsymbol{G}_s'(\omega) < 0 \qquad \forall\omega\in\boldsymbol{R}^{++} \tag{2.4}$$

the subscript s denoting the half-range Fourier sine transform. Of course (2.3) and (2.4) are equivalent if $\boldsymbol{G}'(s)$ is a symmetric tensor for any value of s.

Especially in the fifties and sixties, much attention was devoted to a property that, in a sense, was regarded as a condition of thermodynamic character. Such a property is named *dissipativity* and can be stated as follows [3]. The inequality

$$w(\boldsymbol{E}^t) = \int_{-\infty}^t \mathscr{T}(\boldsymbol{E}^\tau)\cdot\dot{\boldsymbol{E}}(\tau)\,\mathrm{d}\tau \geqslant 0 \tag{2.5}$$

holds for any C^1-function $\boldsymbol{E}(t)$, on $\boldsymbol{R}$, starting from $\boldsymbol{E}(-\infty)=0$. In words, work must be done to deform a solid from the undeformed configuration.

The connection between the second law and the dissipativity inequality is established through the following theorem with the additional requirement that $\boldsymbol{G}'$ be symmetric on $\boldsymbol{R}^+$.

Theorem 1. *If the relaxation function* $\boldsymbol{G}$ *is symmetric and satisfies the thermodynamic inequalities* (2.2) *and* (2.4) *then, for any* $\boldsymbol{E}^t\in H^1(\boldsymbol{R}^+)$, *the dissipativity inequality holds.*

Proof. Following [2], p. 56, for any history $E^t \in H^1(R^+)$ we let

$$e(\tau) = \begin{cases} E(\tau) & \tau \leq t \\ E(t) & \tau > t. \end{cases}$$

Moreover, let $\qquad \overline{G}(s) = G(|s|) - G_\infty \qquad s \in R.$

Then by the symmetry of G we have

$$w(E^t) = \frac{1}{2} E(t) \cdot G_\infty E(t) + \frac{1}{2} \int_{-\infty}^{\infty} \dot{e}(u) \cdot \int_{-\infty}^{\infty} \overline{G}(u-s)\, \dot{e}(s)\, ds\, du .$$

Hence by Plancherel's theorem, we have

$$w(E^t) = \frac{1}{2} E(t) \cdot G_\infty E(t) + \frac{1}{4\pi} \int_{-\infty}^{\infty} \dot{e}_F^* \cdot \overline{G}_F(\omega)\, \dot{e}_F(\omega)\, d\omega$$

where the subscript F means Fourier transform and * means complex conjugate. Then, letting the subscript c denote the half-range Fourier cosine transform, we can write

$$w(E^t) = \frac{1}{2} E(t) \cdot G_\infty E(t) + \frac{1}{2\pi} \int_0^{\infty} [\dot{e}_c(\omega) \cdot \overline{G}_c \dot{e}_c(\omega) + \dot{e}_s(\omega) \cdot \overline{G}_c(\omega)\, \dot{e}_s(\omega)]\, d\omega .$$

Because of the identity $\qquad \overline{G}_c(\omega) = -\dfrac{G_s'(\omega)}{\omega}$

by (1.2), (2.2), and (2.4) we have the desired result.

Accordingly, in linear viscoelasticity, the dissipativity is a consequence of thermodynamics.

3 - Assumptions involved in stability problems

Observe that upon the change of variable $s \to \tau = t - s$ in (1.1) we have

$$T(t) = G_0 E(t) + \int_{-\infty}^{t} G'(t-\tau) E(\tau)\, d\tau . \tag{3.1}$$

Suppose that u (and then E) vanishes in $\mathscr{R} \times (-\infty, 0)$. Then (3.1) reduces to

$$T(t) = G_0 E(t) + \int_0^t G'(t-\tau) E(\tau)\, d\tau .$$

This is just the form considered in [1], the only significant difference being that, in [1], the dependence of $\boldsymbol{G}'$ on t, τ is not allowed in the particular, though standard, form $t-\tau$. Moreover $\boldsymbol{G}_0$ is taken to depend on t. Two assumptions are made (in [1]). The first one is that there is $b>0$ such that $\boldsymbol{G}_0(\boldsymbol{x}, t)$ satisfies

$$(3.2)\qquad -\int_{\mathscr{R}} \boldsymbol{E}\cdot\dot{\boldsymbol{G}}_0\boldsymbol{E}\,\mathrm{d}x \geqslant b\int_{\mathscr{R}} \boldsymbol{E}\cdot\boldsymbol{E}\,\mathrm{d}x$$

for any time t and any symmetric, second-order tensor $\boldsymbol{E}$. Incidentally, this condition rules out the possibility of a constant instantaneous elastic modulus. Sometimes (3.2) is introduced as a generalization of an analogous condition in [5] § 36, which is motivated by the use of a time-dependent reference configuration. In fact, Knops and Wilkes investigate elastic solids and prove that if the elasticity coefficients are positive definite and their time-derivatives are negative definite then the null solution is stable with respect to an appropriate measure.

The second assumption is that, for any $\boldsymbol{x}\in\mathscr{R}$ and $t\in\boldsymbol{R}^+$,

$$(3.3)\qquad \int_{\mathscr{R}}\int_0^t \dot{\boldsymbol{E}}(\tau)\cdot\int_0^\tau \boldsymbol{G}'(\tau-s)\boldsymbol{E}(s)\,\mathrm{d}s\,\mathrm{d}\tau\,\mathrm{d}x \geqslant 0$$

for any strain function $\boldsymbol{E}$ on $\mathscr{R}\times\boldsymbol{R}^+$. At first sight, (3.3) resembles the dissipativity condition (2.5) but there are two conceptual differences. First, relative to (2.5), the inequality (3.3) does not involve the *elastic* part $\dot{\boldsymbol{E}}\cdot\boldsymbol{G}_0\boldsymbol{E}$. Second, the assumption (3.3) does not require that the initial value of $\boldsymbol{E}$ vanish while the dissipativity (2.5) does. In words, (2.5) holds only for strain functions starting from the zero value, while (3.3) holds for every initial value. Seemingly, the motivation for (3.2) and (3.3) is of technical character in connection with stability problems. To the author's knowledge, no mechanical or thermodynamic arguments have been given to support (3.2) and (3.3). Indeed, the next Theorem 2 makes a physical motivation of (3.3) quite unlikely.

For simplicity, let the material function $\boldsymbol{G}'$ and the strain $\boldsymbol{E}$ be independent of the position $\boldsymbol{x}\in\mathscr{R}$. Then (3.3) reduces to

$$(3.4)\qquad \int_0^t \dot{\boldsymbol{E}}(\tau)\cdot\int_0^\tau \boldsymbol{G}'(\tau-s)\boldsymbol{E}(s)\,\mathrm{d}s\,\mathrm{d}\tau \geqslant 0\,.$$

Now we derive a consequence of the inequality (3.4).

Theorem 2. *If $G' \in C(\boldsymbol{R}^+)$ then the inequality* (3.4) *holds for any continuous, piecewise differentiable, function $\boldsymbol{E}$ on $\boldsymbol{R}^+$ only if $G' = 0$ on $\boldsymbol{R}^+$.*

Proof. For any $t \in \boldsymbol{R}^{++}$ and $\varepsilon \in (0, t)$ choose the function $\boldsymbol{E}$ on $\boldsymbol{R}^+$ as

$$\boldsymbol{E}(\tau) = \begin{cases} \boldsymbol{A} & \tau \in [0, t-\varepsilon) \\ \boldsymbol{A} + (\tau - t + \varepsilon)\dfrac{\boldsymbol{B}}{\varepsilon} & \tau \in [t-\varepsilon, t) \\ \boldsymbol{A} + \boldsymbol{B} & \tau \in [t, \infty) \end{cases}$$

$\boldsymbol{A}$, $\boldsymbol{B}$ being arbitrary, symmetric, second-order tensors. Substitution in (3.4) yields

$$0 \leqslant \frac{1}{\varepsilon} \int_{t-\varepsilon}^{t} \boldsymbol{B} \cdot \Big[\int_0^{t-\varepsilon} \boldsymbol{G}'(\tau - s)\, \mathrm{d}s \boldsymbol{A} + \int_{t-\varepsilon}^{\tau} \boldsymbol{G}'(\tau - s)\Big(\boldsymbol{A} + \frac{s - t + \varepsilon}{\varepsilon}\boldsymbol{B}\Big)\, \mathrm{d}s \Big]\, \mathrm{d}\tau .$$

By the continuity of $\boldsymbol{G}'$ we have $\boldsymbol{B} \cdot \int_0^t \boldsymbol{G}'(\xi)\, \mathrm{d}\xi \boldsymbol{A} + O(\varepsilon) \geqslant 0$. Since ε can be taken as small as we please, the arbitrariness of $\boldsymbol{A}$ and $\boldsymbol{B}$ yields

$$\int_0^t \boldsymbol{G}'(\xi)\, \mathrm{d}\xi = 0 \qquad \forall t \in \boldsymbol{R}^{++}$$

whence $\boldsymbol{G}'(t) = 0$ $\forall t \in \boldsymbol{R}^+$.

The vanishing of $\boldsymbol{G}'$ on $\boldsymbol{R}^+$ makes the constitutive functional (3.1) reduce to the function $\boldsymbol{T}(t) = \boldsymbol{G}_0 \boldsymbol{E}(t)$ of (anisotropic) elasticity.

Of course, if (3.4) holds only for $\boldsymbol{E}(0) = 0$ (dissipativity) then Theorem 2 does not apply.

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Summary

The paper investigates some features of models of linear viscoelastic solids. Two main results are exhibited. First, the second law of thermodynamics and a dissipativity inequality are shown to be different from an inequality sometimes used in stability problems. Second, such an inequality is proved to be overly restrictive in that makes the viscoelastic solid be elastic.

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